# ME 230 Kinematics and Dynamics

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# **Lecture 6: Particle Kinetics**

#### • Kinetics of a particle (Chapter 13)

- <u>13.4-13.6</u>





# **Chapter 13: Objectives**

• State Newton's laws of motion and gravitational attraction. Define mass and weight

• <u>To analyse the accelerated motion</u> of a particle using the equation of motion with different coordinate systems



# **Material covered**

# Kinetics of a particle

- Equations of motion: Rectangular coordinate system
- Equations of motion: n-t coordinates
- Equations of motion:Cylindrical coordinates

...Next lecture...solving problems, <u>Design Project</u> and starting Ch. 14

# Objectives

#### Students should be able to:

- 1. Apply Newton's second law to determine forces and accelerations for particles in rectilinear motion
- 2. Apply the equation of motion using normal and tangential coordinates
- 3. Analyze the kinetics of a particle using cylindrical coordinates









W. Wang

# **Applications**



If a man is pushing a 100 lb crate, how large a force F must he exert to start moving the crate?

Objects that move in any fluid have a drag force acting on them. This drag force is a function of velocity.

If the ship has an initial velocity  $v_o$ and the magnitude of the opposing drag force at any instant is half the velocity, how long it would take for the ship to come to a stop if its engines stop? W. Wang



### **Rectangular coordinates**

The equation of motion, F = m a, is best used when the problem requires finding forces (especially forces perpendicular to the path), accelerations, velocities or mass. Remember, unbalanced forces cause acceleration!

Three scalar equations can be written from this vector equation. The equation of motion, being a vector equation, may be expressed in terms of its three components in the Cartesian (rectangular) coordinate system as:

$$\sum \mathbf{F} = \mathbf{m}\mathbf{a}$$
 or  $\sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = \mathbf{m}(\mathbf{a}_x \mathbf{i} + \mathbf{a}_y \mathbf{j} + \mathbf{a}_z \mathbf{k})$ 

or, as scalar equations,  $\sum F_x = ma_x$ ,  $\sum F_y = ma_y$ , and  $\sum F_z = ma_z$ 

## Theory: EOM in Rectangular Coordinates (13.4)

expressing the vector EOM in rectangular coordinates turn out to be exactly as you expect:

$$\sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = m \left( a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \right)$$
  
$$\sum F = m \tilde{a} \quad (x, y, z)$$



# **Procedure of analysis I**

#### **Free Body Diagram (FBD)**

Establish your coordinate system and draw the particle's free body diagram showing <u>only</u> external forces. These external forces usually include the weight, normal forces, friction forces, and applied forces. Show the 'm*a*' vector (sometimes called the inertial force) on a separate diagram



Make sure any friction forces act opposite to the direction of motion! If the particle is connected to an elastic spring, a spring force equal to ks should Wenincluded on the FBD

# **Procedure of analysis II**

#### **Equations of Motion**

If the forces can be resolved directly from the free-body diagram (often the case in 2-D problems), use the scalar form of the equation of motion. In more complex cases (usually 3-D), a Cartesian vector is written for every force and a vector analysis is often best

A Cartesian vector formulation of the second law is:

 $\sum F = ma$  or

$$\sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = m(a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k})$$

Three scalar equations can be written from this vector equation. You may only need two equations if the motion is in 2-D

# **Procedure of analysis III**

#### <u>Kinematics!!!</u>

The second law only provides solutions for forces and accelerations. If velocity or position have to be found, kinematics equations are used once the acceleration is found from the equation of motion (e.g. in cylindrical coordinate:

Position:  $r_P \doteq ru_r + zu_z$ Velocity:  $v_P = ru_r + r\theta u_\theta + zu_z$ Acceleration:  $a_P \doteq (r - r\theta^2)u_r + (r\theta + 2r\theta)\ddot{u}_\theta + zu_z$ 

Any of the tools learned in <u>Chapter 12</u> may be needed to solve a problem. Make sure you use consistent positive coordinate directions as used in the equation of motion part of the problem!



#### EXAMPLE



**Given:** The motor winds in the cable with a constant acceleration such that the 20-kg crate moves a distance s = 6 m in 3 s, starting from rest.  $\mu_k = 0.3$ . **Find:** The tension developed in

#### **Plan:**

1) Draw the free-body and kinetic diagrams of the crate.

2) Using a kinematic equation, determine the acceleration of the crate.

the cable.

3) Apply the equation of motion to determine the cable tension.



#### Solution:

1) Draw the free-body and kinetic diagrams of the



Since the motion is up the incline, rotate the x-y axes so the x-axis aligns with the incline. Then, motion occurs only in the x-direction.

There is a friction force acting between the surface and the crate. Why is it in the direction shown on the FBD?







3) Apply the equations of motion

+\ ∑ F<sub>y</sub> = 0 ⇒ -20 g (cos30°) + N = 0 => N = 169.9 N +-∑ F<sub>x</sub> = m a ⇒ T - 20g(sin30°) -0.3 N = 20 a ⇒ T = 20 (981) (sin30°) + 0.3(169.9) + 20 (1.333) ⇒ T = 176 N



# n-t coordinates (13.5)

When a particle moves along a curved path, it may be more convenient to write the equation of motion in terms of normal and tangential coordinates



The normal direction (n) <u>always</u> points toward the path's center of curvature. In a circle, the center of curvature is the center of the circle

The tangential direction (t) is tangent to the path, usually set as positive in the direction of motion of the particle <sup>W. Wang</sup>

# **Equations of motion**



Since the equation of motion is a vector equation,  $\sum \mathbf{F} = m\mathbf{a}$ ,

it may be written in terms of the n & t coordinates as;

 $\sum F_t \boldsymbol{u}_t + \sum F_n \boldsymbol{u}_n = m\boldsymbol{a}_t + m\boldsymbol{a}_n$ 

Here  $\sum F_t \& \sum F_n$  are the sums of the force components acting in the t & n directions, respectively

This vector equation will be satisfied provided the individual components on each side of the equation are equal, resulting in the two scalar equations:  $\Sigma F_t = ma_t$  and  $\Sigma F_n = ma_n$ 

<u>Since</u> there is no motion in the binormal (b) direction, we can also write  $\sum F_b = 0$ 

#### Theory: EOM in n-t Coordinates (13.5)

expressing the EOM in n-t coordinates also turn out exactly as you expect:

$$\sum F_{t}\mathbf{u}_{t} + \sum F_{n}\mathbf{u}_{n} + \sum F_{b}\mathbf{u}_{b} = m (a_{t}\mathbf{u}_{t} + a_{n}\mathbf{u}_{n})$$
  

$$t: \Sigma F_{t} = ma_{t}$$
  

$$h: \Sigma F_{n} = ma_{n}$$
  

$$b: \Sigma F_{b} = 0$$
  

$$EoM$$
  

$$L$$
, kinematics  
Inertial coordinate system

# Normal and tangential accelerations

The tangential acceleration,  $a_t = dv/dt$ , represents the time rate of change in the magnitude of the velocity. Depending on the direction of  $\sum F_t$ , the particle's speed will either be increasing or decreasing

The normal acceleration,  $a_n = v^2/\rho$ , represents the time rate of change in the direction of the velocity vector. Remember,  $a_n$  always acts toward the path's center of curvature. Thus,  $\sum F_n$  will always be directed toward the center of the path

Recall, if the path of motion is defined as y = f(x), the radius of curvature at any point can be obtained from:



# Solving problems with n-t coordinates

- Use n-t coordinates when a particle is moving along a known, curved path
- Establish the n-t coordinate system on the particle
- Draw free-body and kinetic diagrams of the particle. The normal acceleration (a<sub>n</sub>) always acts "inward" (the positive n-direction). The tangential acceleration (a<sub>t</sub>) may act in either the positive or negative t direction
- Apply the equations of motion in scalar form and solve
  - It may be necessary to employ the kinematic relations:

$$a_t = dv/dt = v dv/ds$$
  $a_n = v^2/\rho$ 

#### EXAMPLE



Given: The 10-kg ball has a velocity of 3 m/s when it is at A, along the vertical path.

Find: The tension in the cord and the increase in the speed,  $a_{t_i}$  of the ball.

- Plan: 1) Since the problem involves a curved path and requires finding the force perpendicular to the path, use n-t coordinates. Draw the ball's free-body and kinetic diagrams.
  - 2) Apply the equation of motion in the n-t directions.



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#### Solution:

 The n-t coordinate system can be established on the ball at Point A, thus at an angle of 45°. Draw the free-body and kinetic diagrams of the ball.





Kinetic diagram





2) Apply the equations of motion in the n-t directions.

(a) 
$$\Sigma F_n = ma_n \implies T - W \sin 45^\circ = m a_n$$
  
Using  $a_n = v^2/\rho = 3^2/2$ ,  $W = 10(9.81)$  N, and  $m = 10$  kg  
 $\Rightarrow T - 98.1 \sin 45^\circ = (10) (3^2/2)$   
 $\Rightarrow T = 114$  N

(b) 
$$\Sigma F_t = ma_t \implies W \cos 45^\circ = ma_t$$
  
 $\implies 98.1 \cos 45^\circ = 10 a_t$   
 $\implies a_t = (dv/dt) = 6.94 \text{ m/s}^2$ 



### Some more theory...





# **Cylindrical coordinates (13.6)**

This approach to solving problems has some external similarity to the normal & tangential method just studied. However, the path may be more complex or the problem may have other attributes that make it desirable to use cylindrical coordinates



Equilibrium equations or "Equations of Motion" in cylindrical coordinates (using r,  $\theta$ , and z coordinates) may be expressed in scalar form as:

$$\sum F_{r} = ma_{r} = m(\ddot{r} - \dot{r}\dot{\theta}^{2})$$
$$\sum F_{\theta} = ma_{\theta} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$
$$\sum F_{z} = ma_{z} = m\ddot{z}$$

# Cylindrical coordinates (13.6) continued...

If the particle is constrained to move only in the  $r - \theta$  plane (i.e., the z coordinate is constant), then only the first two equations are used (as shown below). The coordinate system in such a case becomes a polar coordinate system. In this case, the path is only a function of  $\theta$ .

 $\sum F_{r} = ma_{r} = m(\ddot{r} - \dot{r}\dot{\theta}^{2})$  $\sum F_{\theta} = ma_{\theta} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ 

<u>Note</u> that a fixed coordinate system is used, not a "body-centered" system as used in the n - t approach

#### **Tangential and normal forces**

If a force P causes the particle to move along a path defined by  $r = f(\theta)$ , the normal force N exerted by the path on the particle is always perpendicular to the path's tangent. The frictional force F always acts along the tangent in the opposite direction of motion. The directions of N and F can be specified relative to the radial coordinate by using the angle  $\psi$ 





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# Determination of angle $\boldsymbol{\psi}$

The angle  $\psi$ , defined as the angle between the extended radial line ar the tangent to the curve, can be required to solve some problems. can be determined from the follow relationship;

$$\tan \Psi = \frac{r \, d\theta}{dr} = \frac{r}{dr/d\theta}$$





If  $\psi$  is positive, it is measured counterclockwise from the radial line to the tangent. If it is negative, it is measured clockwise



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#### EXAMPLE



**Given:** The 0.2 kg pin (P) is constrained to move in the smooth curved slot, defined by  $r = (0.6 \cos 2\theta)$  m. The slotted arm OA has a constant angular velocity of  $\dot{\theta} = -3$  rad/s. Motion is in the vertical plane.

#### Find:

Force of the arm OA on the pin P when  $\theta = 0^{\circ}$ .

Plan: 1) Draw the FBD and kinetic diagrams.
2) Develop the kinematic equations using cylindrical coordinates.
3) Apply the equation of motion to find the force.

#### **Solution:**

1) Free Body and Kinetic Diagrams Establish the *r*,  $\theta$  coordinate system when  $\theta = 0^{\circ}$ , and draw the free body and kinetic diagrams.





Kinetic diagram



$$\sum F_{r} = ma_{r} = m(\ddot{r} - \dot{r}\dot{\theta}^{2})$$
$$\sum F_{\theta} = ma_{\theta} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

2) Notice that  $r = 0.6 \cos(2\theta)$ , therefore:  $\dot{r} = -1.2 \sin(2\theta) \dot{\theta}$  $\ddot{r} = -2.4 \cos(2\theta) \dot{\theta}^2 - 1.2 \sin(2\theta) \ddot{\theta}$ 

Kinematics: at  $\theta = 0^{\circ}$ ,  $\dot{\theta} = -3 \text{ rad/s}$ ,  $\ddot{\theta} = 0 \text{ rad/s}^2$ .  $r = 0.6 \cos(0) = 0.6 \text{ m}$   $\dot{r} = -1.2 \sin(0) (-3) = 0 \text{ m/s}$  $\ddot{r} = -2.4 \cos(0) (-3)^2 - 1.2 \sin(0)(0) = -21.6 \text{ m/s}^2$ 

Acceleration components are

$$a_{\rm r} = \ddot{r} - r\dot{\theta}^2 = -21.6 - (0.6)(-3)^2 = -27 \text{ m/s}^2$$
$$a_{\theta} = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.6)(0) + 2(0)(-3) = 0 \text{ m/s}^2$$





Equation of motion:  $\theta$  direction  $(+\uparrow) \sum F_{\theta} = ma_{\theta}$  N - 0.2 (9.81) = 0.2 (0) $N = 1.96 N \uparrow$ 

#### **Example II**



**Given:** The smooth particle P is attached to an elastic cord extending from O to P. Due to the slotted arm guide, P moves along a *horizontal circular* path. The mass of P = 0.08 kg. The cord's stiffness is k = 30 N/m, its un-stretched length = 0.25 m,  $r = (0.8 \sin \theta)$  m.

- **Find:** Forces of the guide on the particle when  $\theta = 60^{\circ}$  and  $\dot{\theta} = 5$  rad/s, which is constant.
- Plan: 1) Find the acceleration components using the kinematic equations.
  - 2) Draw free body diagram & kinetic diagram.
  - 3) Apply the equation of motion to find the forces.



#### Solution (continued)

#### **Solution:**

1) Kinematics:  $r = 0.8 (\sin \theta)$   $\dot{r} = 0.8 (\cos \theta)\dot{\theta}$   $\ddot{r} = -0.8 (\sin \theta)\dot{\theta}^2 + 0.8 (\cos \theta)\ddot{\theta}$  **constant** When  $\theta = 60^\circ$ ,  $\dot{\theta} = 5 \text{ rad/s}$  and  $\ddot{\theta} = 0 \text{ rad/s}^2$ . r = 0.6928 m  $\dot{r} = 2 \text{ m/s}$  $\ddot{r} = -17.32 \text{ m/s}^2$ 

Accelerations :  $a_r = \ddot{r} - r\dot{\theta}^2 = -17.32 - (0.6928) 5^2 = -34.64 \text{ m/s}^2$  $a_{\theta} = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.6928) 0 + 2 (2) 5 = 20 \text{ m/s}^2$ 



#### **Solution** (continued)

3) Apply equation of motion :

 $\sum F_r = ma_r \implies -13.28 + N \cos 30^\circ = 0.08 (-34.64)$  $\sum F_\theta = ma_\theta \implies F - N \sin 30^\circ = 0.08 (20)$ N = 12.1 N, F = 7.67 N





$$\sum F_{\theta} = ma_{\theta} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$





# Homework Assignment

#### Chapter13-59, 65, 66, 75, 91, 93, 97, 107 Chapter14-3, 11, 14, 21

Due next Wednesday !!!