ME 230 Kinematics and Dynamics

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Announcement

A. PowerPoint lecture notes and assignment solutions are now posted in:

http://courses.washington.edu/engr100/me230

B. Help session tomorrow in your assigned Recitation section

C. **Homework** to hand in Wednesday!!

Lecture 3: Particle Kinematics

- **Kinematics of a particle** (Chapter 12)

- 12.7-12.8

Objectives

- Concepts such as position, displacement, velocity and acceleration are introduced
- Study the motion of particles along a straight line. Graphical representation
- Investigation of a particle motion along a curved path. Use of different coordinate systems
- Analysis of dependent motion of two particles
- Principles of relative motion of two particles. Use of translating axis

Material covered

- **Kinematics of a particle**

- Curvilinear motion: Normal & tangential components and cylindrical components

- Next lecture; Absolute dependent motion. Analysis of two particles

…and…Relative motion. Analysis of two particles using translating axis

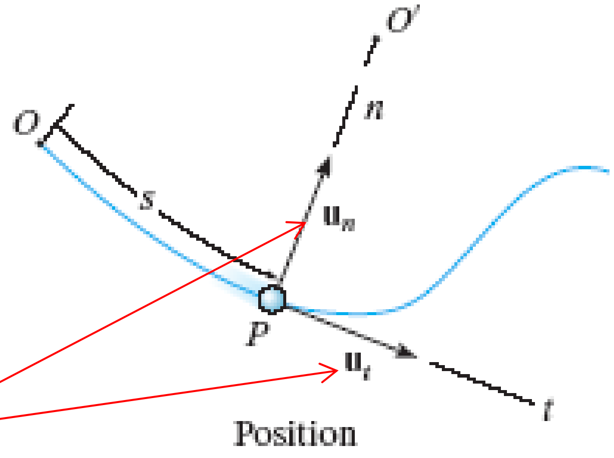
Objectives

<u>Students should be able to</u>:

1. Determine the normal and tangential components of velocity and acceleration of a particle traveling along a curved path .
2. Determine velocity and acceleration components using cylindrical coordinates

Normal and tangential components I

When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian. When the path of motion is known, normal (n) and tangential (t) coordinates are often used

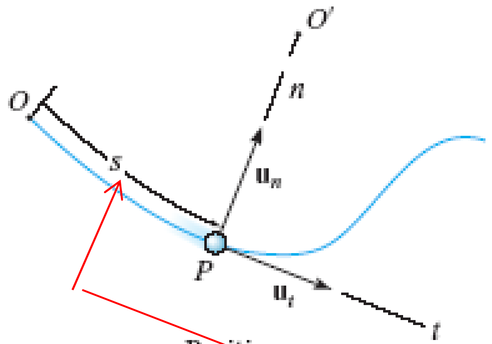


In the n-t coordinate system, the origin is located on the particle (the origin moves with the particle)

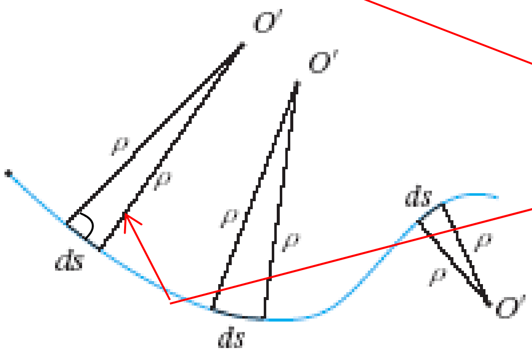
The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle's motion

The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve

Normal and tangential components II



Position



Radius of curvature

The positive n and t directions are defined by the unit vectors $\boldsymbol{u}_n$ and $\boldsymbol{u}_t$, respectively

The center of curvature, O', always lies on the concave side of the curve.

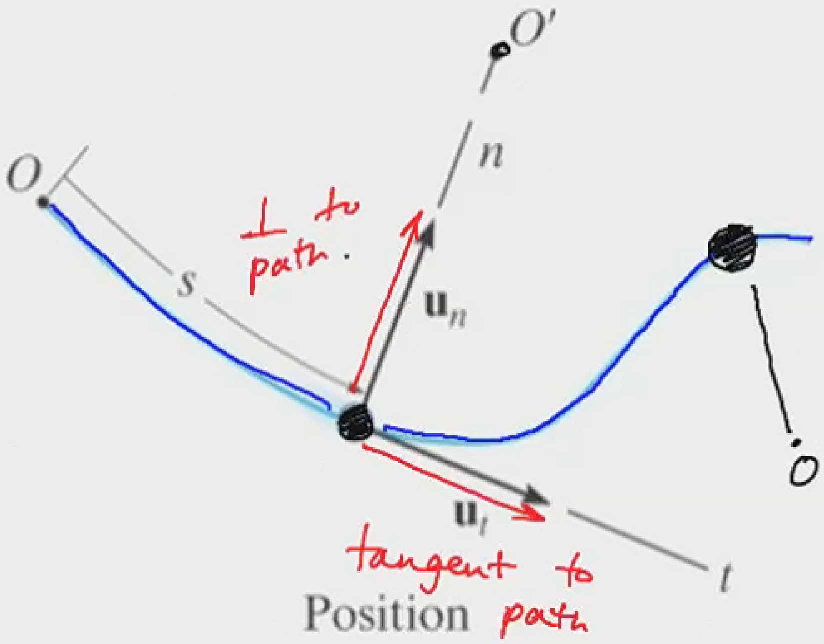
The radius of curvature, ρ, is defined as the perpendicular distance from the curve to the center of curvature at that point

The position of the particle at any instant is defined by the distance, s, along the curve from a fixed reference point (here O).

- if the path traveled by a particle is known, then it may be convenient to use a path coordinate plus normal and tangential vectors to represent kinematic quantities

curved path.

Q: why is it important to know the path?

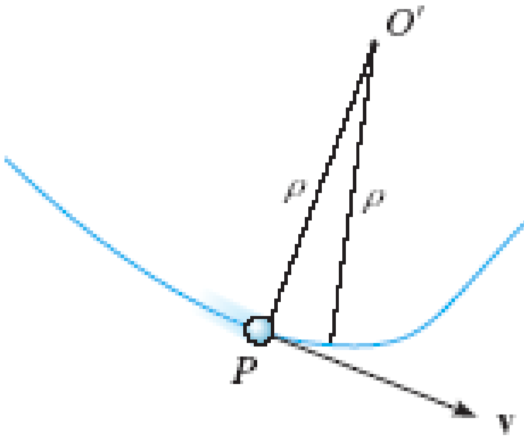


(a)

fig12_24a.jpg

- definitions:
  - O' : center of curvature
  - s : path coordinate
  - u_n } unit vectors in n-t
  - u_t

Velocity in the n-t coordinate system



Velocity

The velocity vector is always tangent to the path of motion (t-direction)

The magnitude is determined by taking the time derivative of the path function, s(t)

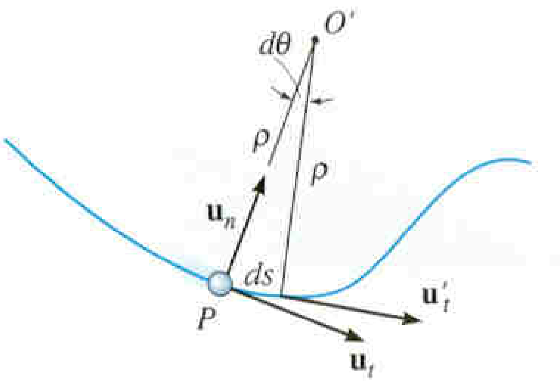
$\boldsymbol{v} = v\boldsymbol{u}_t$ where $v = ds/dt$

Here v defines the magnitude of the velocity (speed) and (unit vector) $\boldsymbol{u}_t$ defines the direction of the velocity vector.

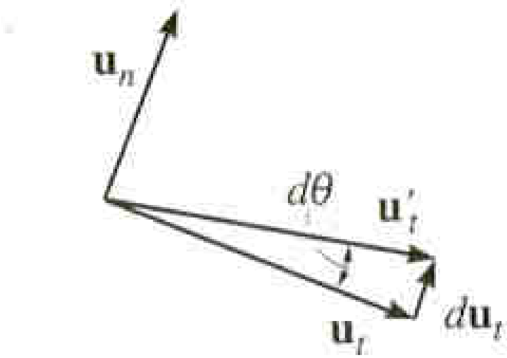
Acceleration in the n-t coordinate system I

Acceleration is the time rate of change of velocity:

$$\boldsymbol{a} = d\boldsymbol{v}/dt = d(v\boldsymbol{u}_t)/dt = \dot{v}\boldsymbol{u}_t + v\,\dot{\boldsymbol{u}}_t$$



Here $\dot{v}$ represents the **change in the magnitude** of velocity and $\dot{\boldsymbol{u}}_t$ represents the rate of **change in the direction** of $\boldsymbol{u}_t$.



After mathematical manipulation, the acceleration vector can be expressed as:

How?

$$\boldsymbol{a} = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n$$

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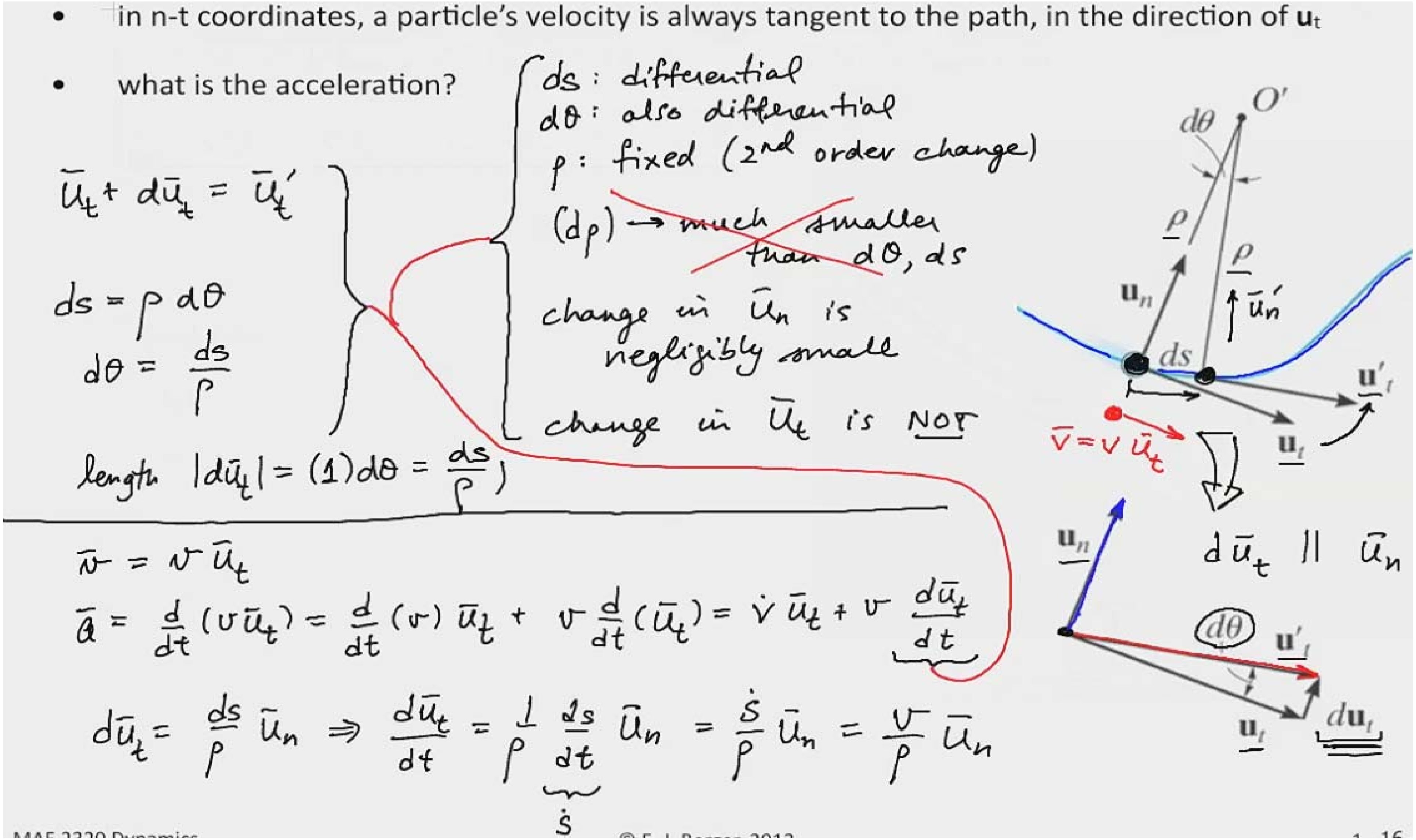
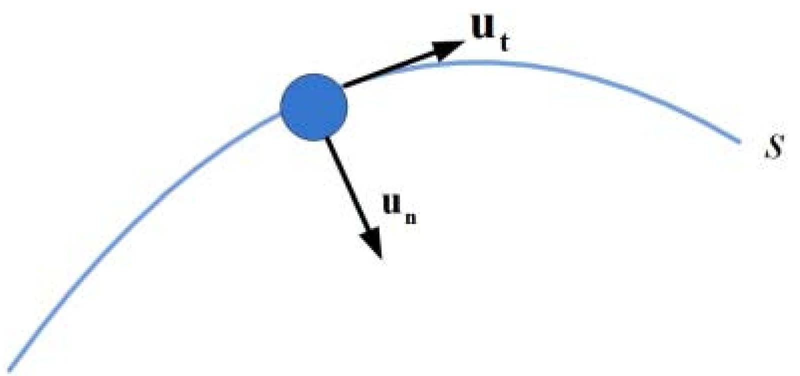


$$\boldsymbol{a} = d\boldsymbol{v}/dt = d(v\boldsymbol{u}_t)/dt = \dot{v}\boldsymbol{u}_t + v\,\dot{\boldsymbol{u}}_t$$

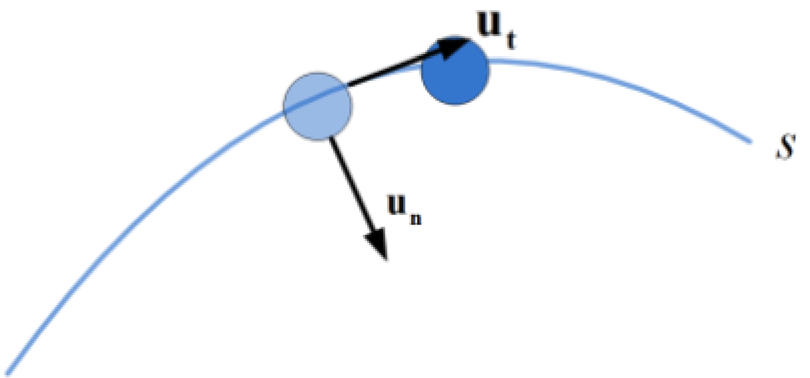
$$\boldsymbol{a} = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n$$

Derivation of tangential and normal acceleration

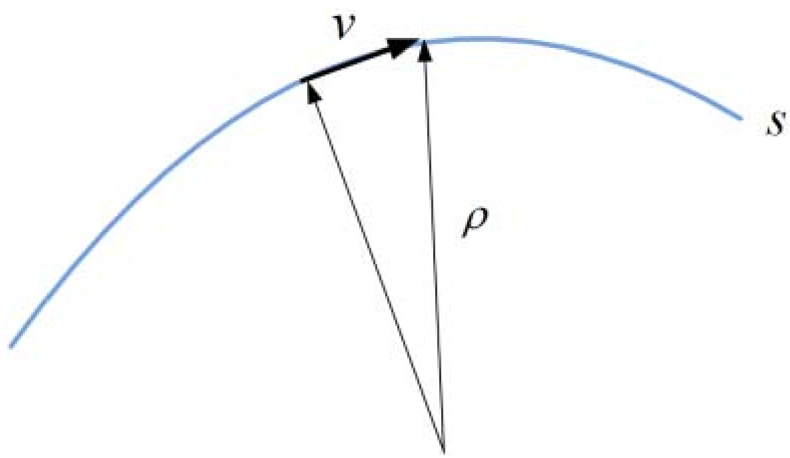
We set up the coordinate system with the tangential unit direction tangent to the direction of motion and the normal direction point towards the center of the curve,



The particle moves from an initial position to a new position,



Derivation of tangential and normal acceleration

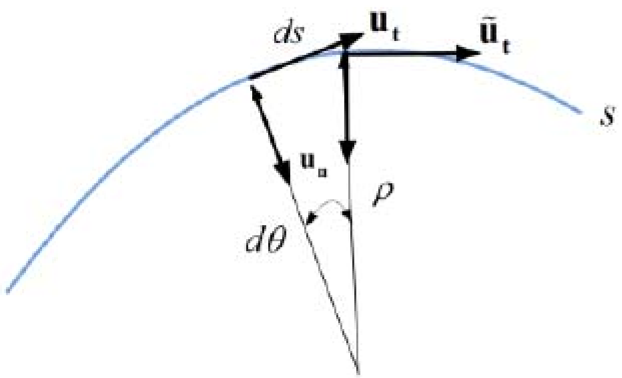


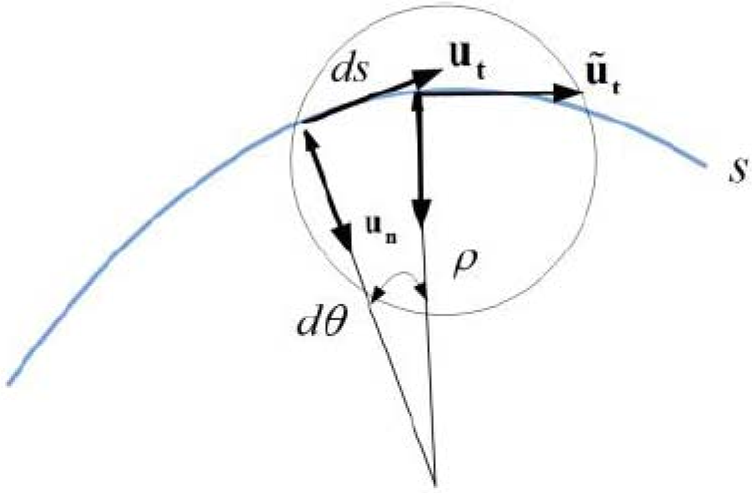
The velocity is the change in position over time, so the velocity is (by definition) tangent to the curve:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\hat{\mathbf{u}}_t$$

$$\therefore \mathbf{v} = v\hat{\mathbf{u}}_t$$

Derivation of tangential and normal acceleration





Acceleration is the time rate of change of velocity. Using the product rule, we get:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} v\hat{\mathbf{u}}_t$$

$$\therefore \mathbf{a} = \dot{v}\hat{\mathbf{u}}_t + v\dot{\hat{\mathbf{u}}}_t$$

??

$\dot{v}$ is the rate at which the speed is increasing or decreasing, or the tangential acceleration:

$$\dot{v} = a_t$$

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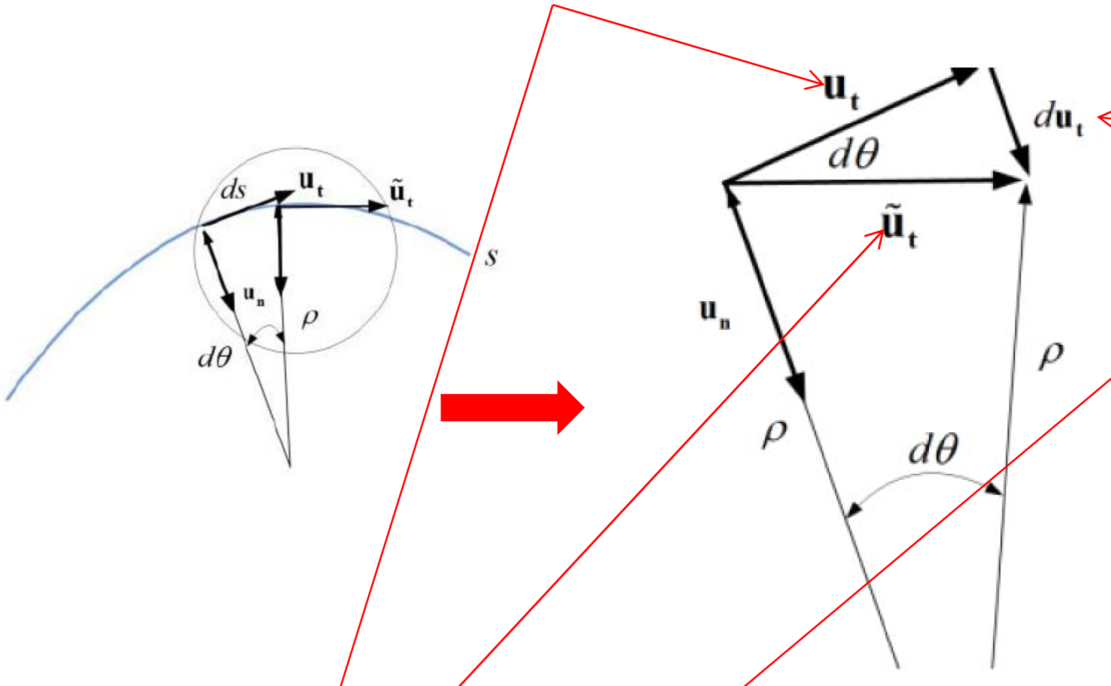
Derivation of tangential and normal acceleration



(c)

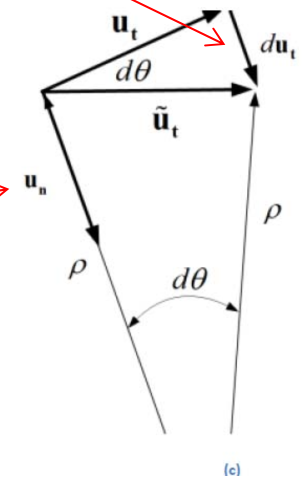
$d\rho$ is small

We also need $\dot{\hat{\mathbf{u}}}_t$. We can determine the differential change in the tangential by looking at the geometry of the curve. Remember, we are approximating the arc length ds by a circle. From Figure and vector addition we can see the new normal tangential direction is:

$$\tilde{\mathbf{u}}_t = \hat{\mathbf{u}}_t + d\hat{\mathbf{u}}_t$$

Derivation of tangential and normal acceleration

We approximate the differential change $d\hat{\mathbf{u}}_t$ by an arc. The length of an arc is $s = r\theta$, but in our case the radius is $|\hat{\mathbf{u}}_t| = 1$. Note the direction of change is the normal direction $\hat{\mathbf{u}}_n$.



$$\therefore d\hat{\mathbf{u}}_t = 1 d\theta \hat{\mathbf{u}}_n$$

We want the differential change with respect to time, therefore we have:

$$\dot{\hat{\mathbf{u}}}_t = \dot{\theta}\hat{\mathbf{u}}_n$$

du_t is going in the same direction as u_n

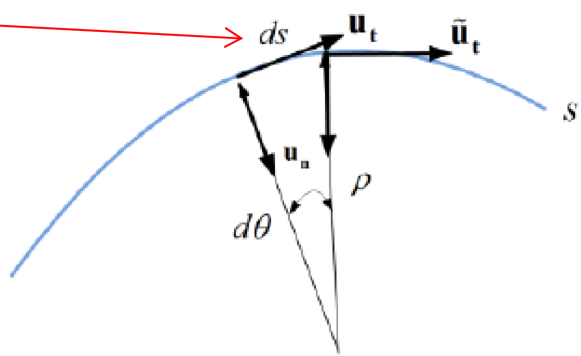
Finally, we have:

$$\mathbf{a} = \dot{v}\hat{\mathbf{u}}_t + v\dot{\theta}\hat{\mathbf{u}}_n$$

du_t, magnitude change ~ $d\theta$

In normal-tangential coordinates, we typically have a path and a velocity, so we want to get ride of $\dot{\theta}$.
We are approximating the curve by a arc:

$$ds = \rho d\theta$$



$$\therefore \frac{ds}{dt} = \rho \frac{d\theta}{dt}$$

$$v = \rho\dot{\theta}$$

Or,

Derivation of tangential and normal acceleration

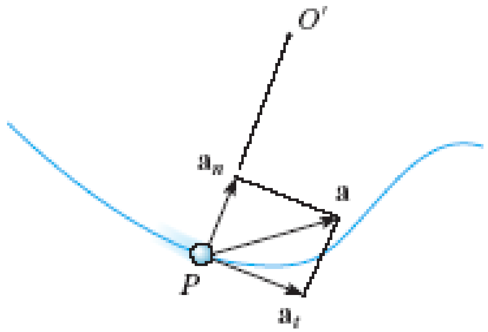
$$\dot{\theta} = \frac{v}{\rho}$$

So acceleration in normal and tangential coordinates is,

$$\mathbf{a} = a_t \mathbf{u_t} + a_n \mathbf{u_n} = \dot{v} \mathbf{u_t} + \frac{v^2}{\rho} \mathbf{u_n}$$

a_n is the centripetal acceleration and a_t is the tangential acceleration (or increase/decrease in speed).

Acceleration in the n-t coordinate system II



Acceleration

There are two components to the acceleration vector:

$$\boldsymbol{a} = a_t \boldsymbol{u}_t + a_n \boldsymbol{u}_n$$

The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.

$$a_t = \dot{v} \quad \text{or} \quad a_t\, ds = v\, dv$$

The normal or centripetal component is **always directed toward the center of curvature** of the curve, $a_n = v^2/\rho$.

The magnitude of the acceleration vector is

$$a = [(a_t)^2 + (a_n)^2]^{0.5}$$

Acceleration in the n-t coordinate system II

The tangential component of acceleration is constant, $a_t = (a_t)_c$.

In this case,

$$s = s_o + v_o t + (1/2)(a_t)_c t^2$$

$$v = v_o + (a_t)_c t$$

$$v^2 = (v_o)^2 + 2(a_t)_c (s - s_o)$$

$$\boldsymbol{a} = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n$$

As before, s_o and v_o are the initial position and velocity of the particle at $t = 0$

Special cases of motion I

There are some special cases of motion to consider

1) The particle moves along a straight line.

$$\rho \sim \infty \quad \Rightarrow \quad a_n = v^2/\rho = 0 \quad \Rightarrow \quad a = a_t = \dot{v}$$

The tangential component represents the time rate of change in the magnitude of the velocity.

2) The particle moves along a curve at constant speed.

$$a_t = \dot{v} = 0 \quad \Rightarrow \quad a = a_n = v^2/\rho$$

The normal component represents the time rate of change in the direction of the velocity.

Special cases of motion II

3) The tangential component of acceleration is constant, $a_t = (a_t)_c$.

In this case,

$$s = s_o + v_o t + (1/2)(a_t)_c t^2$$

$$v = v_o + (a_t)_c t$$

$$v^2 = (v_o)^2 + 2(a_t)_c(s - s_o)$$

As before, s_o and v_o are the initial position and velocity of the particle at $t = 0$

4) The particle moves along a path expressed as $y = f(x)$.
The radius of curvature, ρ, at any point on the path can be calculated from

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{\left| d^2y/dx^2 \right|}$$

Three dimensional motion

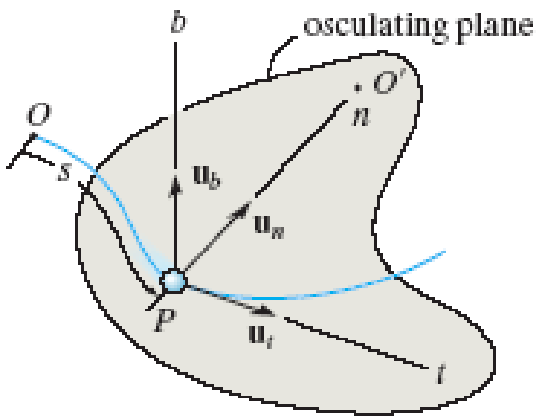


Fig. 12–26

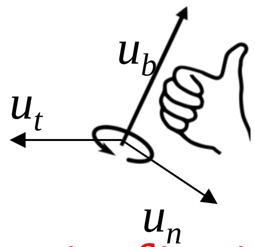
If a particle moves along a space curve, the n and t axes are defined as before. At any point, the t-axis is tangent to the path and the n-axis points toward the center of curvature. The plane containing the n and t axes is called the osculating plane.

A third axis can be defined, called the binomial axis, b. The binomial unit vector, $\boldsymbol{u}_b$, is directed perpendicular to the osculating plane, and its sense is defined by the cross product $\boldsymbol{u}_b = \boldsymbol{u}_t \times \boldsymbol{u}_n$.

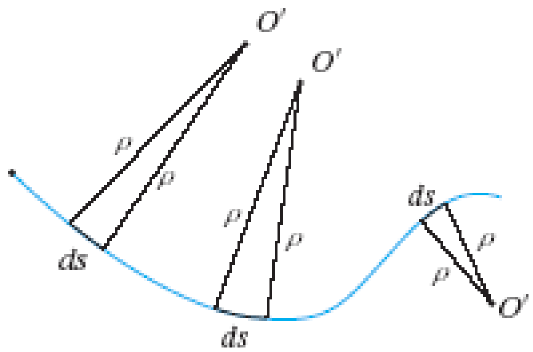
There is no motion, thus no velocity or acceleration, in the binomial direction.

Normal and tangential components III

b= bi-normal direction (perpendicular to both t and n direction through a right hand rule)



- 2-D planar motion, unit vector $\boldsymbol{u}_b$ is fix in space
- 3D motion unit vector $\boldsymbol{u}_b$ is not fix in space



2D, bi-normal direction is perpendicular to the screen

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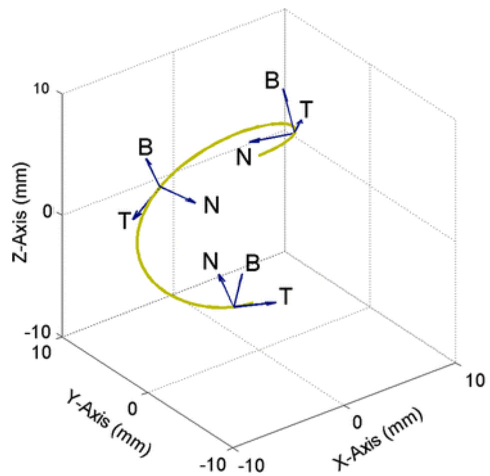
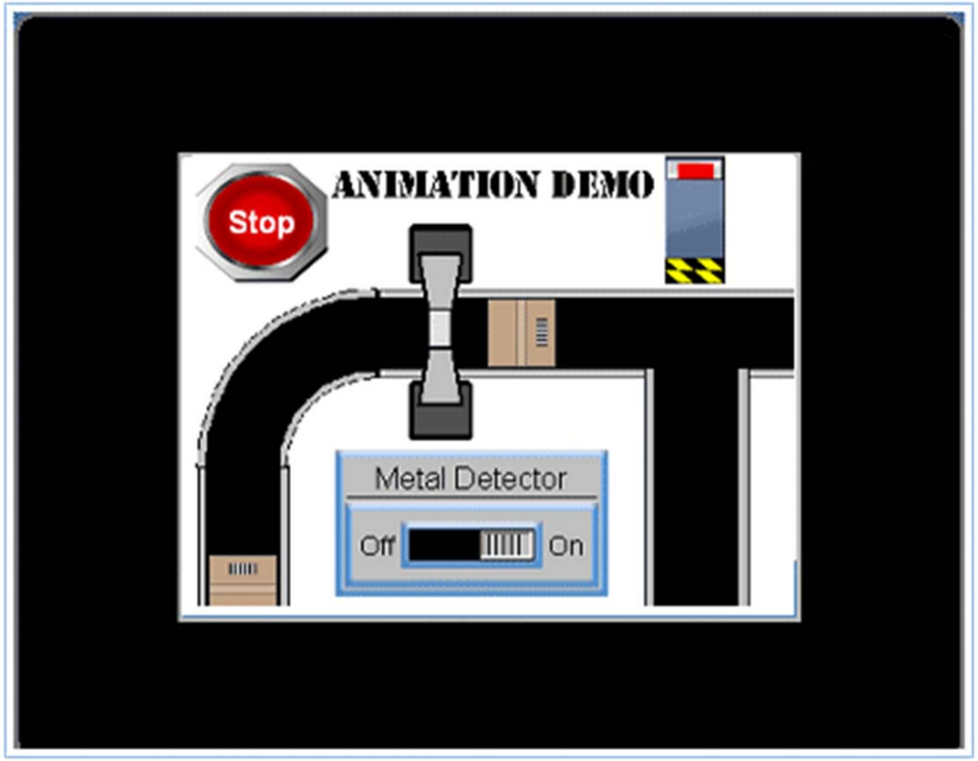


3D

Curvilinear motion:
Normal & tangential components



…Example 12.16 !!!!

http://c-more.automationdirect.com/images/sample_box_conveyor.gif

Curvilinear motion:

Cylindrical components (12.8)

Applications

Slide

The cylindrical coordinate system is used in cases where the particle moves along a 3-D curve.

(spiral motion)

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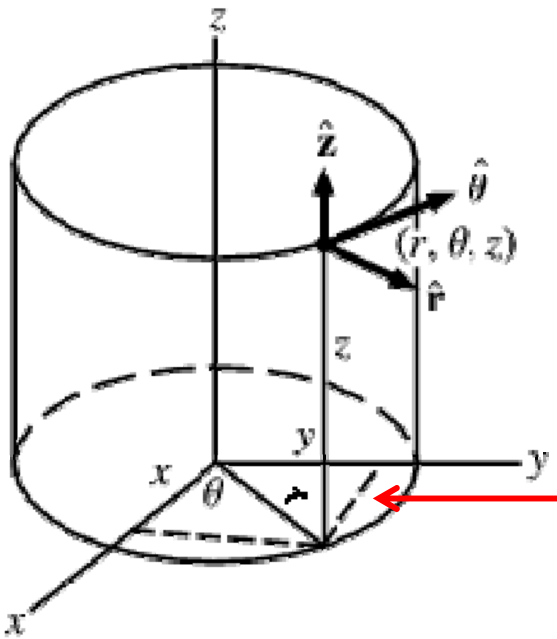
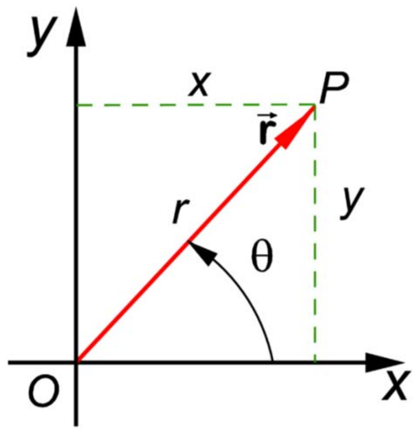
Cylindrical and Polar coordinates



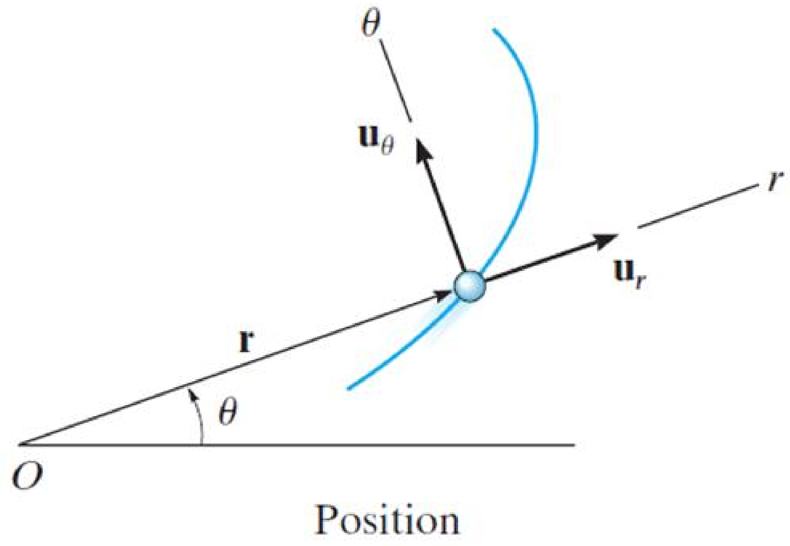
Cylindrical Coordinates



Polar coordinates

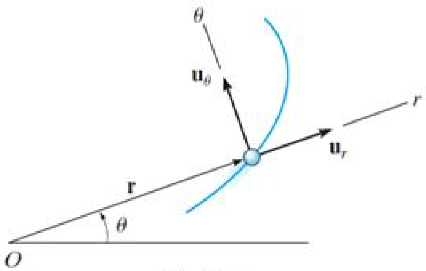
Cylindrical coordinates are a generalization of two-dimensional polar coordinates to three dimensions by superposing a height in (Z) axis.

Cylindrical components



We can express the location of P in polar coordinates as $\boldsymbol{r} = r\boldsymbol{u}_r$. Note that the radial direction, r, extends outward from the fixed origin, O, and the transverse coordinate, θ, is measured counter-clockwise (CCW) from the horizontal.

Velocity (Polar coordinates)



Position

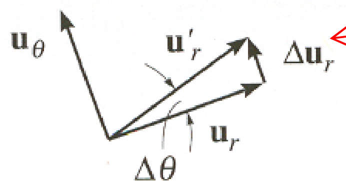
The instantaneous velocity is defined as:

$$\boldsymbol{v} = d\boldsymbol{r}/dt = d(r\boldsymbol{u}_r)/dt$$

$$\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\,\frac{d\boldsymbol{u}_r}{dt}$$

One simple way to look at this is that Δu_r basically going in the same direction as theta $\Delta\theta$ and its magnitude change is cause only by the angle change.

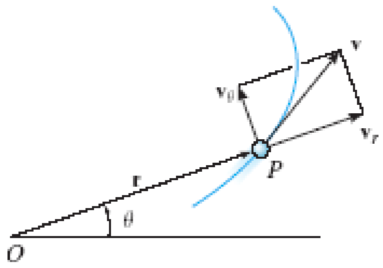
Using the chain rule:



$$d\boldsymbol{u}_r/dt = (d\boldsymbol{u}_r/d\theta)(d\theta/dt)$$

We can prove that $d\boldsymbol{u}_r/d\theta = \boldsymbol{u}_\theta$ so $d\boldsymbol{u}_r/dt = \dot{\theta}\boldsymbol{u}_\theta$

Therefore: $\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$



Velocity

Thus, the velocity vector has two components: $\dot{r}$, called the radial component, and $r\dot{\theta}$, called the transverse component. The speed of the particle at any given instant is the sum of the squares of both components or

$$v = \sqrt{(r\dot{\theta})^2 + (\dot{r})^2}$$

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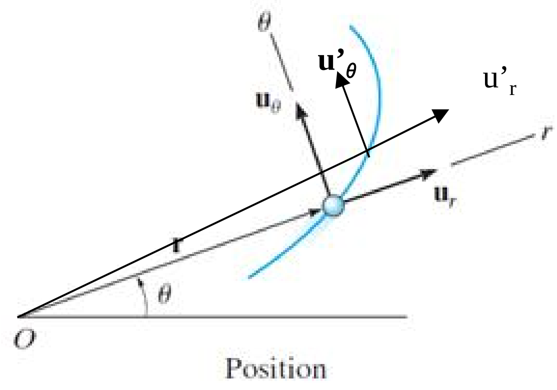
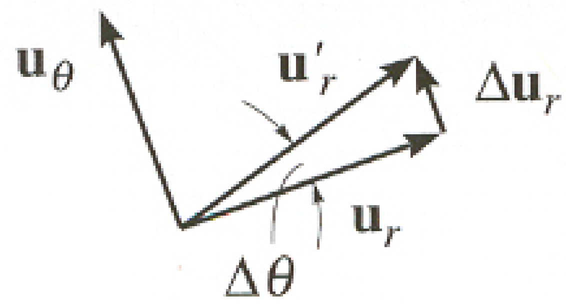


d$\boldsymbol{u}_r$/dt = $\dot{\theta}\boldsymbol{u}_\theta$?



Key points: u_r only change its direction with respect to time. Hence during the time interval Δt, a change of Δr will not cause a change in the direction of $\boldsymbol{u}_r$. However, a change Δθ will cause $\boldsymbol{u}_r$ to change to $\boldsymbol{u'}_r$, where $u'_r = u_r + \Delta u_r$. Figure shows the time change in $\boldsymbol{u}_r$ is Δu_r. For a small changing angle Δθ,
this vector has a magnitude of $\Delta u_r \sim 1(\Delta\theta)$, and going in the $\boldsymbol{u}_\theta$

direction. Therefore $\dot{u}_r = \lim_{\Delta t \to \infty} \frac{\Delta u_r}{\Delta t} = \left(\lim_{\Delta t \to \infty} \frac{\Delta\theta}{\Delta t}\right) = \dot{\theta}\boldsymbol{u}_\theta$

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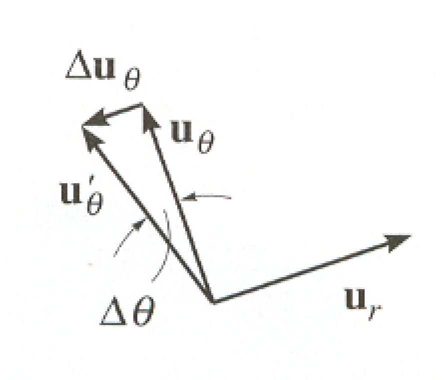
Therefore: $\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$

Acceleration (Polar coordinates)



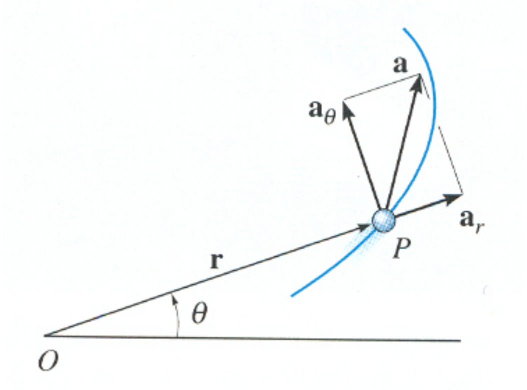
The instantaneous acceleration is defined as:

$$\boldsymbol{a} = d\boldsymbol{v}/dt = (d/dt)(\dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta)$$

After manipulation, the acceleration can be expressed as

How? see Next page

$$\boldsymbol{a} = (\ddot{r} - \underline{r\dot{\theta}^2})\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta$$

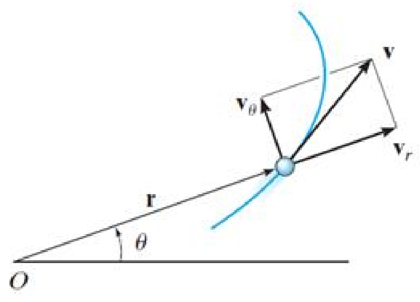


Acceleration

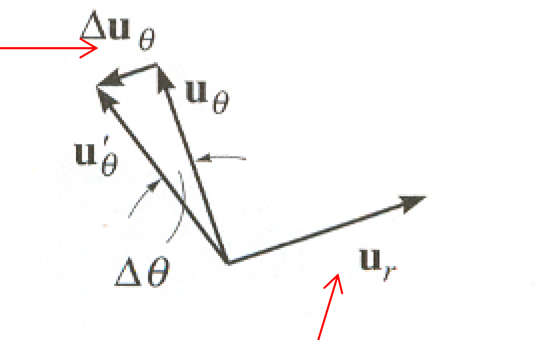
The term $(\ddot{r} - r\dot{\theta}^2)$ is the radial acceleration or a_r.

The term $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the transverse acceleration or a_θ

The magnitude of acceleration is $a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$



Velocity



Velocity vector is defined as, $v = r\dot{u_\theta} + \dot{r}u_\theta$

Therefore, acceleration is $a = \dot{v} = \ddot{r}u_r + \dot{r}\dot{u_r} + \dot{r}\dot{u_r}u_\theta + r\ddot{\theta}u_\theta + r\dot{\theta}\dot{u}_\theta$

Since we know $\dot{u}_r = \dot{\theta}\boldsymbol{u_\theta}$, ***the only unknown in the equation is*** $\dot{u}_\theta$ ***so that's what we need to find.***

Fortunately, $\dot{u}_\theta$ only change its direction with respect to time. Hence during the time interval Δt, a change of Δr will not cause a change in the direction of $\boldsymbol{u_\theta}$. However, a change Δθ will cause $\boldsymbol{u_\theta}$ to change to $\boldsymbol{u'_\theta}$, where $u'_\theta = u_\theta + \Delta u_\theta$. Figure shows the time change in $\boldsymbol{u_\theta}$ is Δu_θ or a small changing angle Δθ, this vector has a magnitude of $\Delta u_\theta \sim 1(\Delta\theta)$, and going in the -$\boldsymbol{u_r}$ direction.

Therefore $\dot{u}_\theta = \lim_{\Delta t \to \infty} \frac{\Delta u_\theta}{\Delta t} = -\left(\lim_{\Delta t \to \infty} \frac{\Delta \theta}{\Delta t}\right) = -\dot{\theta}\ \boldsymbol{u_r}$

$Then\ accleration\ in\ polar\ coordinates$:

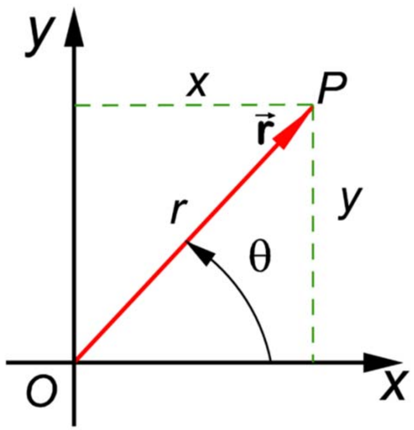
$$a = \dot{v} = \ddot{r}u_r + \dot{r}\dot{u}_r + \dot{r}\dot{\theta}u_\theta + r\ddot{\theta}u_\theta + r\dot{\theta}\dot{u}_\theta$$

$$= \ddot{r}u_r + \dot{r}\dot{\theta}u_\theta + \dot{r}\dot{\theta}u_\theta + r\ddot{\theta}u_\theta - r\dot{\theta}\dot{\theta}u_r$$

$$= (\ddot{r}u_r - r\dot{\theta}\dot{\theta}u_r) + (2\dot{r}\dot{\theta}u_\theta + r\ddot{\theta}u_\theta)$$

$$= (\ddot{r} - r\dot{\theta}\dot{\theta})\ \boldsymbol{u_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\ \boldsymbol{u_\theta}$$

Polar coordinates

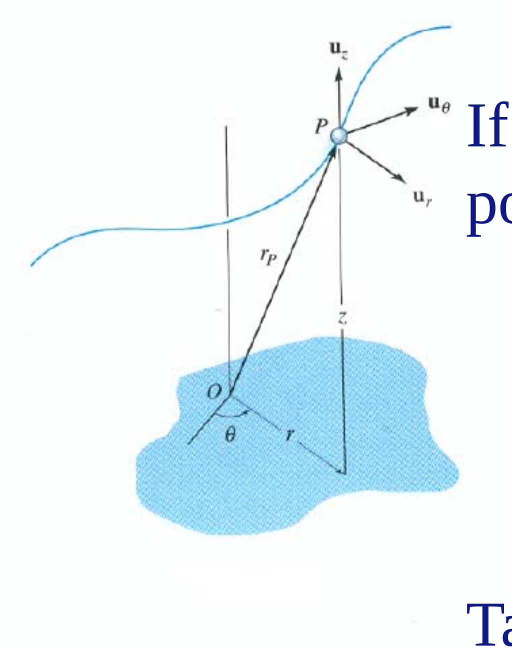


Position $\boldsymbol{r} = r\boldsymbol{u}_r$

Velocity: $\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$

Acceleration: $a = \dot{v} = (\ddot{r} - r\dot{\theta}\dot{\theta})\ \boldsymbol{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\ \boldsymbol{u}_\theta$

Cylindrical coordinates



If the particle P moves along a space curve, its position can be written as

$$\boldsymbol{r_P} = r\boldsymbol{u_r} + z\boldsymbol{u_z}$$

Taking time derivatives and using the chain rule:

Velocity: $\boldsymbol{v_P} = \dot{r}\boldsymbol{u_r} + r\dot{\theta}\boldsymbol{u_\theta} + \dot{z}\boldsymbol{u_z}$

Acceleration: $\boldsymbol{a_P} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u_\theta} + \ddot{z}\boldsymbol{u_z}$

Two Types of Problems

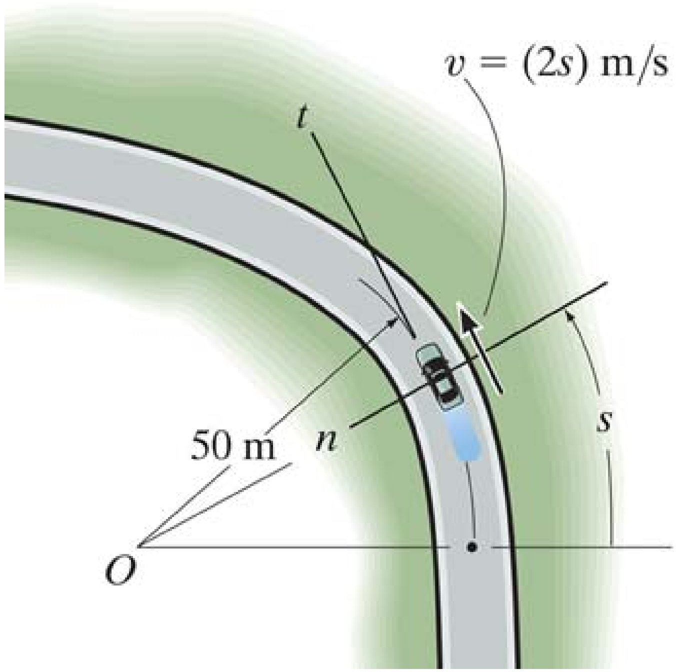
1. Given $\underset{\sim}{r} = \underset{\sim}{r}(t)$. Find velocity & Acceleration.

   $\Longrightarrow$ Differentiation with chain rule.

2. Given $\underset{\sim}{a}(t)$. Find velocity and position.

   $\Longrightarrow$ Integration.

EXAMPLE I



Given: A car travels along the road with a speed of v = (2s) m/s, where s is in meters.

ρ = 50 m

Find: The magnitudes of the car's acceleration at s = 10 m.

Plan:

1) Calculate the velocity when s = 10 m using v(s).
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

Acceleration in the n-t coordinate system II

The tangential component of acceleration is constant, $a_t = (a_t)_c$.

In this case,

$$s = s_o + v_o t + (1/2)(a_t)_c t^2$$

$$v = v_o + (a_t)_c t$$

$$v^2 = (v_o)^2 + 2(a_t)_c(s - s_o)$$

$$\boldsymbol{a} = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n$$

As before, s_o and v_o are the initial position and velocity of the particle at $t = 0$

EXAMPLE I
(continued)

Solution:

1) The velocity vector is $\boldsymbol{v} = v\,\boldsymbol{u}_t$, where the magnitude is given by v = (2s) m/s.

When s = 10 m: v = 20 m/s

2) The acceleration vector is $\mathbf{a} = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n$

Tangential component:

Since $a_t = \dot{v} = dv/dt = (dv/ds)\,(ds/dt) = v\,(dv/ds)$

where v = 2s $\Rightarrow$ $a_t = d(2s)/ds\,(v) = 2\,v$

Use chain rule

At s = 10 m: $a_t = 40\ m/s^2$

Normal component: $a_n = v^2/\rho$

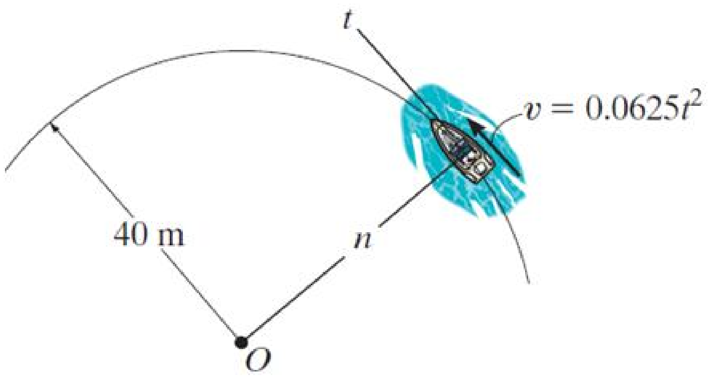
No need to worry about normal acceleration because it's just a function of velocity

When s = 10 m: $a_n = (20)^2 / (50) = 8\ m/s^2$

The magnitude of the acceleration is

$$a = [(a_t)^2 + (a_n)^2]^{0.5} = [(40)^2 + (8)^2]^{0.5} = 40.8\ m/s^2$$

EXAMPLE II



Given: A boat travels around a circular path, ρ = 40 m, at a speed that increases with time, v = (0.0625 t^2) m/s.

Find: The magnitudes of the boat's velocity and acceleration at the instant t = 10 s.

Plan:

The boat starts from rest (v = 0 when t = 0).

1) Calculate the velocity at t = 10 s using v(t).
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

EXAMPLE II
(continued)

Solution:

1) The velocity vector is $\mathbf{v} = v\,\mathbf{u}_t$, where the magnitude is given by v = (0.0625t^2) m/s. At t = 10s:

$$v = 0.0625\,t^2 = 0.0625\,(10)^2 = 6.25 \text{ m/s}$$

2) The acceleration vector is $\mathbf{a} = a_t\mathbf{u}_t + a_n\mathbf{u}_n = \dot{v}\mathbf{u}_t + (v^2/\rho)\mathbf{u}_n$.

Tangential component: $a_t = \dot{v} = d(.0625\,t^2)/dt = 0.125\,t$ m/s^2

At t = 10s: $a_t = 0.125t = 0.125(10) =$ 1.25 m/s^2

Normal component: $a_n = v^2/\rho$ m/s^2

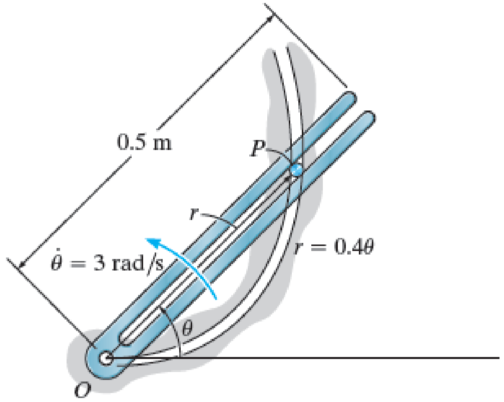
At t = 10s: $a_n = (6.25)^2 / (40) =$ 0.9766 m/s^2

The magnitude of the acceleration is

$a = [(a_t)^2 + (a_n)^2]^{0.5} = [(1.25)^2 + (0.9766)^2]^{0.5} =$ 1.59 m/s^2

EXAMPLE II

The slotted link is pinned at O, and as a result of the constant angular velocity $\dot{\theta} = 3$ rad/s it drives the peg P for a short distance along the spiral guide $r = (0.4\theta)$ m, where θ is in radians. Determine the velocity and acceleration of the particle at the instant it leaves the slot in the link, i.e., when $r = 0.5$ m.



$r = 0.4\,\theta$

$\dot{r} = 0.4\,\dot{\theta}$

$\ddot{r} = 0.4\,\ddot{\theta}$

$\dot{\theta} = 3$

$\ddot{\theta} = 0$

At $r = 0.5$ m,

$$\theta = \frac{0.5}{0.4} = 1.25 \text{ rad}$$

$\dot{r} = 1.20$

$\ddot{r} = 0$

$v_r = \dot{r} = 1.20$ m/s

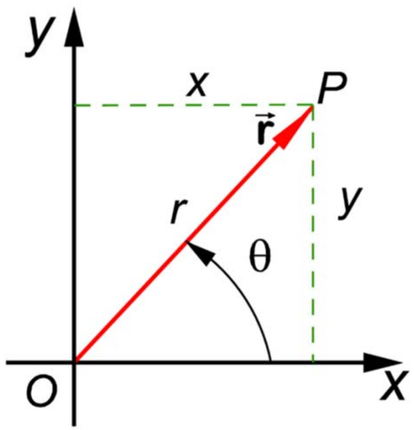
$v_\theta = r\,\dot{\theta} = 0.5(3) = 1.50$ m/s

$a_r = \ddot{r} - r(\dot{\theta})^2 = 0 - 0.5(3)^2 = -4.50 \text{ m/s}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 + 2(1.20)(3) = 7.20 \text{ m/s}^2$

Magnitude of the velocity and acceleration vectors : $v = [(v_r)^2 + (v_\theta)^2]^{0.5}$ and $a = [(a_r)^2 + (a_\theta)^2]^{0.5}$

Polar coordinates



Position $\boldsymbol{r} = r\boldsymbol{u}_r$

Velocity: $\boldsymbol{v} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta$

Acceleration: $a = \dot{v} = (\ddot{r} - r\dot{\theta}\dot{\theta})\ \boldsymbol{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\ \boldsymbol{u}_\theta$

Homework Assignment

Chapter 12: 10, 22, 24, 26, 28, 32, 37, 62, 71, 92, 98, 112, 120, 122, 144, 163, 175, 179

Due Wednesday !!!

W. Wang

Lecture 4: Particle Kinematics

- **Kinematics of a particle** (Chapter 12)

- 12.9-12.10

W. Wang

Kinematics of a particle: Objectives

- Concepts such as position, displacement, velocity and acceleration are introduced
- Study the motion of particles along a straight line. Graphical representation
- Investigation of a particle motion along a curved path. Use of different coordinate systems
- Analysis of dependent motion of two particles
- Principles of relative motion of two particles. Use of translating axis

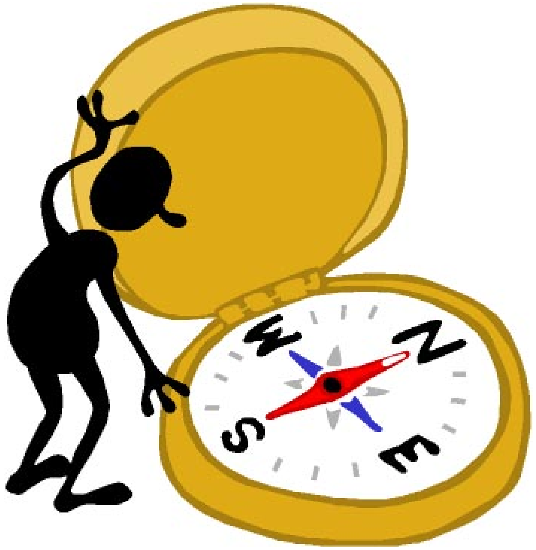
Material covered

- **Kinematics of a particle**

- Absolute dependent motion analysis of two particles

- Relative motion analysis of two particles using translating axis

- Next lecture; Starting Chapter 13…

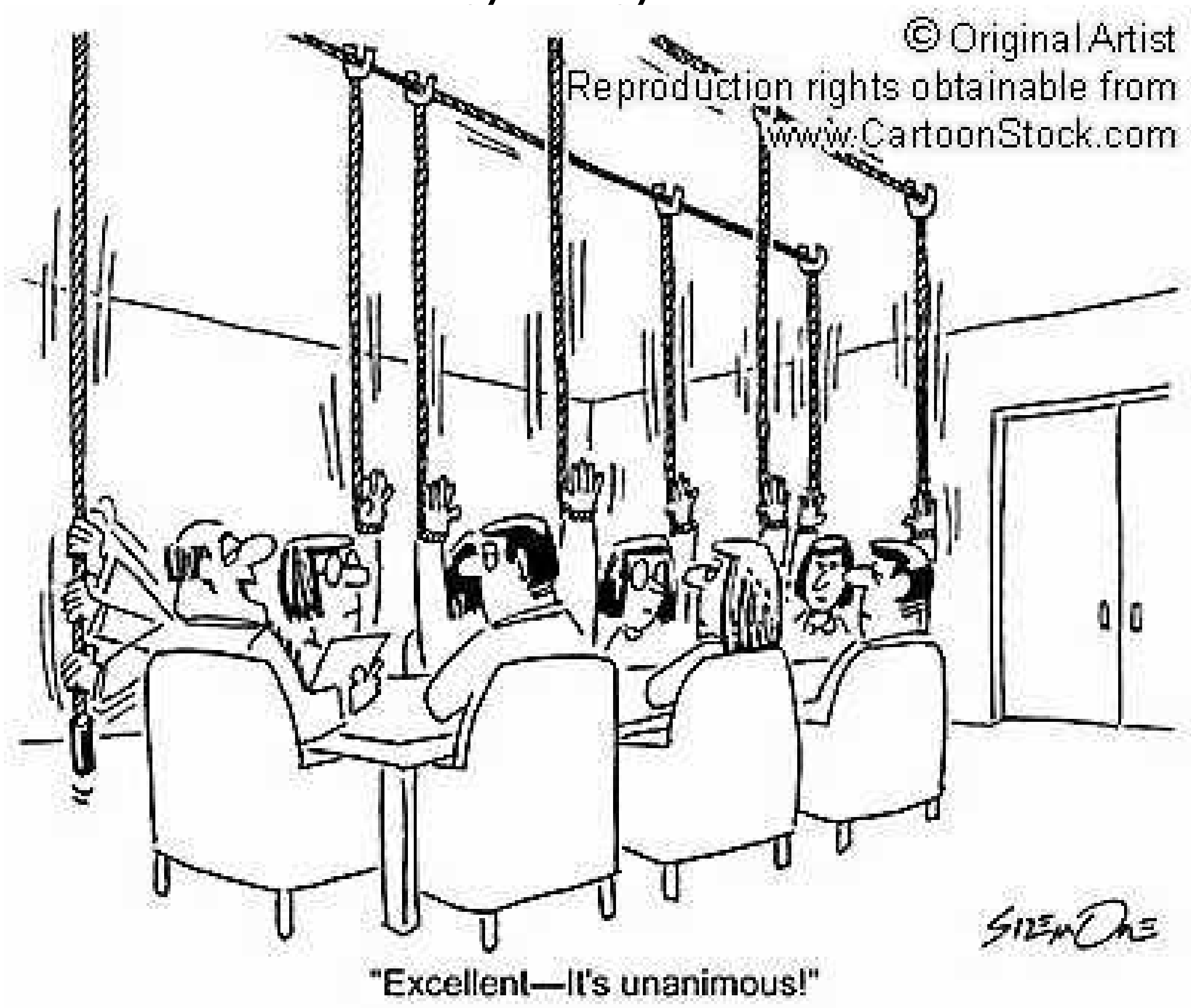


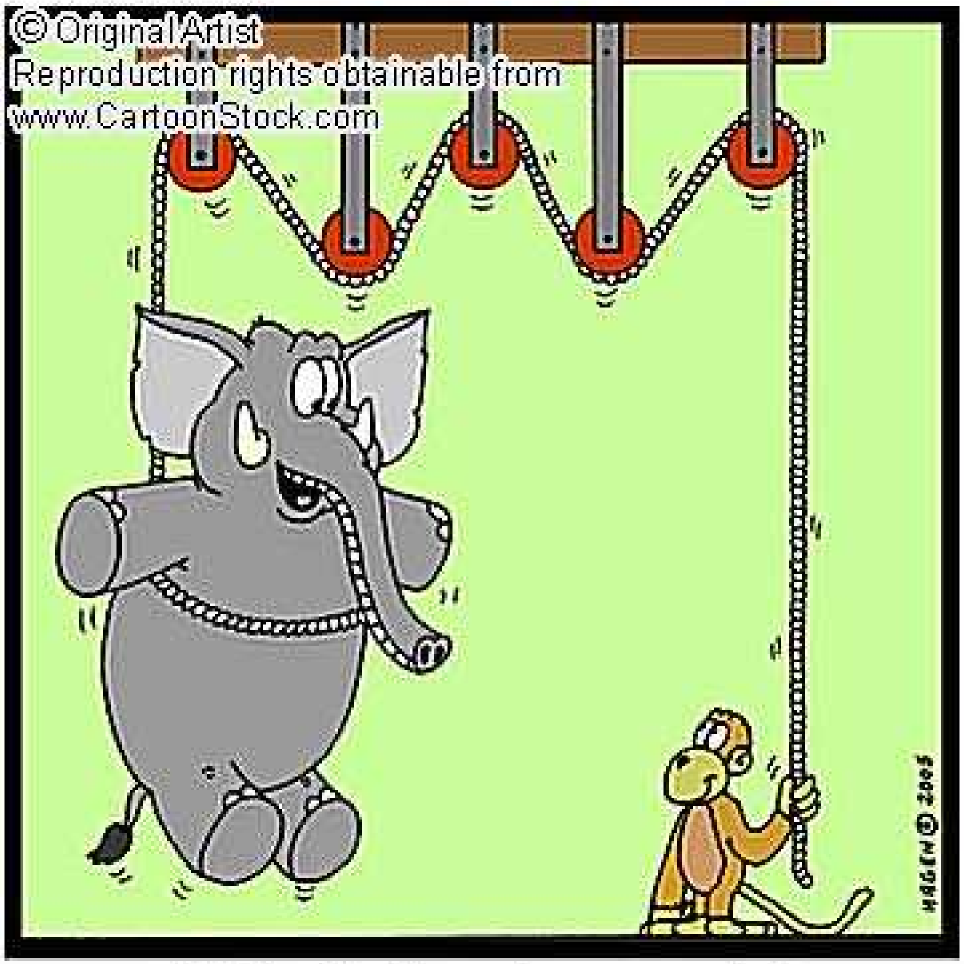
Objectives

Students should be able to:

1. Relate the positions, velocities, and accelerations of particles undergoing dependent motion
2. Understand translating frames of reference
3. Use translating frames of reference to analyze relative motion

Pulley Systems

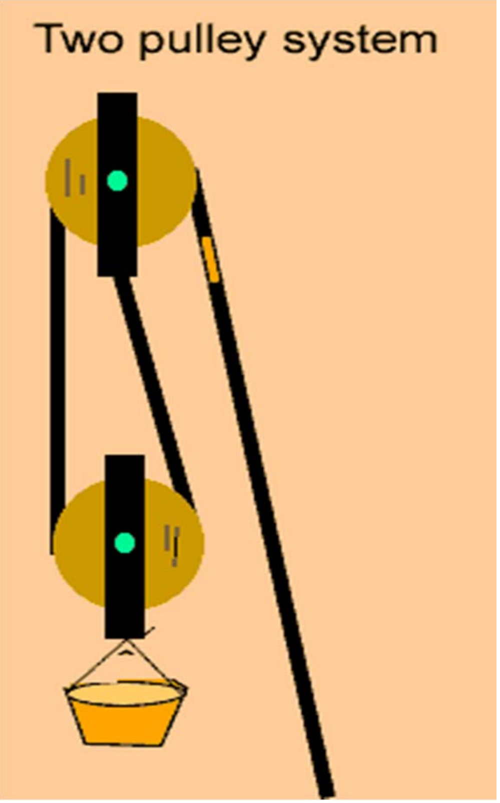




Alright, alright, you've won your bet:
You can lift me with one hand...

Pulley

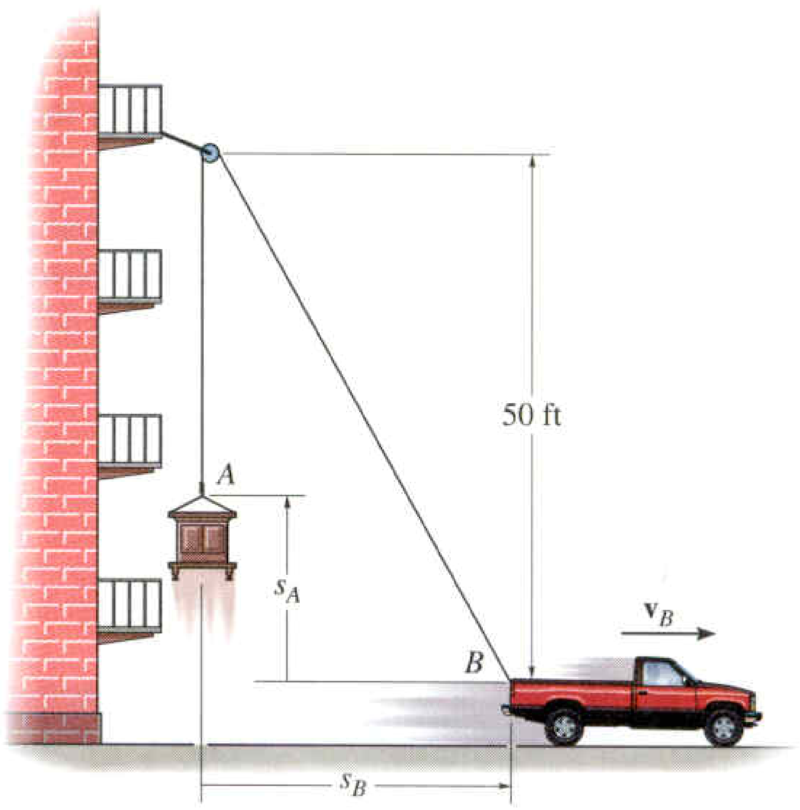
- The more pulleys we have the easier it is to lift heavy objects. As rope is pulled from the top pulley wheel, the load and the bottom pulley wheel are lifted. If 2 metres of rope are pulled through the bucket (load) will only rise 1 metre (there are two ropes holding the bucket and both have to shorten).



Applications I

The cable and pulley system shown here can be used to modify the speed of block B relative to the speed of the motor. It is important to relate the various motions in order to determine the power requirements for the motor and the tension in the cable

Applications II

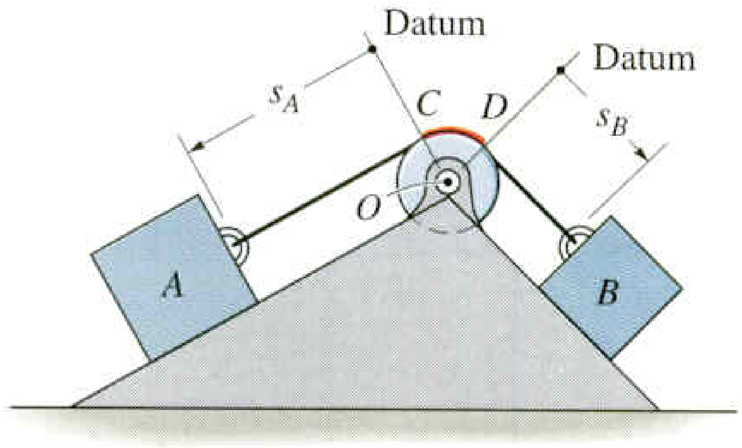


Rope and pulley arrangements are often used to assist in lifting heavy objects. The total lifting force required from the truck depends on the acceleration of the cabinet

Dependent motion

In many kinematics problems, the motion of one object will depend on the motion of another object

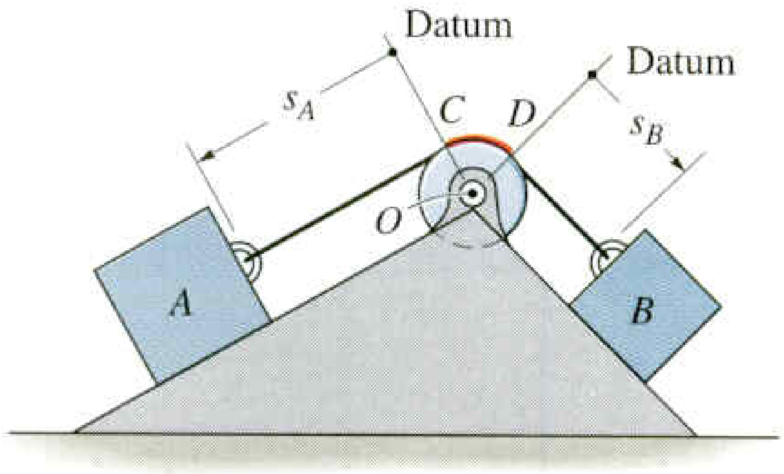
Datum- a fixed starting point



The blocks in this figure are connected by an inextensible cord wrapped around a pulley. If block A moves downward along the inclined plane, block B will move up the other incline

The motion of each block can be related mathematically by defining position coordinates, s_A and s_B. Each coordinate axis is defined from a fixed point or datum line, measured positive along each plane in the direction of motion of each block

Dependent motion (continued)



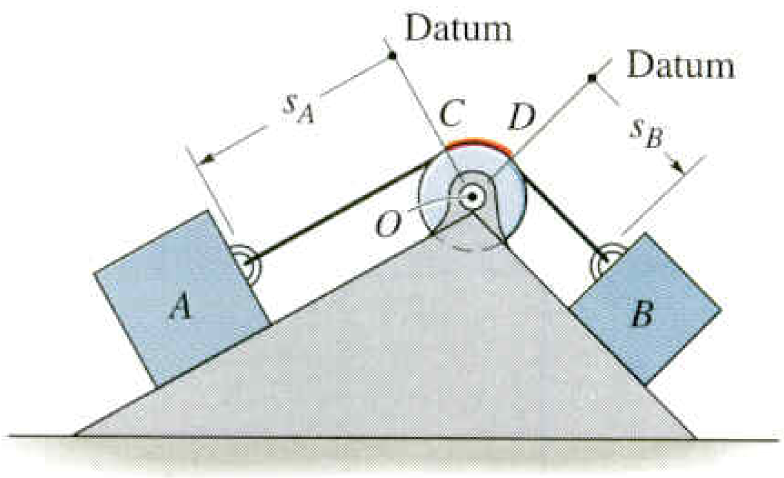
In this example, position coordinates s_A and s_B can be defined from fixed datum lines extending from the center of the pulley along each incline to blocks A and B

If the cord has a fixed length, the position coordinates s_A and s_B are related mathematically by the equation

$$s_A + l_{CD} + s_B = l_T$$

Here l_T is the total cord length and l_{CD} is the length of cord passing over arc CD on the pulley

Dependent motion (continued)



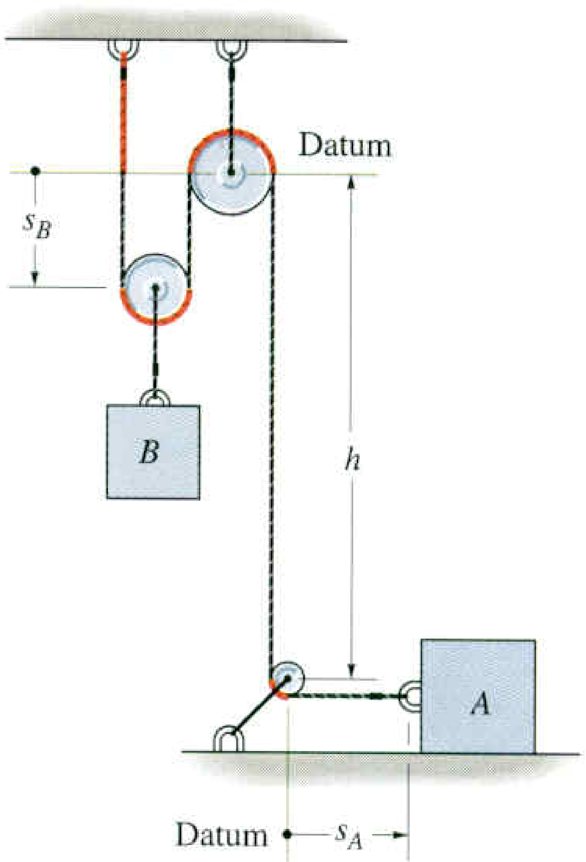
The velocities of blocks A and B can be related by differentiating the position equation. Note that l_{CD} and l_T remain constant, so $dl_{CD}/dt = dl_T/dt = 0$

$$ds_A/dt + ds_B/dt = 0 \quad \Rightarrow \quad v_B = -v_A$$

The negative sign indicates that as A moves down the incline (positive s_A direction), B moves up the incline (negative s_B direction)

Accelerations can be found by differentiating the velocity expression

Example

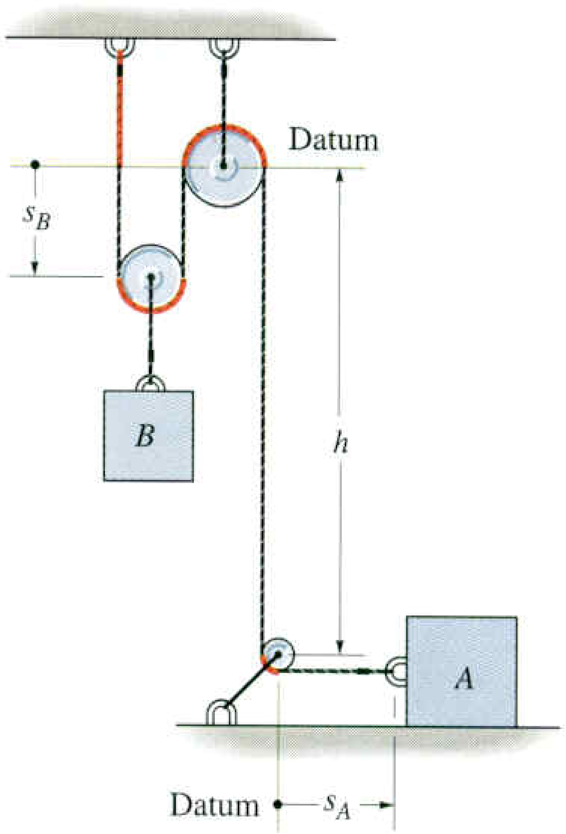


Consider a more complicated example. Position coordinates (s_A and s_B) are defined from fixed datum lines, measured along the direction of motion of each block

Note that s_B is only defined to the center of the pulley above block B, since this block moves with the pulley. Also, h is a constant

The red colored segments of the cord remain constant in length during motion of the blocks

Example (continued)



The position coordinates are related by the equation

$2s_B + h + s_A = l$

Where l is the total cord length minus the lengths of the red segments

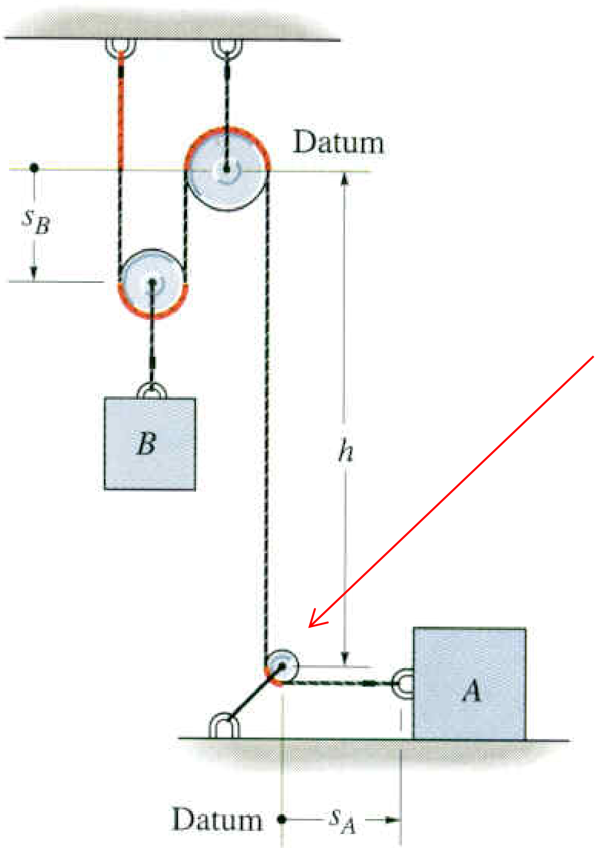
Since l and h remain constant during the motion, the velocities and accelerations can be related by two successive time derivatives:

$2v_B = -v_A$ and $2a_B = -a_A$

When block B moves downward ($+s_B$), block A moves to the left ($-s_A$).

W. Wang Remember to be consistent with the sign convention!

Example (continued)



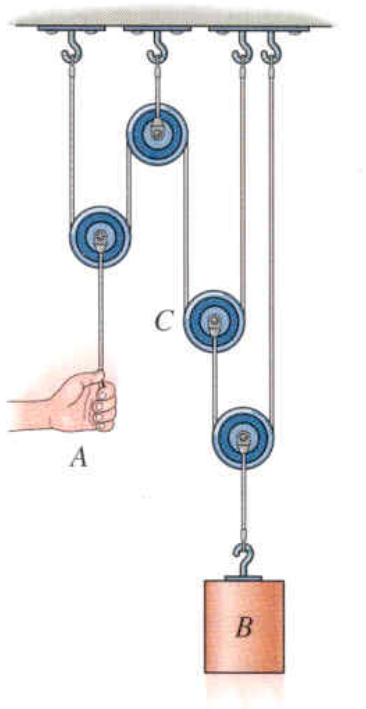
This example can also be worked by defining the position coordinate for B (s_B) from the bottom pulley instead of the top pulley

The position, velocity, and acceleration relations then become

$$2(h - s_B) + h + s_A = l$$

and $2v_B = v_A$ $2a_B = a_A$

Example



Given: In the figure on the left, the cord at A is pulled down with a speed of 8 ft/s

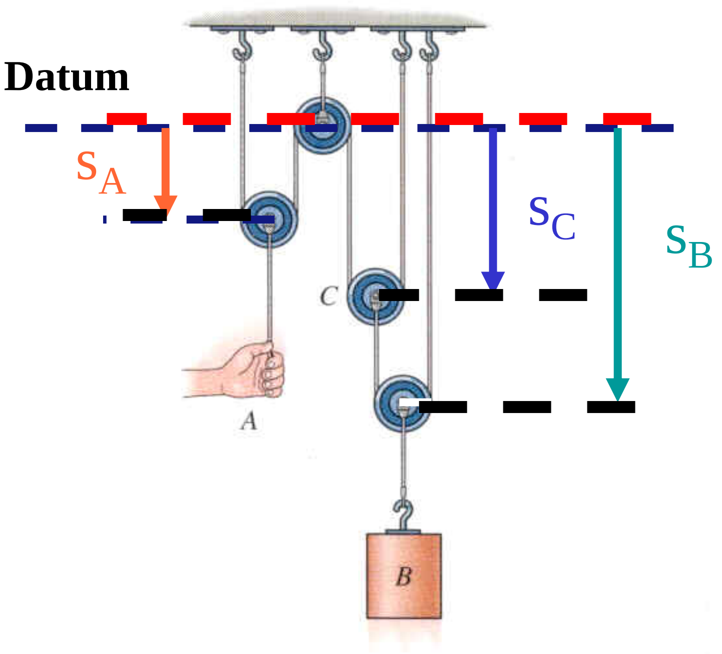
Find: The speed of block B

Plan: There are two cords involved in the motion in this example. The position of a point on one cord must be related to the position of a point on the other cord. There will be two position equations (one for each cord)

Example (continued)

Solution:

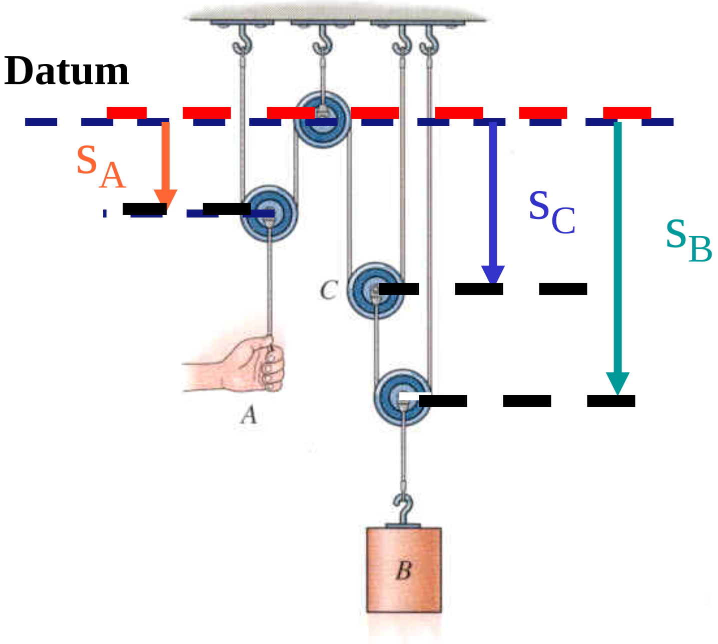
1) Define the position coordinates from a fixed datum line. Three coordinates must be defined: one for point A (s_A), one for block B (s_B), and one relating positions on the two cords. Note that pulley C relates the motion of the two cords



- Define the datum line through the top pulley (which has a fixed position).
- s_A can be defined to the center of the pulley above point A.
- s_B can be defined to the center of the pulley above B.
- s_C is defined to the center of pulley C.
- All coordinates are defined as positive down and along the direction of motion of each point/object.

Example (continued)

2) Write position/length equations for each cord. Define l_1 as the length of the first cord, minus any segments of constant length. Define l_2 in a similar manner for the second cord:



Cord 1: $2s_A + 2s_C = l_1$

Cord 2: $s_B + (s_B - s_C) = l_2$

3) Eliminating s_C between the two equations, we get

$$2s_A + 4s_B = l_1 + 2l_2$$

4) Relate velocities by differentiating this expression. Note that l_1 and l_2 are constant lengths.

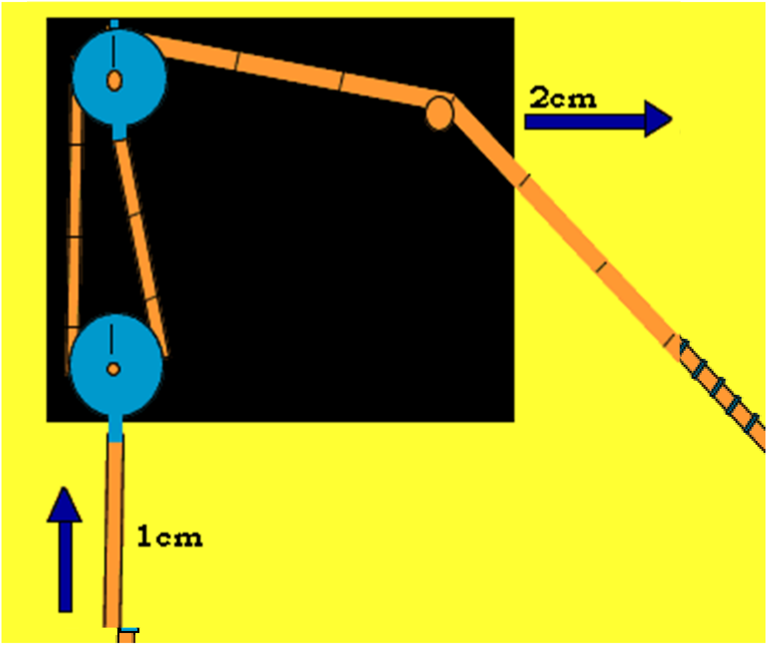
$2v_A + 4v_B = 0 \quad \Rightarrow \quad v_B = -0.5v_A = -0.5(8) = -4$ ft/s

The velocity of block B is 4 ft/s up (negative s_B direction).

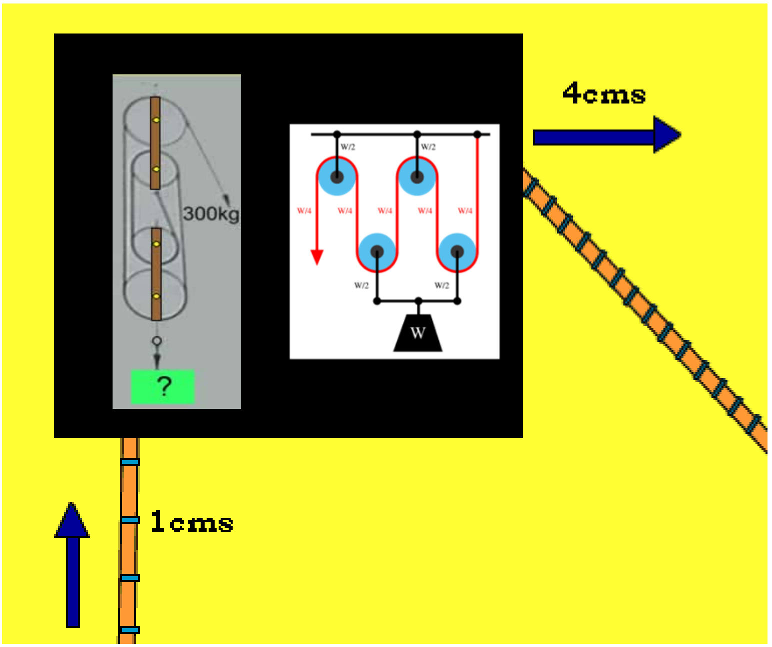
Dependent motion: Procedures for analysis

These procedures can be used to relate the dependent motion of particles moving along rectilinear paths (only the magnitudes of velocity and acceleration change, not their line of direction)

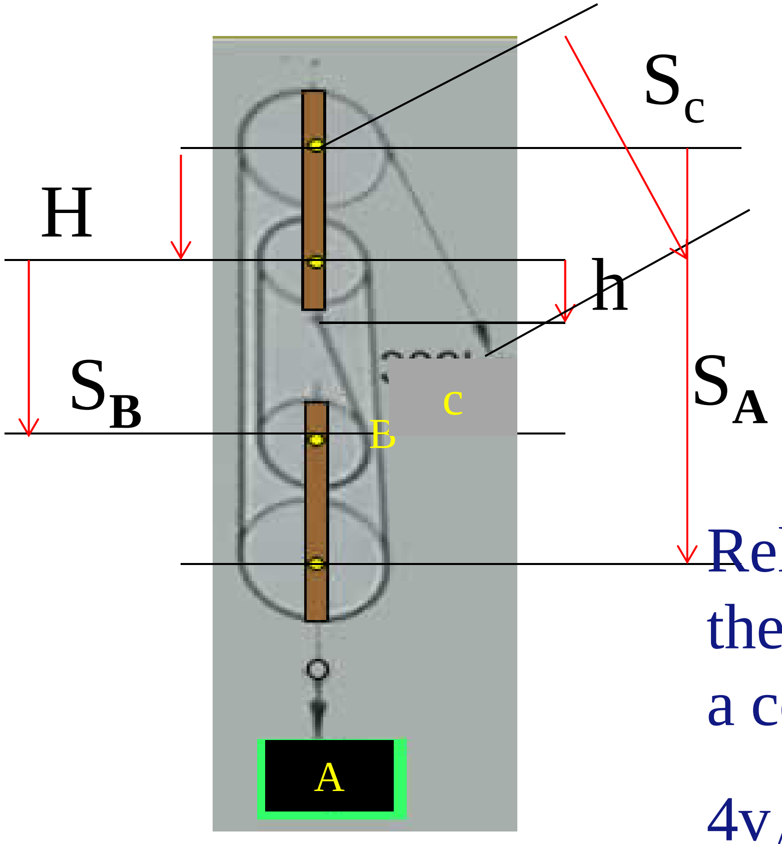
1) Define position coordinates from fixed datum lines, along the path of each particle. Different datum lines can be used for each particle
2) Relate the position coordinates to the cord length. Segments of cord that do not change in length during the motion may be left out
3) If a system contains more than one cord, relate the position of a point on one cord to a point on another cord. Separate equations are written for each cord
4) Differentiate the position coordinate equation(s) to relate velocities and accelerations. Keep track of signs!



Describe a pulley system that allows the string on the right to be pulled through 10cms while the string at the bottom is pulled up 5cms.



Describe a pulley system that allows the string on the right to be pulled through 40cms while the string at the bottom is pulled up 10cms.



Define l as the length of the cord, minus any segments of constant length.

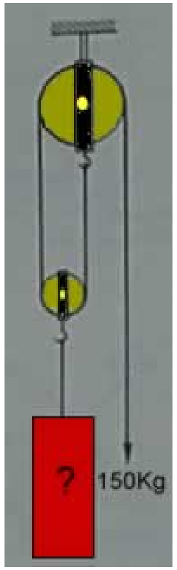
Cord : $2s_A - H + 2s_B - h + s_C = l$

$s_B = s_A$ **(Move same amount of distance)**

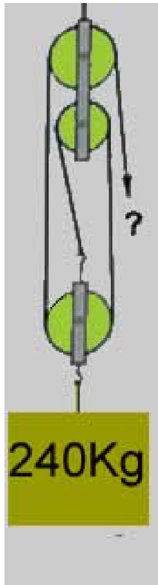
Relate velocities by differentiating the above expression. Note that l is a constant length.

$4v_A + v_c = 0 \quad \Rightarrow \quad v_c = -0.25v_A$

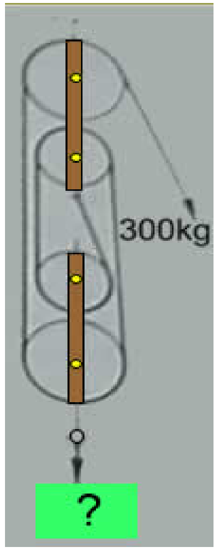
Pulley System



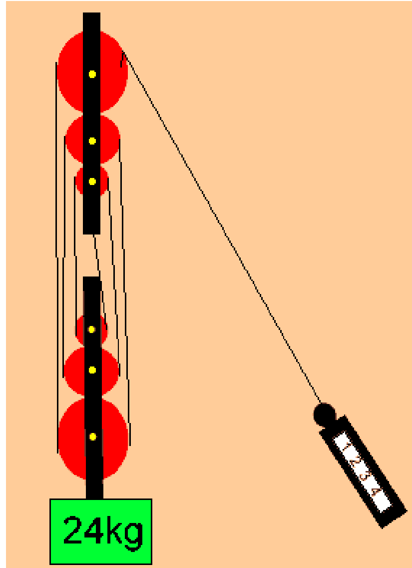
Two



Three



four



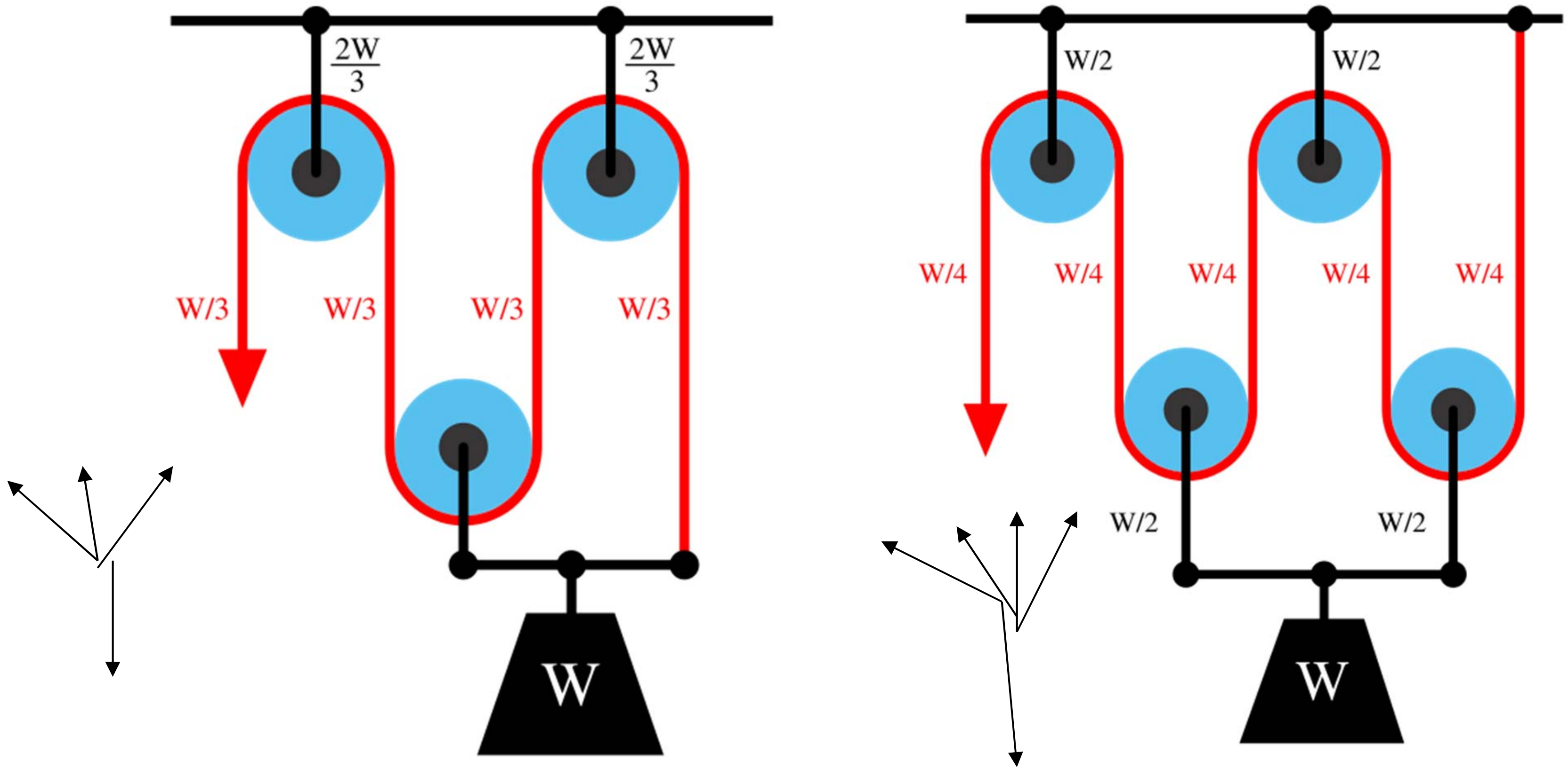
six

Mechanical advantage and is calculated by dividing the load by the effort (load/effort).

Lift height/pull length = 1/(load/effort) (Cons. Of energy)

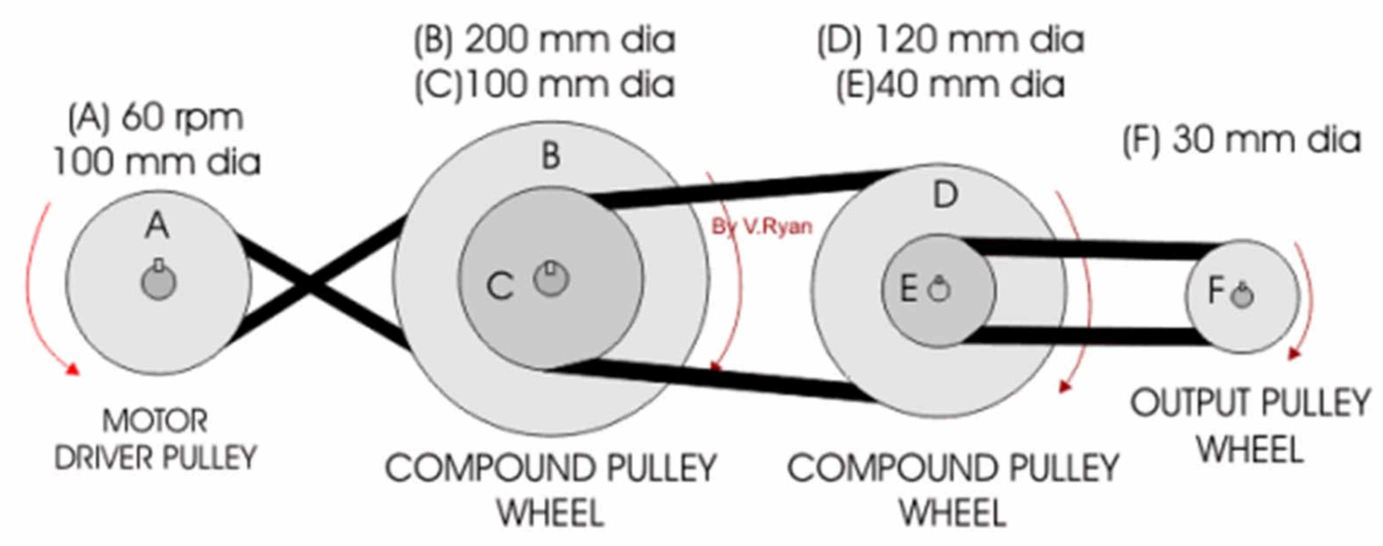
$Work = Force \cdot distance$

Fix and movable pulleys



Look at direction of supporting forces elative to load before summing

Think about it…



What is rotation speed $\dot{\theta}$ at pulley F?

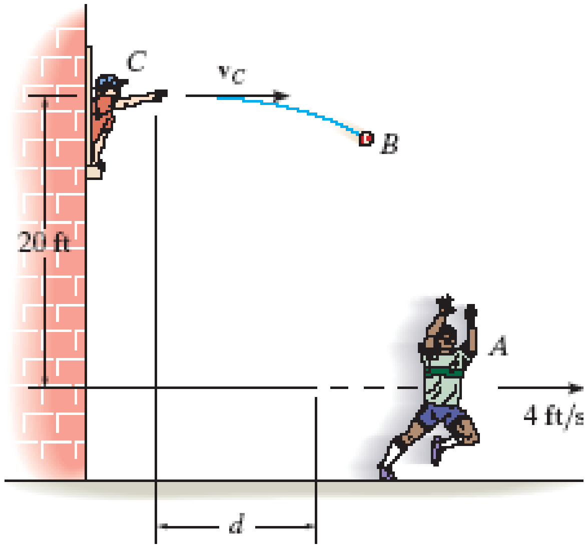
http://www.technologystudent.com/gears1/pulley6.htm

Show Lego gearbox

Now it is time to move to 12.10…

Relative motion analysis of two particles using translating axis

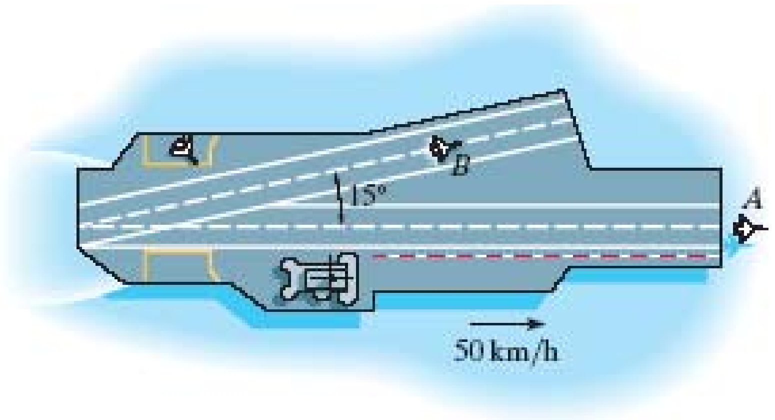
Applications I



When you try to hit a moving object, the position, velocity, and acceleration of the object must be known. Here, the boy on the ground is at d = 10 ft when the girl in the window throws the ball to him

If the boy on the ground is running at a constant speed of 4 ft/s, how fast should the ball be thrown?

Applications II



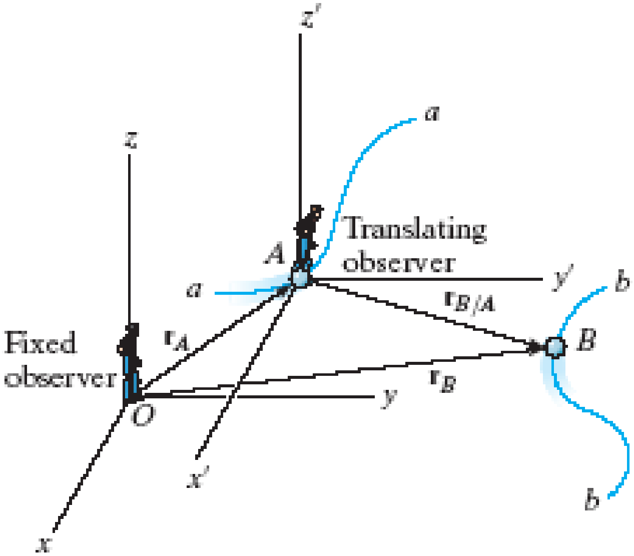
When fighter jets take off or land on an aircraft carrier, the velocity of the carrier becomes an issue

If the aircraft carrier travels at a forward velocity of 50 km/hr and plane A takes off at a horizontal air speed of 200 km/hr (measured by someone on the water), how do we find the velocity of the plane relative to the carrier?

How would you find the same thing for airplane B?

How does the wind impact this sort of situation?

Relative position



The absolute position of two particles A and B with respect to the fixed x, y, z reference frame are given by $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$. The position of B relative to A is represented by

$$\boldsymbol{r}_{B/A} = \boldsymbol{r}_B - \boldsymbol{r}_A$$

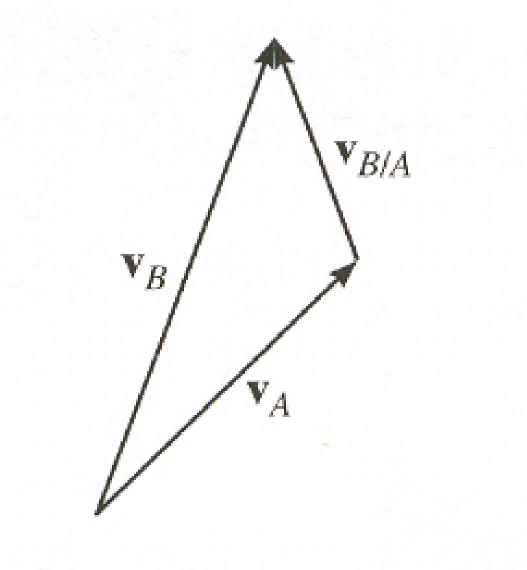
Or
$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$$

Therefore, if $\boldsymbol{r}_B = (10\,\boldsymbol{i} + 2\,\boldsymbol{j})$ m

and $\boldsymbol{r}_A = (4\,\boldsymbol{i} + 5\,\boldsymbol{j})$ m

then $\boldsymbol{r}_{B/A} = (6\,\boldsymbol{i} - 3\,\boldsymbol{j})$ m

Relative velocity



To determine the relative velocity of B with respect to A, the time derivative of the relative position equation is taken.

$$v_{B/A} = v_B - v_A$$

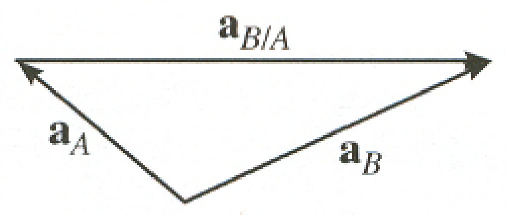
or

$$v_B = v_A + v_{B/A}$$

In these equations, v_B and v_A are called absolute velocities and $v_{B/A}$ is the relative velocity of B with respect to A.

Note that $v_{B/A} = - v_{A/B}$.

Relative acceleration



The time derivative of the relative velocity equation yields a similar vector relationship between the absolute and relative accelerations of particles A and B

$$a_{B/A} = a_B - a_A$$

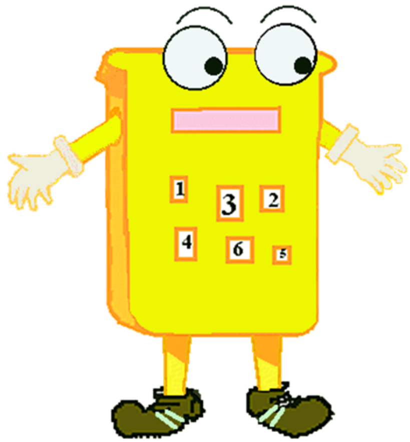
or

$$a_B = a_A + a_{B/A}$$

Solving problems

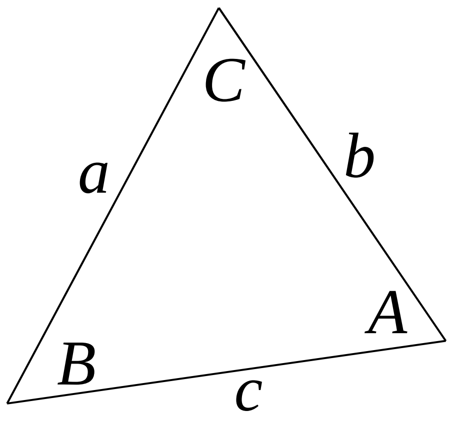
Since the relative motion equations are vector equations, problems involving them may be solved in one of two ways.

For instance, the velocity vectors in $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$ could be written as Cartesian vectors and the resulting scalar equations solved for up to two unknowns.



Alternatively, vector problems can be solved "graphically" by use of trigonometry. This approach usually makes use of the law of sines or the law of cosines.

Laws of sines and cosines



Since vector addition or subtraction forms a triangle, sine and cosine laws can be applied to solve for relative or absolute velocities and accelerations. For review, their formulations are provided below.

Law of Sines: $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

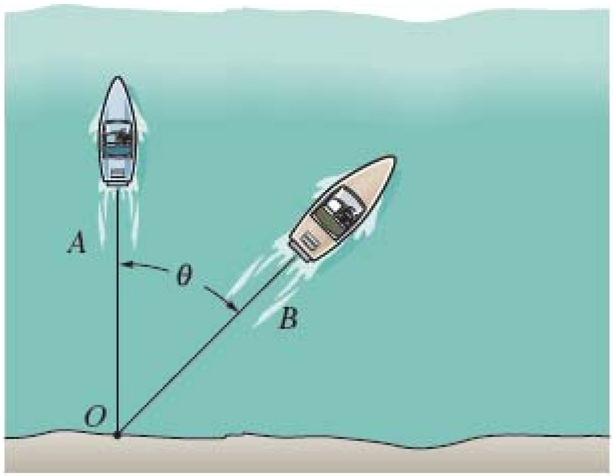
Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example



Both boats A and B leave the shore at O at the same time. If A travels at v_A and B travels at v_B, write a general expression to determine the velocity of A with respect to B.

Relative Velocity:

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$v_A\mathbf{j} = v_B \sin\theta\mathbf{i} + v_B \cos\theta\mathbf{j} + \mathbf{v}_{A/B}$$

$$\mathbf{v}_{A/B} = -v_B \sin\theta\mathbf{i} + (v_A - v_B\cos\theta)\mathbf{j}$$

Thus, the magnitude of the relative velocity $\mathbf{v}_{A/B}$ is

$$v_{A/B} = \sqrt{(-v_B\sin\theta)^2 + (v_A - v_B\cos\theta)^2}$$
$$= \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$

And its direction is

$$\theta = \tan^{-1}\left(\frac{v_A - v_B\cos\theta}{v_B\sin\theta}\right) \measuredangle$$

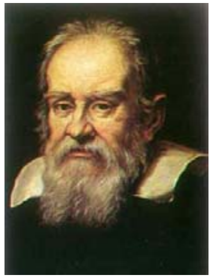
Homework Assignment

Chapter 12- 211, 232, 234, 238
Chapter 13-16, 22, 28, 42, 43, 48

Due next Wednesday !!!

W. Wang

Kinematics of a particle: Introduction



Important contributors

Galileo Galilei, Newton, Euler

Mechanics

Equilibrium of a body that is at rest/moves with constant velocity ← Statics

Dynamics → Accelerated motion of a body

- **Kinematics**: geometric aspects of the motion
- **Kinetics**: Analysis of forces which cause the motion

W. Wang



Introducing Particle Kinetics, Chapter 13

- Chapter 13 introduces the kinetics of a particle
  - kinetics: the study of the relationships between changes in motion of a body and the forces which cause those changes
  - particle: a body which can be modeled as having zero physical dimensions
- Chapter 13 unfolds by gradually increasing the complexity of our view of this topic, considering different kinds of equations of motion in different coordinate systems

EOM

- ☑ Newton's Second Law: F = ma (13.1)
- ☑ the equation of motion: motion of a particle (13.2)
- ☑ EOM for a particle system: motion of a particle system (13.3)
- ☑ EOM: using rectangular coordinate system (13.4)
- ☑ EOM: using normal / tangential coordinate system (13.5)
- ☑ EOM: using cylindrical coordinate system (13.6)

W. Wang

Lecture 5: Particle Kinetics

- **Kinetics of a particle** (Chapter 13)

- 13.1-13.3

W. Wang

Chapter 13: Objectives

- State Newton's laws of motion and gravitational attraction. Define mass and weight

- To analyze the accelerated motion of a particle using the equation of motion with different coordinate systems

Material covered

- **Kinetics of a particle**

- Newton's laws of motion

- The equation of motion

- Equation of motion for a system of particles

- Next lecture; Equations of motion: Different coordinate systems

Objectives

Students should be able to:

1. Write the equation of motion for an accelerating body.
2. Draw the free-body and kinetic diagrams for an accelerating body

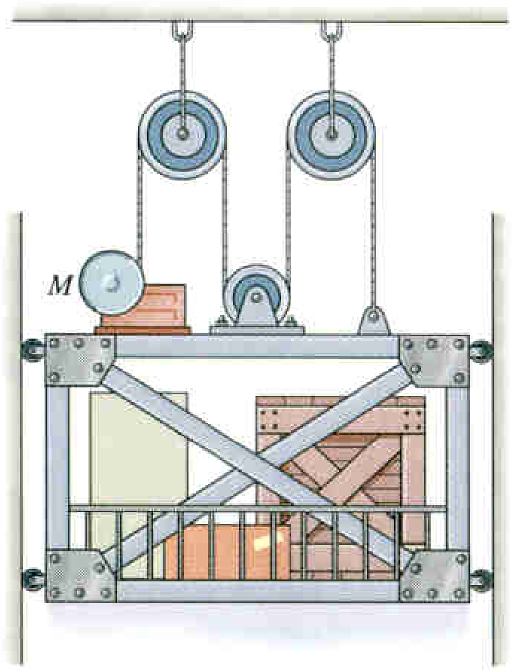
Applications I

The motion of an object depends on the forces acting on it

A parachutist relies on the atmospheric drag resistance force to limit his velocity

Q: Knowing the drag force, how can we determine the acceleration or velocity of the parachutist at any point in time?

Applications II



A freight elevator is lifted using a motor attached to a cable and pulley system as shown

Q1: How can we determine the tension force in the cable required to lift the elevator at a given acceleration?

Q2: Is the tension force in the cable greater than the weight of the elevator and its load?

Newton's laws of motion 1

The motion of a particle is governed by Newton's three laws of motion

First Law: A particle originally at rest, or moving in a straight line at constant velocity, will remain in this state if the resultant force acting on the particle is zero

$$\sum \boldsymbol{F} = 0,$$

Second Law: If the resultant force on the particle is not zero, the particle experiences an acceleration in the same direction as the resultant force. This acceleration has a magnitude proportional to the resultant force.

$$\sum \boldsymbol{F} = m\boldsymbol{a},$$

Third Law: Mutual forces of action and reaction between two particles are equal, opposite, and collinear.

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$$

Newton's laws of motion 2

The first and third laws were used in developing the concepts of statics. Newton's second law forms the basis of the study of dynamics.

Mathematically, Newton's second law of motion can be written:

$$\boldsymbol{F_R} = m\boldsymbol{a}$$

where $\boldsymbol{F_R}$ is the resultant unbalanced force acting on the particle, and $\boldsymbol{a}$ is the acceleration of the particle. The positive scalar m is called the mass of the particle.

Newton's second law **cannot** be used when the particle's speed approaches the speed of light m(object)=m(stationary)/((1-v^2/c^2)^(1/2)) where m(object) is relativistic mass, v is the object's velocity relative to the stationary observer and c is the speed of light, or if the size of the particle is extremely small (~ size of an atom)

Newton's law of gravitational attraction

Any two particles or bodies have a mutually attractive gravitational force acting between them. Newton postulated the law governing this gravitational force as;

$$F = G(m_1 m_2 / r^2)$$

where F = force of attraction between the two bodies,

G = universal constant of gravitation ,

m_1, m_2 = mass of each body, and

r = distance between centers of the two bodies.

When near the surface of the earth, the only gravitational force having any sizable magnitude is that between the earth and the body. This force is called the weight of the body

Distinction between mass and weight

It is important to understand the difference between the mass and weight of a body!

Mass is an absolute property of a body. It is independent of the gravitational field in which it is measured. The mass provides a measure of the resistance of a body to a change in velocity, as defined by Newton's second law of motion ($m = \boldsymbol{F}/\boldsymbol{a}$)

The weight of a body is not absolute, since it depends on the gravitational field in which it is measured. Weight is defined as

$$W = mg$$

where g is the acceleration due to gravity

SI system vs FPS system

SI system: In the SI system of units, mass is a base unit and weight is a derived unit. Typically, mass is specified in kilograms (kg), and weight is calculated from W = mg. If the gravitational acceleration (g) is specified in units of m/s^2, then the weight is expressed in newtons (N). On the earth's surface, g can be taken as g = 9.81 m/s^2.

$$W\ (N) = m\ (kg)\ g\ (m/s^2) \Rightarrow N = kg \cdot m/s^2$$

FPS system: In the FPS system of units, weight is a base unit and mass is a derived unit. Weight is typically specified in pounds (lb), and mass is calculated from m = W/g. If g is specified in units of ft/s^2, then the mass is expressed in slugs. On the earth's surface, g is approximately 32.2 ft/s^2.

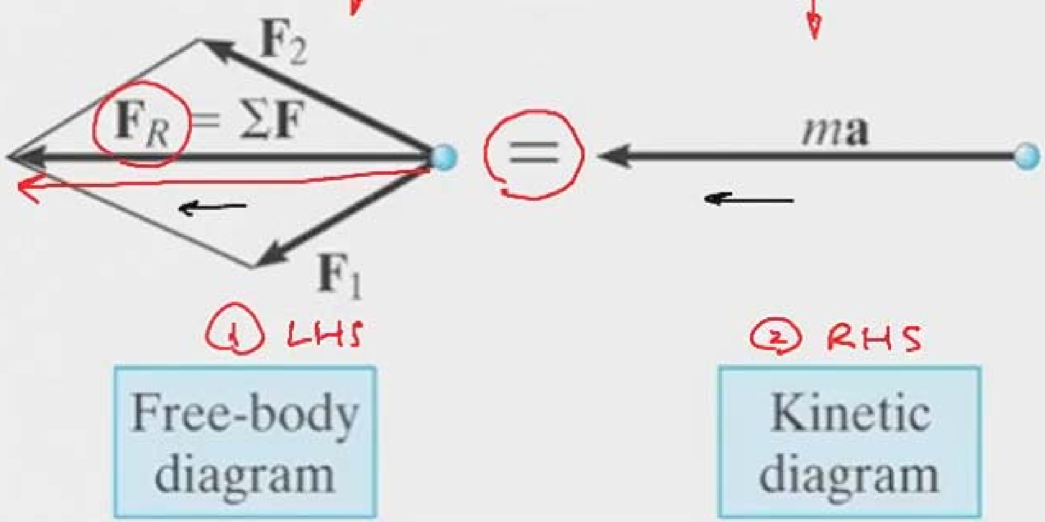
$$m\ (slugs) = W\ (lb)/g\ (ft/s^2) \Rightarrow slug = lb \cdot s^2/ft$$

Theory: The Equation of Motion (13.2)

- Newton's Second Law in vector form:

$$\sum \mathbf{F} = m\mathbf{a}$$

- we typically view kinetics using a kinetics diagram, which is an extended version of a FBD



Inertial frame of reference

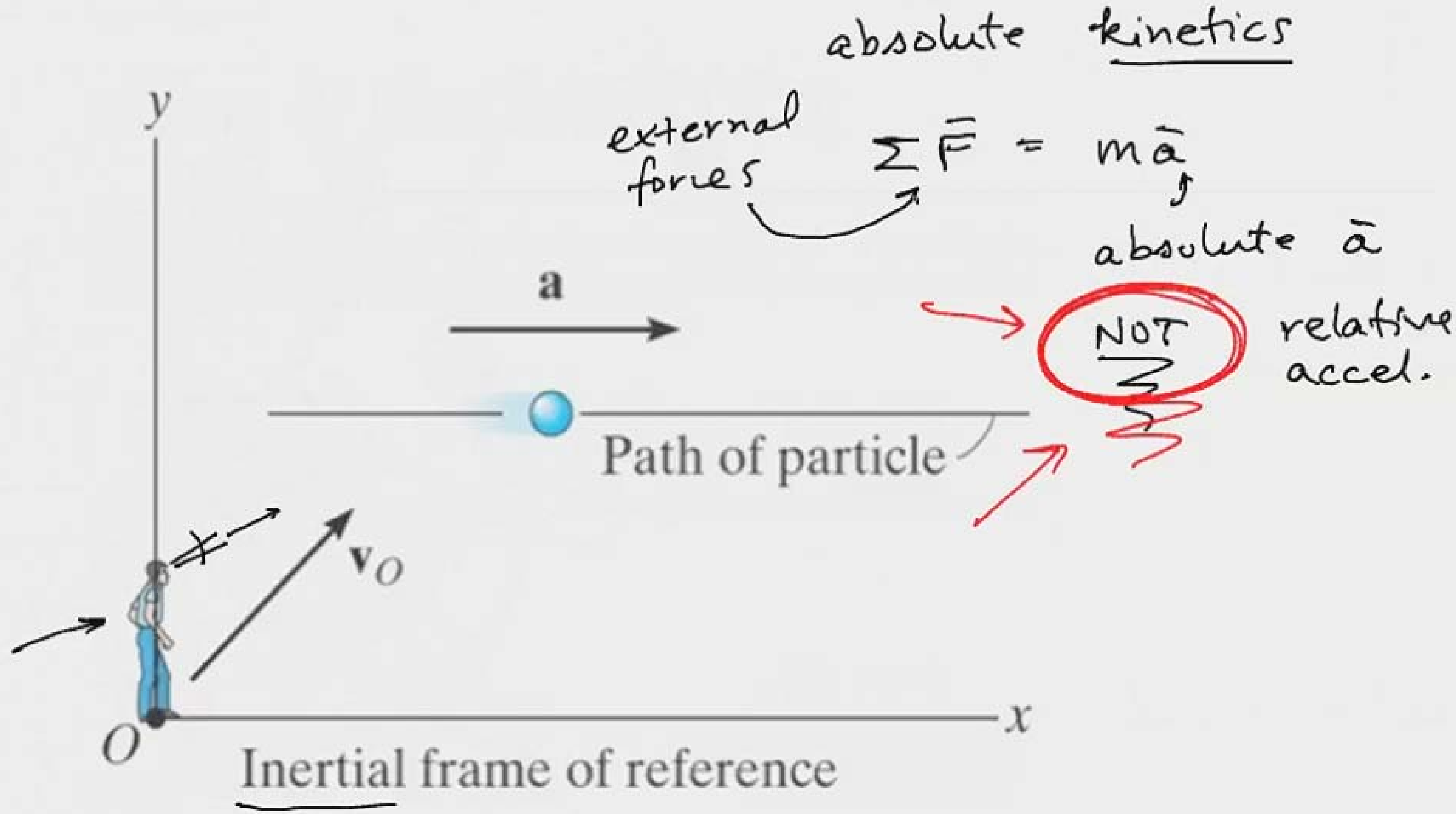
This equation of motion is only valid if the acceleration is measured in a Newtonian or inertial frame of reference. What does this mean?(Means that coordinate system does not rotate and is either fixed or translates with constant velocity)

For problems concerned with motions at or near the earth's surface, we typically assume our "inertial frame" to be fixed to the earth. We neglect any acceleration effects from the earth's rotation.

For problems involving satellites or rockets, the inertial frame of reference is often fixed to the stars.

Theory: Inertial Reference Frame

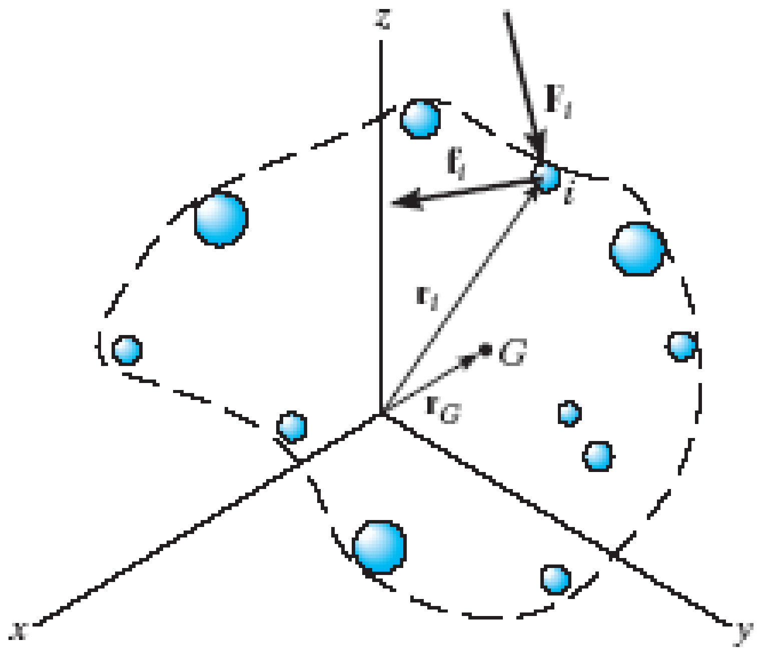
- consider an observer watching the motion of a particle



Equation of motion for a system of particles

The equation of motion can be extended to include systems of particles. This includes the motion of solids, liquids, or gas systems.

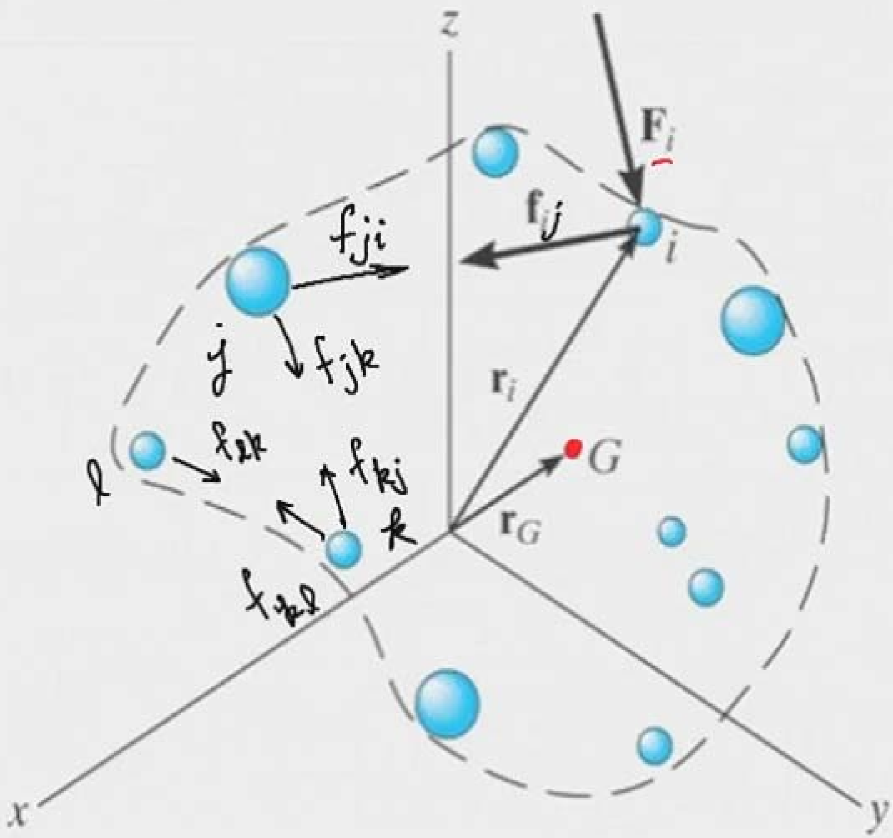
As in statics, there are internal forces and external forces acting on the system. What is the difference between them?



Inertial coordinate system

Theory: A System of Particles (13.3)

- for a system of particles, which contains both external forces F_i and internal (equal and opposite) forces of interaction f_i between particles:



F_i: external forces

f_i = internal forces

$$\bar{f}_{ij} = -(\bar{f}_{ji})$$

internal forces cancel

$$\Sigma \bar{f}_{ij} = \bar{0}$$

$\Rightarrow$ only use external forces in $F = ma$

Important points !!

1) Newton's second law is a "Law of Nature"--experimentally proven and not the result of an analytical proof.

2) Mass (property of an object) is a measure of the resistance to a change in velocity of the object.

3) Weight (a force) depends on the local gravitational field. Calculating the weight of an object is an application of F = ma, i.e., W = m g.

4) Unbalanced forces cause the acceleration of objects. This condition is fundamental to all dynamics problems!

How to analyze problems that involve the equation of motion

1) Select a convenient inertial coordinate system. Rectangular, normal/tangential, or cylindrical coordinates may be used.

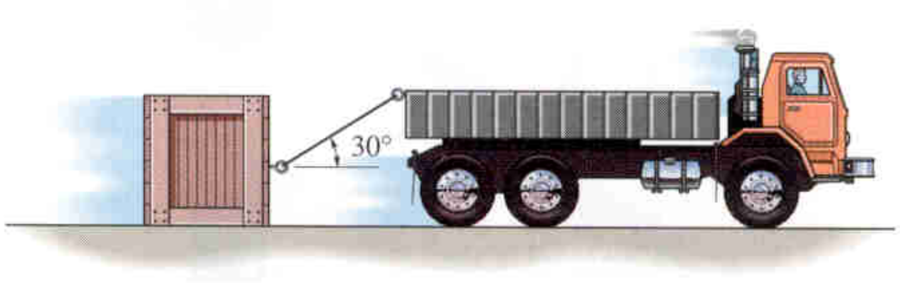
2) Draw a free-body diagram showing all external forces applied to the particle. Resolve forces into their appropriate components.

3) Draw the kinetic diagram, showing the particle's inertial force, m$\boldsymbol{a}$. Resolve this vector into its appropriate components.

4) Apply the equations of motion in their scalar component form and solve these equations for the unknowns.

5) It may be necessary to apply the proper kinematic relations to generate additional equations.

Example



Given: A crate of mass m is pulled by a cable attached to a truck. The coefficient of kinetic friction between the crate and road is μ_k.

Find: Draw the free-body and kinetic diagrams of the crate.

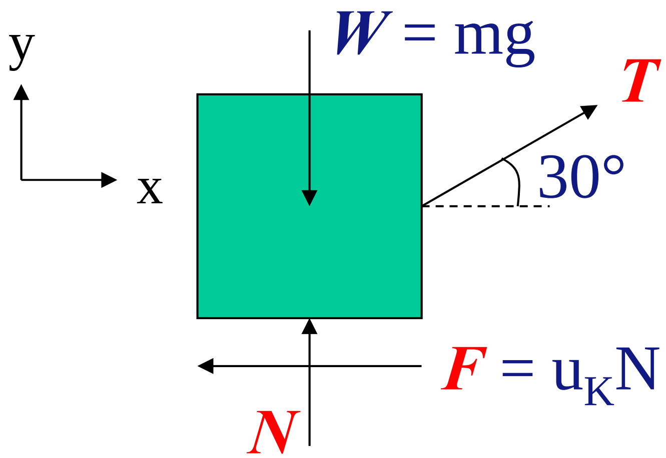
Plan:
1) Define an inertial coordinate system.
2) Draw the crate's free-body diagram, showing all external forces applied to the crate in the proper directions.
3) Draw the crate's kinetic diagram, showing the inertial force vector m***a*** in the proper direction.

Example (continued)

Solution:

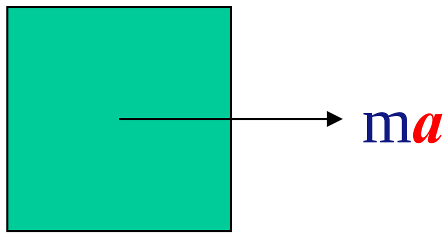
1) An inertial x-y frame can be defined as fixed to the ground.

2) Draw the free-body diagram of the crate:



The weight force (***W***) acts through the crate's center of mass. ***T*** is the tension force in the cable. The normal force (***N***) is perpendicular to the surface. The friction force (***F*** = u_KN) acts in a direction opposite to the motion of the crate.

3) Draw the kinetic diagram of the crate:



The crate will be pulled to the right. The acceleration vector can be directed to the right if the truck is speeding up or to the left if it is slowing down.

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