

ME 230 Kinematics and Dynamics

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Planar kinetics of a rigid body: Force and acceleration

Chapter 17

Chapter objectives

- Introduce the methods used to determine the mass moment of inertia of a body
- To develop the planar kinetic equations of motion for a symmetric rigid body
- To discuss applications of these equations to bodies undergoing translation, rotation about fixed axis, and general plane motion



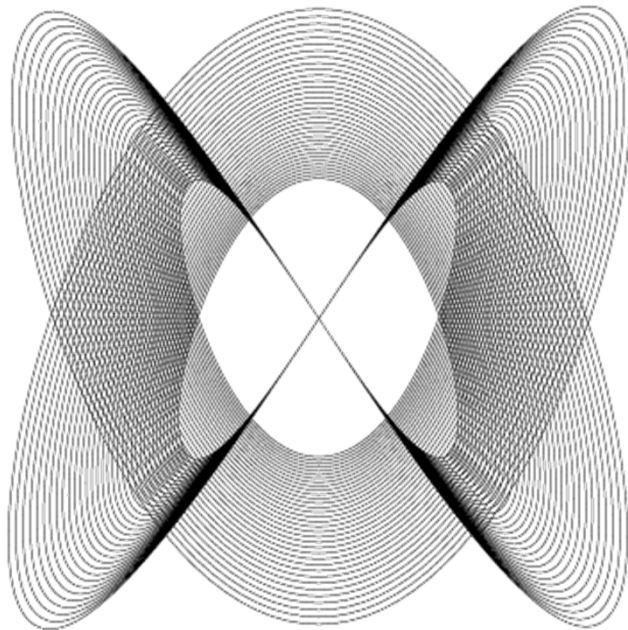
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Lecture 15

- Planar kinetics of a rigid body: Force and acceleration

Moment of Inertia

- 17.1



Material covered

- **Planar kinetics of a rigid body : Force and acceleration**

Moment of inertia

...Next lecture...17.2
and 17.3



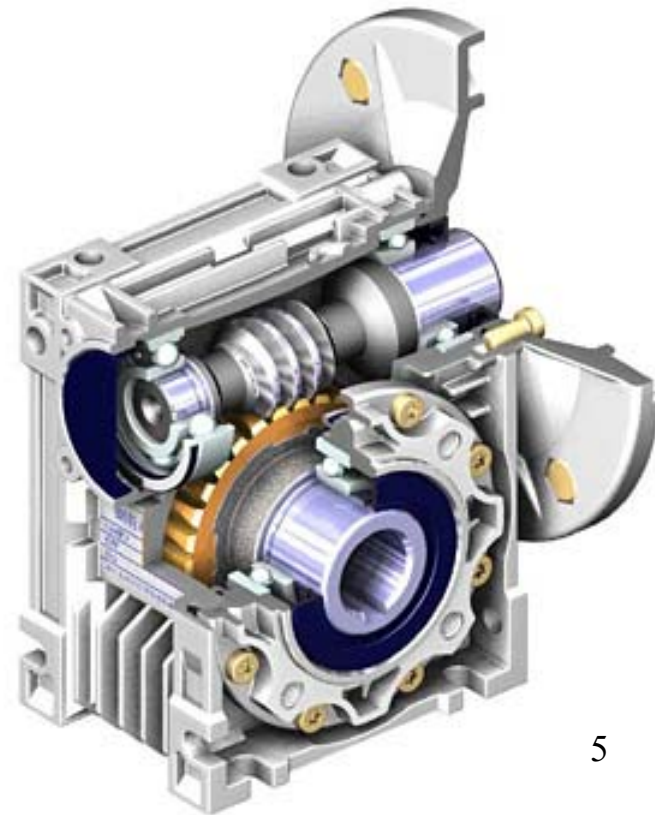
Today's Objectives

Students should be able to:

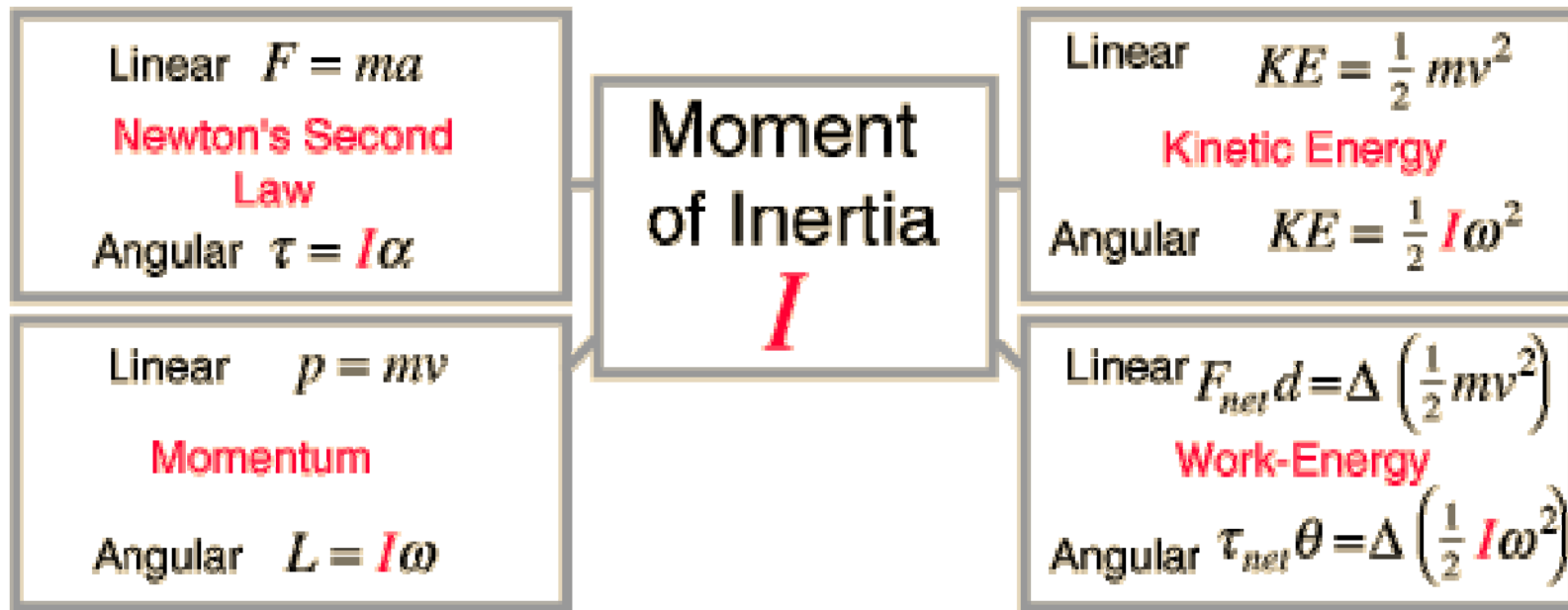
1. Determine the mass moment of inertia of a rigid body or a system of rigid bodies.



W. Wang



Moment of Inertia



Moment of inertia is the name given to rotational inertia, the **rotational analog of mass for linear motion**. For a point mass the moment of inertia is just the mass times the square of perpendicular distance to the rotation axis, $I = mr^2$.



The key idea needed in order to understand why the tightrope walker carries a long pole to aid balance is *moment of inertia*. It has nothing to do with centre of gravity. The long pole increases the tightrope walker's moment of inertia by placing mass far away from the body's centre line (moment of inertia has units of mass times the square of distance).

As a result, any small wobbles about the equilibrium position happen more slowly. They have a longer time period of oscillation (the period of small oscillations about a stable equilibrium increases as the square root of the moment of inertia) and the walker has more time to respond and restore the equilibrium.

Compare how easy it is to balance a one metre ruler on your finger compared with a ten centimetre ruler.

Applications



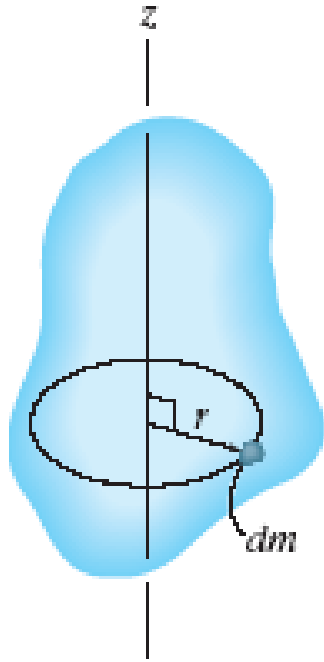
The flywheel on this tractor engine has a large mass moment of inertia about its axis of rotation. Once the flywheel is set into motion, it will be difficult to stop. This tendency will prevent the engine from stalling and will help it maintain a constant power output.

Applications (continues)



The crank on the oil-pump rig undergoes rotation about a fixed axis that is not at its mass center. The crank develops a kinetic energy directly related to its mass moment of inertia. As the crank rotates, its kinetic energy is converted to potential energy and vice versa.

Moment of inertia (17.1)



The mass moment of inertia is a measure of an **object's resistance to rotation**. Thus, the object's mass and how it is distributed both affect the mass moment of inertia.

Mathematically, it is the integral

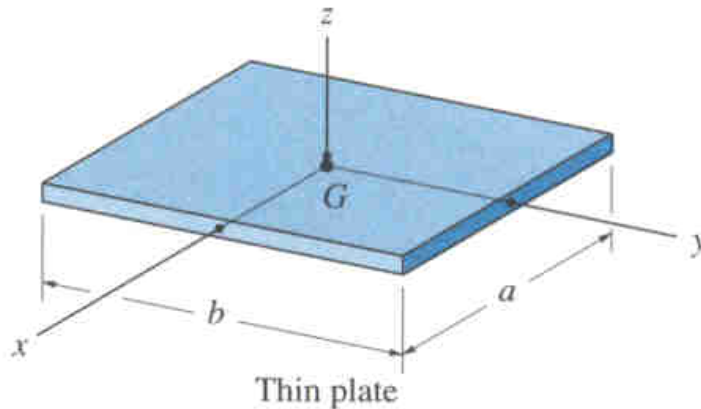
$$I = \int r^2 dm = \int r^2 \rho dV$$

In this integral, r acts as the moment arm of the mass element and ρ is the density of the body. Thus, the value of I **differs for each axis** about which it is computed.

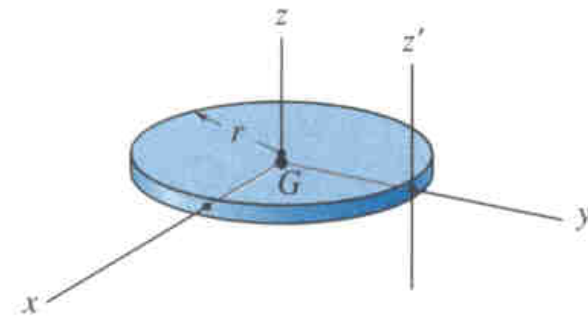
In Section 17.1, the focus is on obtaining the mass moment of inertia via integration!!

Moment of inertia (continues)

The figures below show the mass moment of inertia formulations for two flat plate shapes commonly used when working with three dimensional bodies. The shapes are often used as the **differential element** being integrated over the entire body.



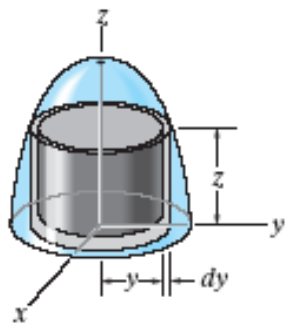
$$I_{xx} = \frac{1}{12}mb^2 \quad I_{yy} = \frac{1}{12}ma^2 \quad I_{zz} = \frac{1}{12}m(a^2 + b^2)$$



$$I_{xx} = I_{yy} = \frac{1}{4}mr^2 \quad I_{zz} = \frac{1}{2}mr^2 \quad I_{z'z'} = \frac{3}{2}mr^2$$

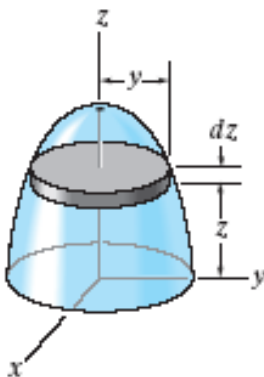
Procedure of analysis

When using direct integration, only symmetric bodies having surfaces generated by revolving a curve about an axis will be considered.



Shell element

- If a shell element having a height z , radius $r = y$, and thickness dy is chosen for integration, then the volume element is $dV = (2\pi y)(z)dy$.
- This element may be used to find the moment of inertia I_z since the entire element, due to its thinness, lies at the same perpendicular distance y from the z -axis.



Disk element

- If a disk element having a radius y and a thickness dz is chosen for integration, then the volume $dV = (\pi y^2)dz$.
- Using the moment of inertia of the disk element, we can integrate to determine the moment of inertia of the entire body.

Moment of Inertia: Cylinder

Using the general definition for moment of inertia:

$$I = \int_0^M r^2 dm$$

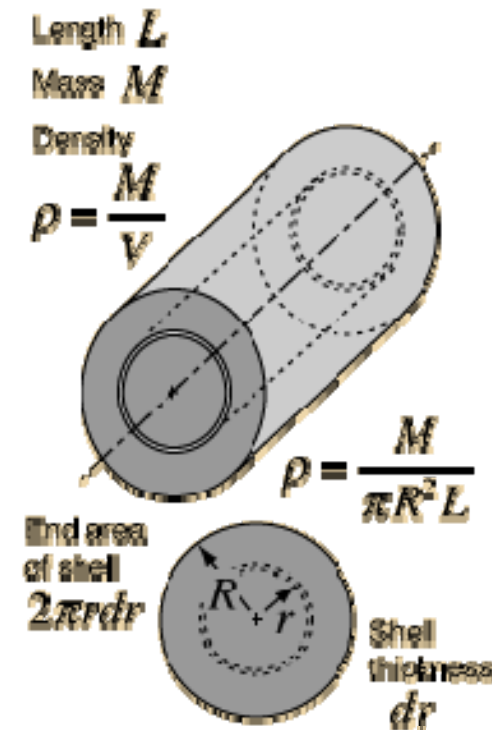
The mass element can be expressed in terms of an infinitesimal radial thickness dr by

Shell element $\longrightarrow dm = \rho dV = \rho L 2\pi r dr$

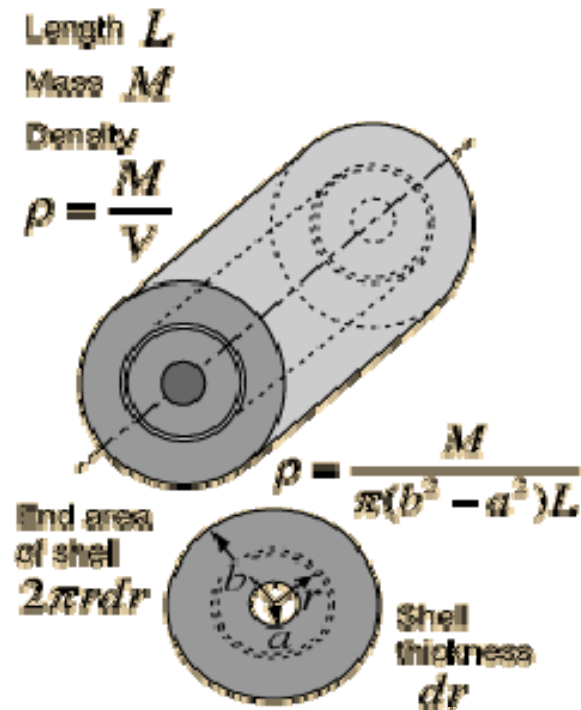
Substituting gives a polynomial form integral:

$$I = 2\pi\rho L \int_0^R r^3 dr = 2\pi\rho L \frac{R^4}{4}$$

$$I = 2\pi \left[\frac{M}{\pi R^2 L} \right] L \frac{R^4}{4} = \frac{1}{2} MR^2$$



Moment of Inertia: Hollow Cylinder



The expression for the **moment of inertia** of a hollow cylinder or hoop of finite thickness is obtained by the same process as that for a **solid cylinder**. The process involves adding up the moments of infinitesimally thin **cylindrical shells**. The only difference from the solid cylinder is that the integration takes place from the inner radius a to the outer radius b :

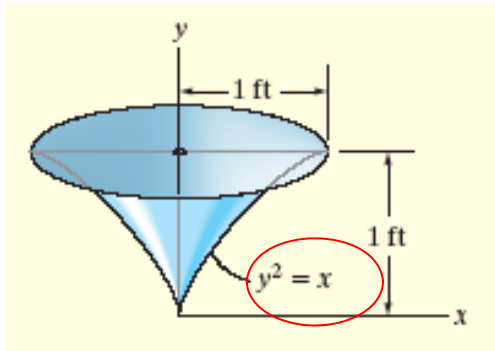
Still use Shell element $\longrightarrow dm = \rho dV = \rho L 2\pi r dr$

$$I = 2\pi\rho L \int_a^b r^3 dr = 2\pi\rho L \left[\frac{b^4}{4} - \frac{a^4}{4} \right]$$

$$I = \frac{\pi}{2} \left[\frac{M}{\pi(b^2 - a^2)L} \right] L [(b^2 - a^2)(b^2 + a^2)]$$

$$I = \frac{1}{2} M (b^2 + a^2)$$

Example 1



Given: The volume shown with $\rho = 5$ slug/ft³.

Find: The mass moment of inertia of this body about the y-axis.

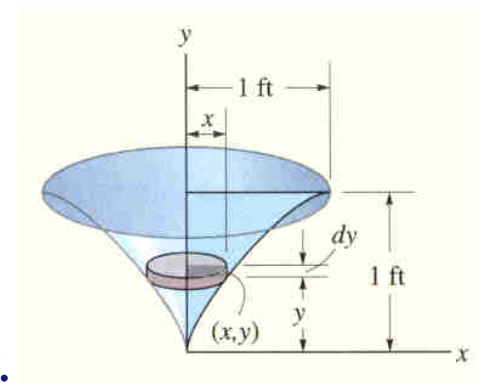
Plan: Find the mass moment of inertia of a disk element about the y-axis, dI_y , and integrate.

Solution: The moment of inertia of a disk about an axis perpendicular to its plane is $I = 0.5 m r^2$. Thus, for the **disk element**, we have

$$dI_y = 0.5 (dm) x^2$$

where the differential mass $dm = \rho dV = \rho \pi x^2 dy$.

$$I_y = \int_0^1 \frac{\rho \pi x^4}{2} dy = \frac{\rho \pi}{2} \int_0^1 y^8 dy = \frac{\pi(5)}{18} = 0.873 \text{ slug} \cdot \text{ft}^2$$



Parallel-Axis theorem

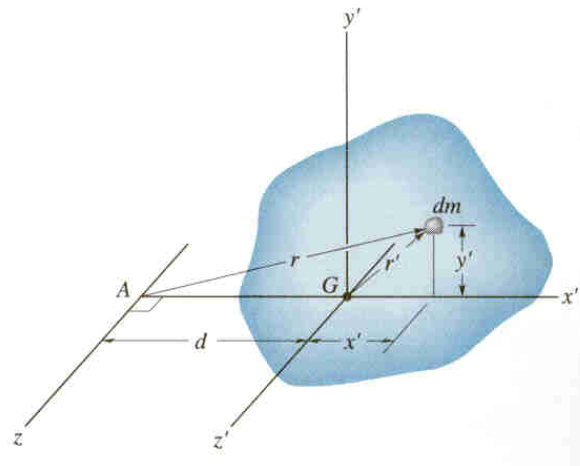
If the mass moment of inertia of a body about an axis passing through the body's mass center is known, then the moment of inertia about any other **parallel axis** may be determined by using the parallel axis theorem,

$$I = I_G + md^2 \quad \leftarrow$$

where I_G = mass moment of inertia about the body's mass center

m = mass of the body

d = **perpendicular** distance between the parallel axes



Parallel-Axis theorem (continues)

Radius of Gyration

The mass moment of inertia of a body about a specific axis can be defined using the radius of gyration (k). The radius of gyration **has units of length** and is a measure of the distribution of the body's mass about the axis at which the moment of inertia is defined.

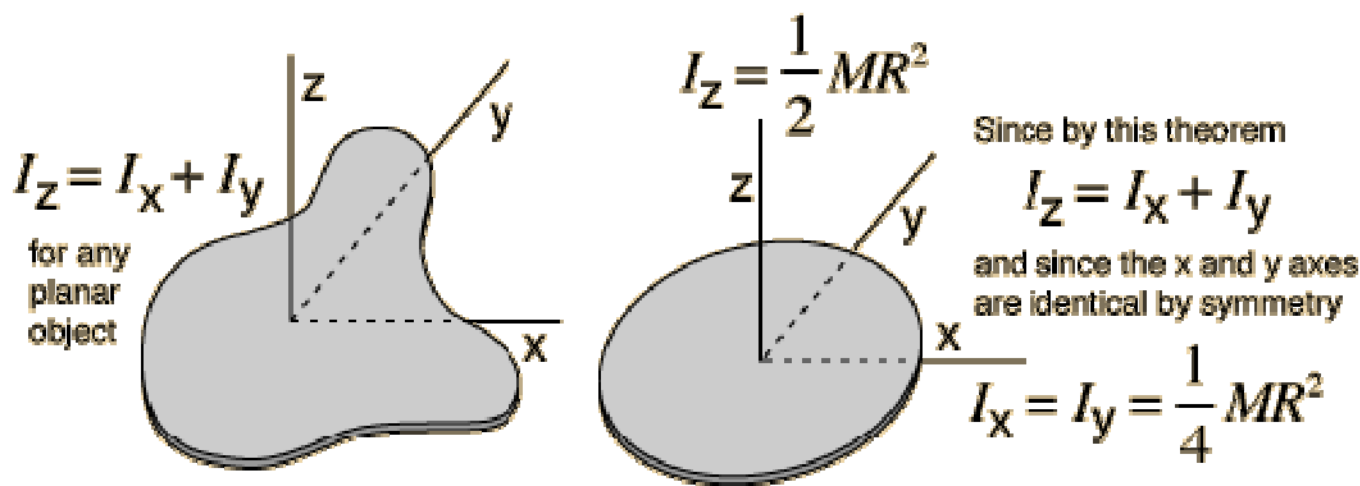
$$I = m k^2 \quad \text{or} \quad k = \sqrt{I/m}$$

Composite Bodies

If a body is constructed of a number of simple shapes, such as disks, spheres, or rods, the mass moment of inertia of the body about any axis can be determined by **algebraically adding** together all the mass moments of inertia, found about the **same axis**, of the different shapes.

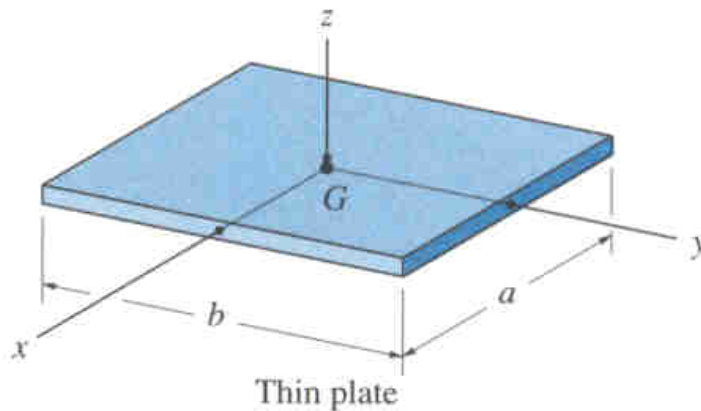
Perpendicular Axis Theorem

For a planar object, the **moment of inertia** about an axis perpendicular to the plane is the sum of the moments of inertia of two perpendicular axes through the same point in the plane of the object. The utility of this theorem goes beyond that of calculating moments of strictly planar objects. It is a valuable tool in the building up of the moments of inertia of three dimensional objects such as **cylinders** by breaking them up into planar disks and summing the moments of inertia of the composite **disks**.

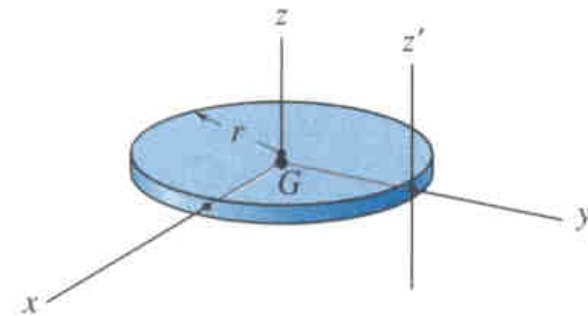


Moment of inertia (continues)

The figures below show the mass moment of inertia formulations for two flat plate shapes commonly used when working with three dimensional bodies. The shapes are often used as the **differential element** being integrated over the entire body.



$$I_{xx} = \frac{1}{12}mb^2 \quad I_{yy} = \frac{1}{12}ma^2 \quad I_{zz} = \frac{1}{12}m(a^2 + b^2)$$



$$I_{xx} = I_{yy} = \frac{1}{4}mr^2 \quad I_{zz} = \frac{1}{2}mr^2 \quad I_{z'z'} = \frac{3}{2}mr^2$$

Perpendicular Axis Theorem

The **perpendicular axis theorem** for planar objects can be demonstrated by looking at the contribution to the three axis **moments of inertia** from an arbitrary mass element. From the **point mass** moment, the contributions to each of the axis moments of inertia are

$$\Delta I_x = \Delta m y^2 \quad \Delta I_y = \Delta m x^2 \quad \Delta I_z = \Delta m r^2$$

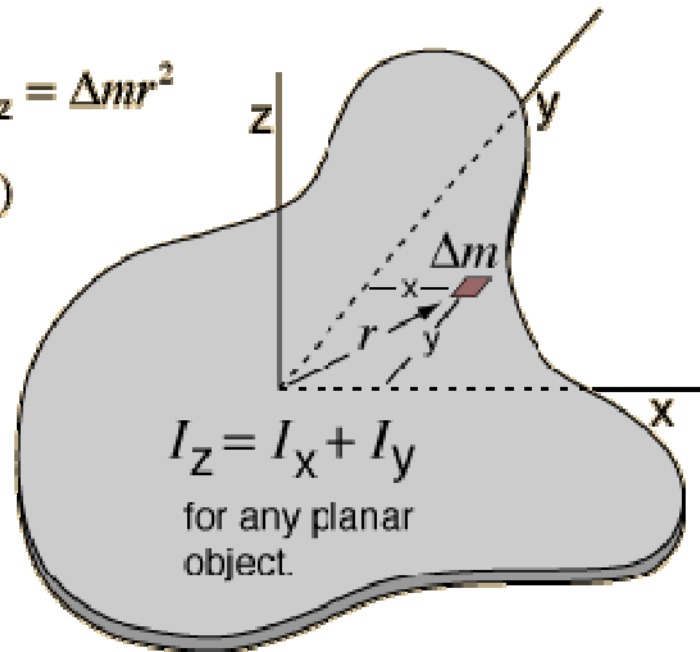
$$\text{Then } \Delta I_x + \Delta I_y = \Delta m(x^2 + y^2)$$

$$\text{but since } r^2 = x^2 + y^2$$

it follows that

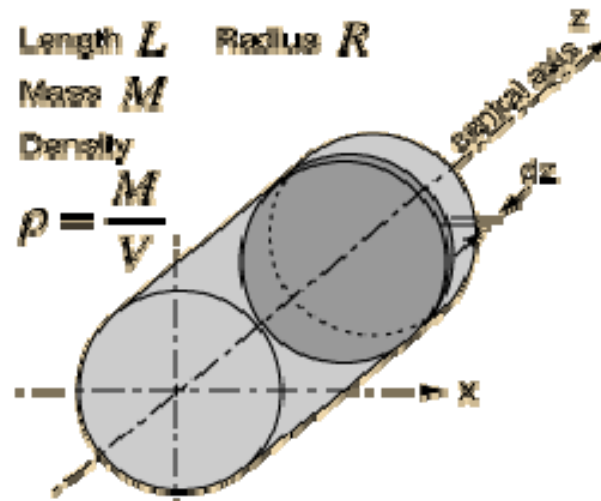
$$\Delta I_x + \Delta I_y = \Delta m r^2 = \Delta I_z$$

Since this is true for any mass element then $I_x + I_y = I_z$



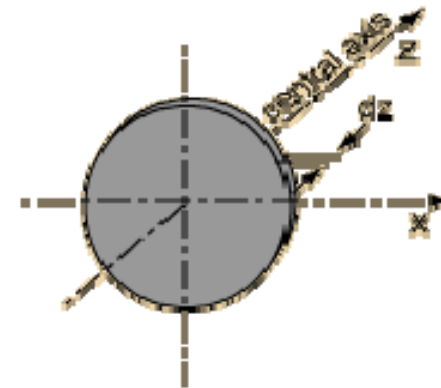
Moment of Inertia: Cylinder About Perpendicular Axis

The development of the expression for the moment of inertia of a cylinder about a diameter at its end (the x-axis in the diagram) makes use of both the **parallel axis theorem** and the **perpendicular axis theorem**. The approach involves finding an expression for a thin disk at distance z from the axis and summing over all such disks.



For a thin disk:

$$dm = \rho dV = \frac{M}{V} A dz = \frac{M}{L} dz$$



For an infinitesimally thin disk of thickness dz , the moment of inertia about the central axis is

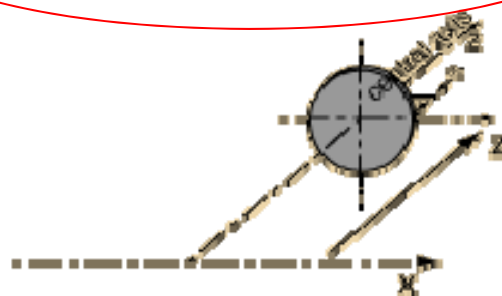
$$dI_z = \frac{1}{2} dm R^2$$

just like any other cylinder about its central axis. But by the perpendicular axis theorem

$$dI_z = dI_x + dI_y$$

Since the x and y moments of inertia must be equal by symmetry, it follows that

$$dI_x = \frac{1}{2} dI_z = \frac{1}{4} dm R^2$$



Obtaining the **moment of inertia** of the full cylinder about a diameter at its end involves summing over an infinite number of thin disks at different distances from that axis. This involves an **integral** from $z=0$ to $z=L$. For any given disk at distance z from the x axis, using the **parallel axis theorem** gives the moment of inertia about the x axis.

$$dI_x = \frac{1}{4}dmR^2 + dmz^2$$

Now expressing the mass element dm in terms of z , we can integrate over the length of the cylinder.

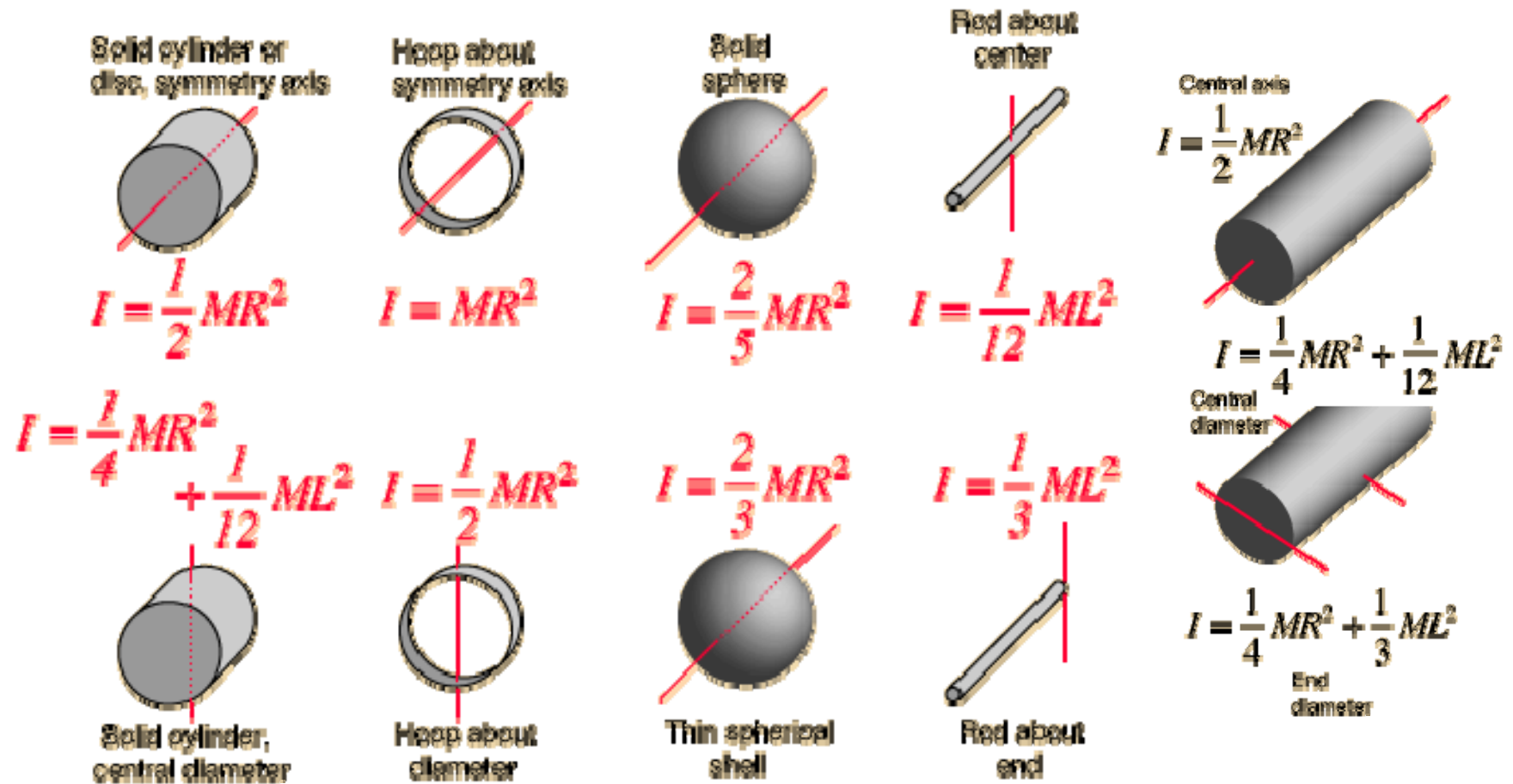
$$I_x = \int_0^L dI_x = \frac{R^2}{4} \frac{M}{L} \int_0^L dz + \frac{M}{L} \int_0^L z^2 dz$$

$$I_x = \frac{1}{4}MR^2 + \frac{1}{3}ML^2$$

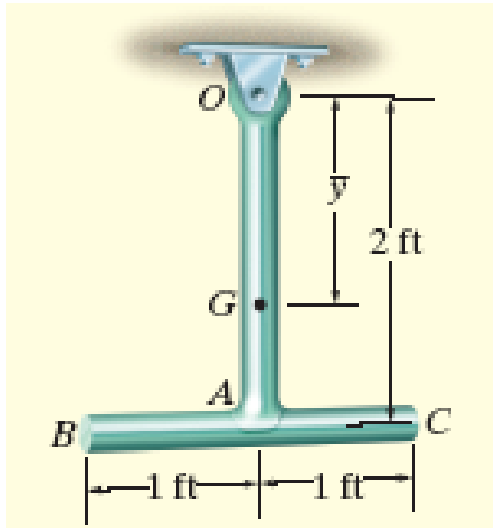
This form can be seen to be plausible. You note that it is the sum of the expressions for a **thin disk** about a diameter plus the expression for a **thin rod** about its end. If you take the limiting case of $R=0$ you get the thin rod expression, and if you take the case where $L=0$ you get the thin disk expression.

The last steps make use of the **polynomial forms** of integrals!

Common Moments of Inertia



Example 2



Given: Two rods assembled as shown, with each rod weighing 10 lb.

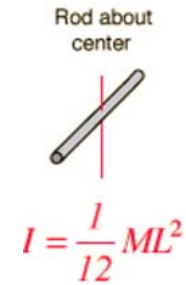
Find: The location of the center of mass G and moment of inertia about an axis passing through G of the rod assembly.

Plan: Find the centroidal moment of inertia for each rod and then use the parallel axis theorem to determine I_G .

Solution: The center of mass is located relative to the pin at O at a distance \bar{y} , where

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i} = \frac{1\left(\frac{10}{32.2}\right) + 2\left(\frac{10}{32.2}\right)}{\frac{10}{32.2} + \frac{10}{32.2}} = 1.5 \text{ ft}$$

Example 2 (continues)



The mass moment of inertia of each rod about an axis passing through its **center of mass** is calculated by using the equation

$$I = (1/12)ml^2 = (1/12)(10/32.2)(2)^2 = 0.104 \text{ slug}\cdot\text{ft}^2$$

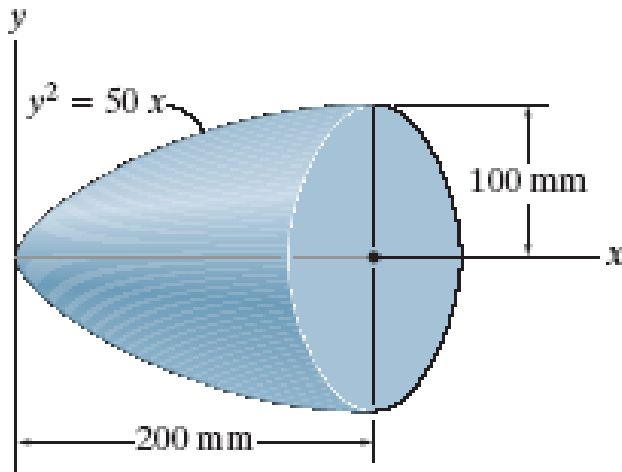
The moment of inertia I_G may then be calculated by using the **parallel axis theorem**.

$$I_G = [I + m(\bar{y}-1)^2]_{OA} + [I + m(2-\bar{y})^2]_{BC}$$

$$I_G = [0.104 + (10/32.2)(0.5)^2] + [0.104 + (10/32.2)(0.5)^2]$$

$$I_G = 0.362 \text{ slug}\cdot\text{ft}^2$$

Example



Given: The density (ρ) of the object is 5 Mg/m^3 .

Find: The radius of gyration, k_y .

Plan: Use a disk element to calculate I_y , and then find k_y .

Solution: Using a disk element (centered on the x-axis) of radius y and thickness dx yields a differential mass dm of

$$dm = \rho \pi y^2 dx = \rho \pi (50x) dx$$

The differential moment of inertia dI_y , about the y-axis passing through the **center of mass** of the element is

$$dI_y = (1/4)y^2 dm = 625 \rho \pi x^2 dx$$

Disk rotate about y-axis

Example (Cont.)

Using the **parallel axis theorem**, the differential moment of inertia about the y-axis is then

$$dI_y = dI_{y'} + dm(x^2) = \rho\pi(625x^2 + 50x^3) dx$$

Integrate to determine I_y :

$$I_y = \int dI_y = \int_0^{200} \rho\pi(625x^2 + 50x^3) dx = \rho\pi\left[\left(\frac{625}{3}\right)(200^3) + \left(\frac{50}{4}\right)(200^4)\right]$$

$$I_y = 21.67 \times 10^9 \rho\pi$$

The mass of the solid is

$$m = \int dm = \int_0^{200} \rho\pi(50x) dx = \rho\pi(25)(200)^2 = 1 \times 10^6 \rho\pi$$

Therefore $I_y = 21.67 \times 10^3 \text{ m}$ and $k_y = \sqrt{I_y/m} = 147.2 \text{ mm}$

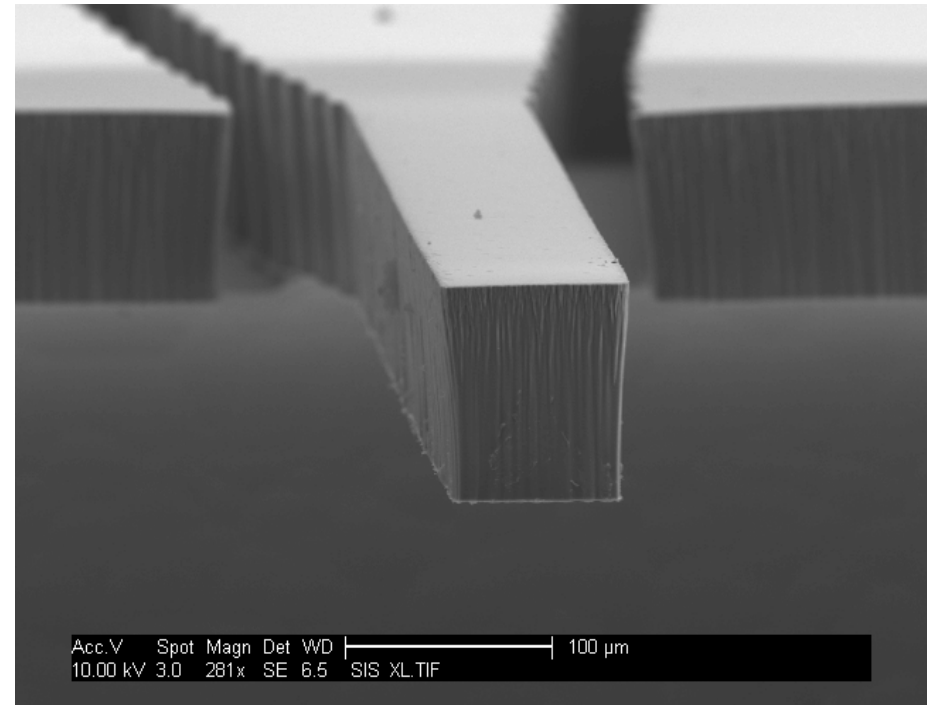
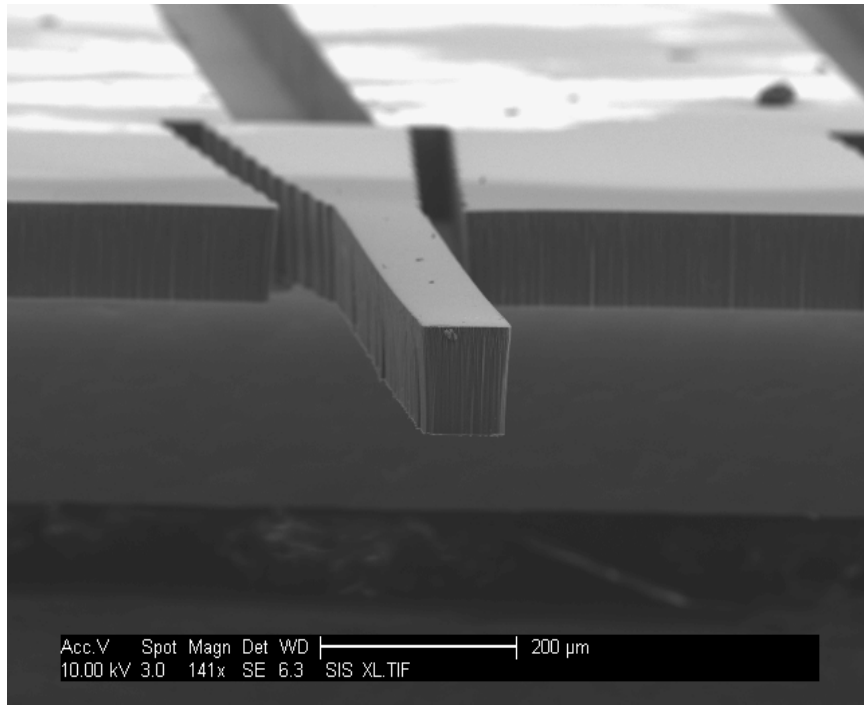
Homework Assignment

Chapter 17- 6, 23, 27, 33, 38, 43, 53, 59, 74, 79, 95,
98, 102, 109

Due next Wednesday !!!

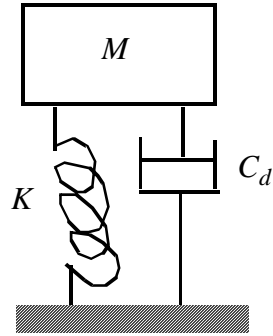
Composite Bodies example

Cantilever based sensors and actuators

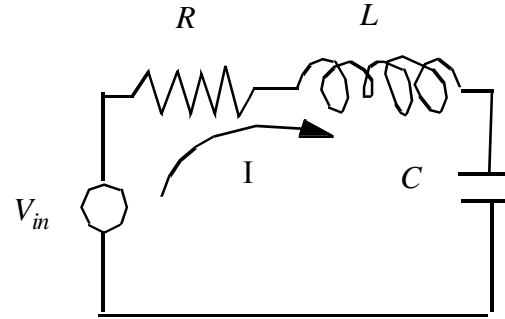


**SU8
Beams**

UWMitech



Simplified Mechanical Model



Equivalent Circuit

$$F(t) = C_d \dot{x} + M\ddot{x} + Kx$$

$$x_{\max} = \frac{F}{(C_d \omega_o)}$$

$$\omega_o = \sqrt{\frac{k}{m}}$$

$$\xi = \frac{1}{2Q} = \frac{C_d}{2\omega_o M} = \frac{C_d}{2\sqrt{KM}}$$

$$\dot{x} \Leftrightarrow I$$

$$F \Leftrightarrow V$$

$$C_d \Leftrightarrow R$$

$$K \Leftrightarrow \frac{1}{C}$$

$$M \Leftrightarrow L$$

$$V_{in} = IR + (1/C) \int Idt + L \frac{dI}{dt}$$

$$I = \frac{V}{R}$$

$$\omega_o = \sqrt{\frac{1}{LC}}$$

$$\xi = \frac{1}{2Q} = \frac{R}{2\omega_o L} = \frac{R}{2} \sqrt{\frac{C}{L}}$$

Resonant frequency

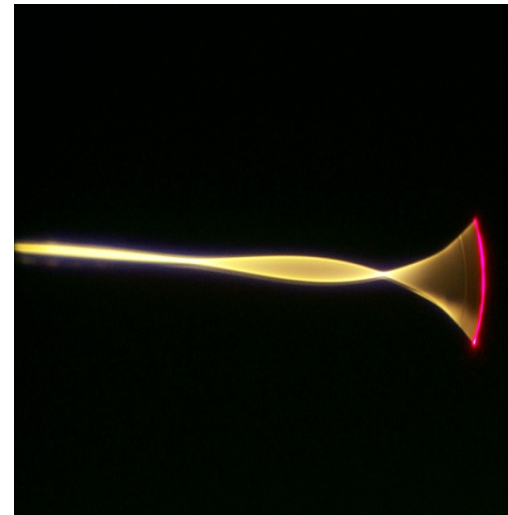
Resonant frequency of a cantilever beam

$$f_o = \frac{1}{2\pi} \left(\frac{1.875}{l} \right)^2 \left(\frac{E_{eq} I_{eq}}{\rho_{eq} A_{eq}} \right)^{0.5}$$

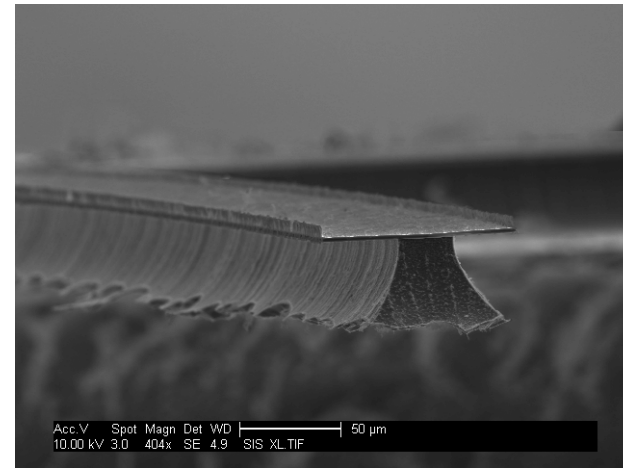
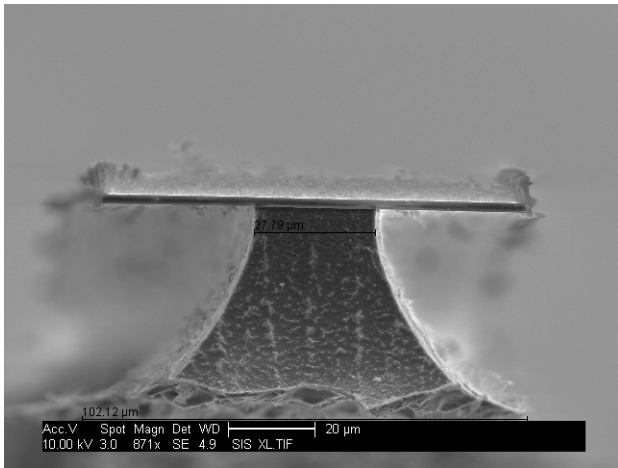
Where the equivalents of E, I, ρ, and A are Young's modulus, moment of inertia ($bh^3/12$), mass density and cross sectional area (bh) of a rectangular beam

b = base of the rectangular cross section and

h = height of the rectangular cross section



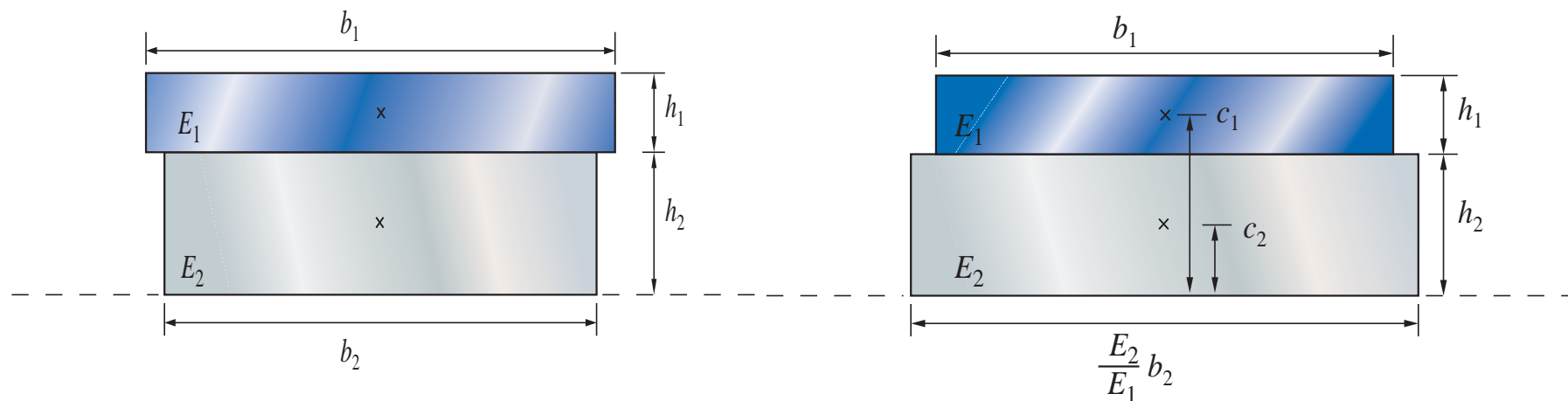
Composite structure



UWMictech

SiO₂/Si composite beam

Composite beam



For a composite beam bending along its horizontal axis, the stiffness of different materials can be handled by selecting one layer as a reference material and then adjusting the widths of the other layers relative to the reference material.

Example: the top layer $E = 70$ GPa and the bottom is 140 GPa & assuming both originally have the same width, if the top layer is selected, the bottom layer needs to be twice as wide to keep the same bending stiffness if the whole beam was made out of just the material of the top layer

Where the equivalents of E , I , ρ , and A are found by using the method of composite beams [13]. To compensate for stiffer or more flexible layers of a composite, this method adjusts the geometry (width only) of each of the layers of the composite beam, as illustrated in Figure 8. This allows one to use the one reference value of Young's modulus (labeled E_1 for this case) for the entire beam by adjusting the beam's geometry to compensate for having a second Young's modulus (E_2). After adjusting the geometry of the beam, the equivalent variables in eq (1) are found by the following formulas: (assuming all layers have rectangular cross sections)

$$I_{eq} = \frac{1}{E_1} \sum_{i=1}^n E_i I_i + \frac{1}{E_1} \sum_{i=1}^n E_i A_i (\bar{c} - c_i)^2$$

where E_i = Young's modulus of layer i
 I_i = moment of inertia of layer i
 A_i = original unadjusted area of layer i
 c_i = centroid of layer i

where centroid of the adjusted beam is found by:

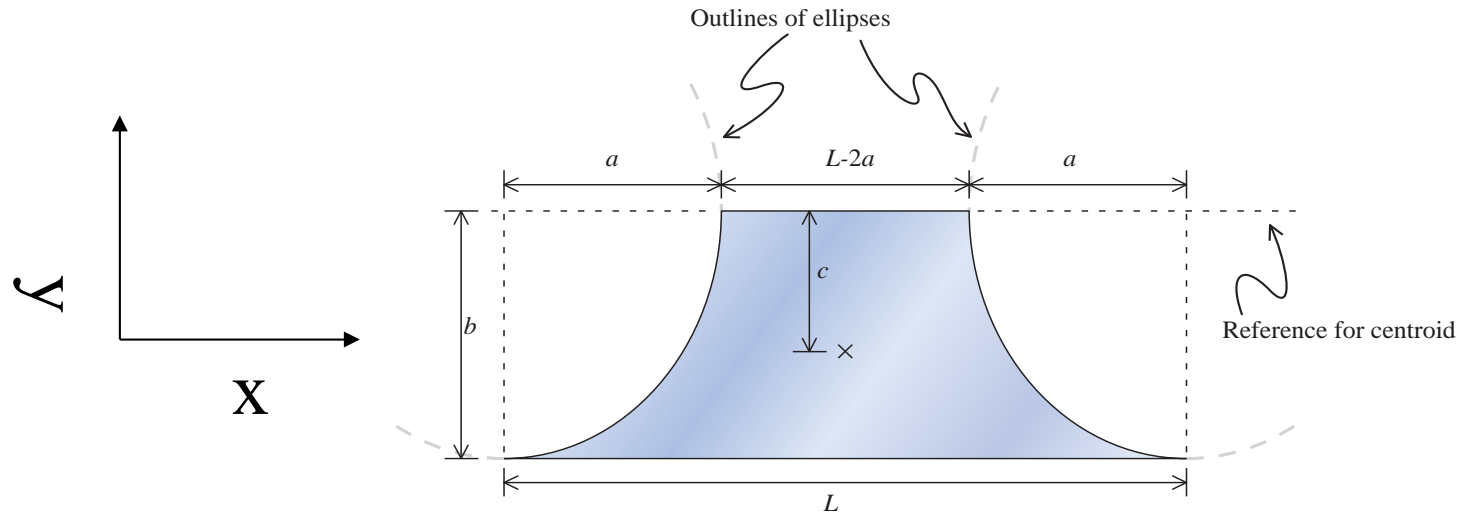
$$\bar{c} = \frac{\sum_{i=1}^n c_i A_i}{\sum_{i=1}^n A_i}$$

The adjusted cross-sectional area is:

$$A_{eq} = \frac{1}{E_1} \sum A_i E_i$$

The mass density is adjusted to maintain the same mass per unit length:

$$\rho_{eq} = \frac{E_i \sum_{i=1}^n \rho_i A_i}{\sum_{i=1}^n E_i A_i}$$



Based on the actual etching profile, an elliptical shape curve (fig 9) is used to approximate the silicon section of the beam. The centroid of the silicon/silicon oxide composite beam is therefore equal to,

$$c_{silicon} = \frac{\int_0^b y(L - 2a\sqrt{1 - \frac{y^2}{b^2}})dy}{(Lb - \frac{\pi ab}{2})} = \frac{b(4a - 3L)}{3(\pi a - 2L)}$$

And the moment of inertia of the composite beam with respect to x axis is,

$$I_x = \int_0^b y^2 dA = \int_0^b y^2 [L - 2(a\sqrt{1 - \frac{y^2}{b^2}})]dy = \frac{1}{24}b^3(8L - 3a\pi)$$

Development of an Optical Waveguide Cantilever Scanner

UWME Dynamics and Vibration Laboratory



Lever #	Thickness/width/length ($\mu\text{m}/\mu\text{m}/\mu\text{m}$)	Observed resonant frequency (kHz)	Calculated resonant frequency (kHz)	Observed Q factor
1	2.2 / 93 / 1500 (SiO_2) 30 / 52 / 1500 (Si)	16.90	16.84	260
2	2.2 / 97 / 1500 (SiO_2) 31 / 50 / 1500 (Si)	17.77	17.36	296
3	2.2 / 97 / 1000 (SiO_2) 30 / 84 / 1000 (Si)	39.62	37.89	514
4	2.2 / 92 / 1000 (SiO_2) 33 / 56 / 1000 (Si)	43.10	41.42	540
5	2.2 / 96 / 490 (SiO_2) 35 / 59 / 490 (Si)	N/A	172.50	N/A

Table1. Resonance frequencies of the SiO_2 /Si composite levers.
W. Wang

Planar kinetics of a rigid body: Force and acceleration

Chapter 17

Chapter objectives

- Introduce the methods used to determine the mass moment of inertia of a body
- To develop the planar kinetic equations of motion for a symmetric rigid body
- To discuss applications of these equations to bodies undergoing translation, rotation about fixed axis, and general plane motion



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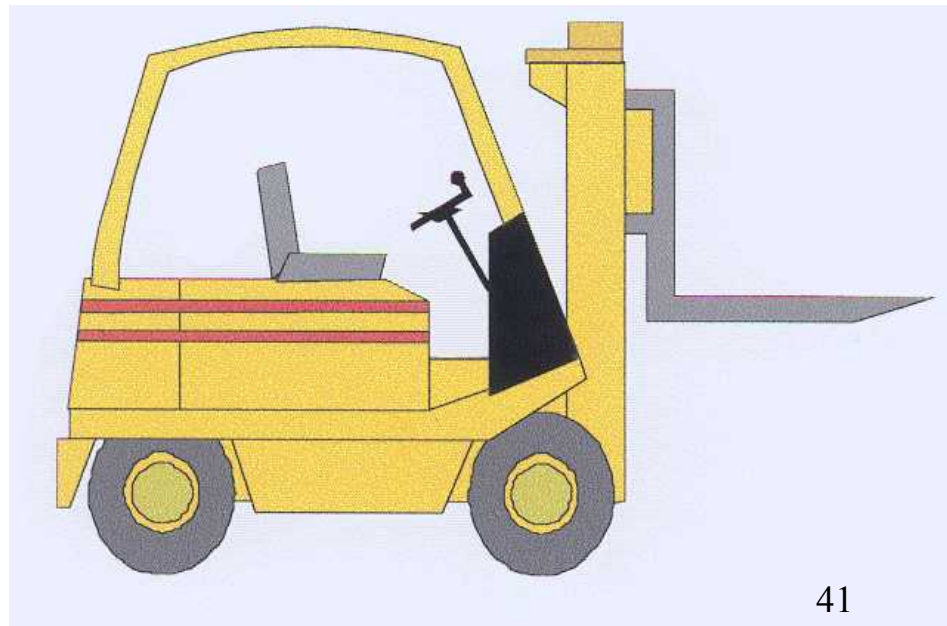
Lecture 16

- **Planar kinetics of a rigid body: Force and acceleration**

Planar kinetic equations of motion

Equations of motion: translation

- 17.2-17.3



Material covered

- Planar kinetics of a rigid body : Force and acceleration

Planar kinetic equations of motion and equations of motion when a rigid body undergoes translation



...Next

lecture...continue with

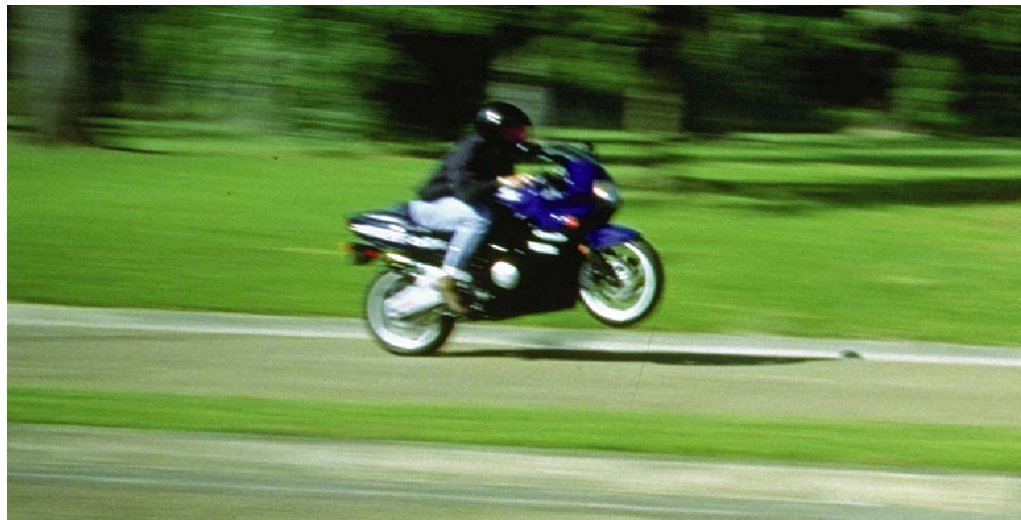
Ch.17

W. Wang

Today's Objectives

Students should be able to:

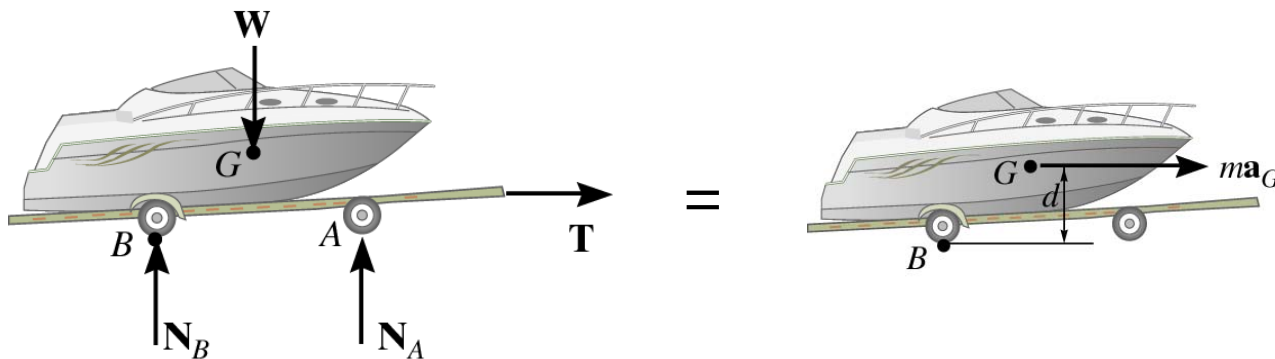
1. Apply the three equations of motion for a rigid body in planar motion.
2. Analyze problems involving translational motion.



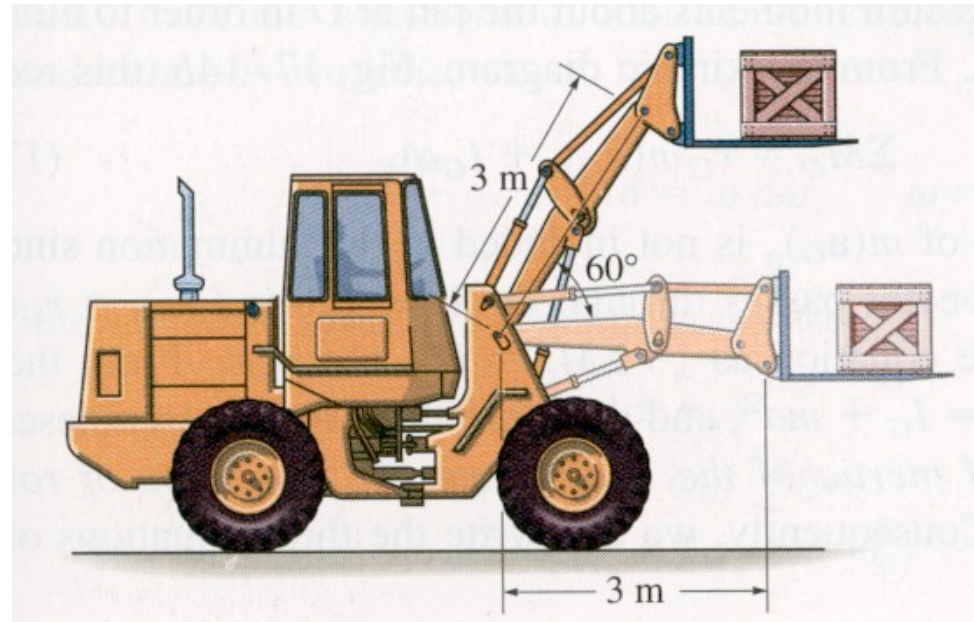
Applications



The boat and trailer undergo rectilinear motion. In order to find the reactions at the trailer wheels and the acceleration of the boat at its center of mass, we need to draw the FBD for the boat and trailer.



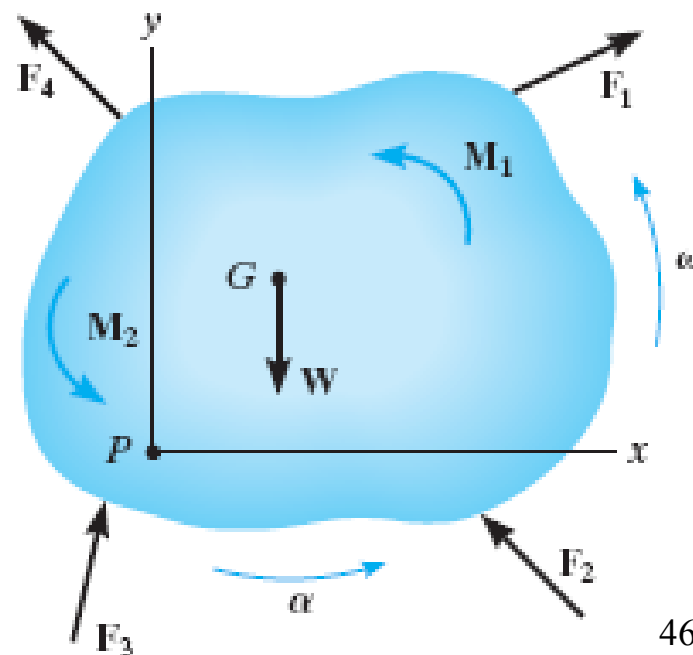
Applications (continues)



As the tractor raises the load, the crate will undergo curvilinear translation if the forks do not rotate.

Planar kinetic equations of motion (17.2)

- We will limit our study of **planar kinetics** to rigid bodies that are symmetric with respect to a fixed reference plane.
- As discussed in Chapter 16, when a body is subjected to general plane motion, it undergoes a combination of **translation** and **rotation**.
- First, a coordinate system with its origin at an arbitrary point P is established. The x - y axes should not rotate and can only either be fixed or translate with constant velocity.

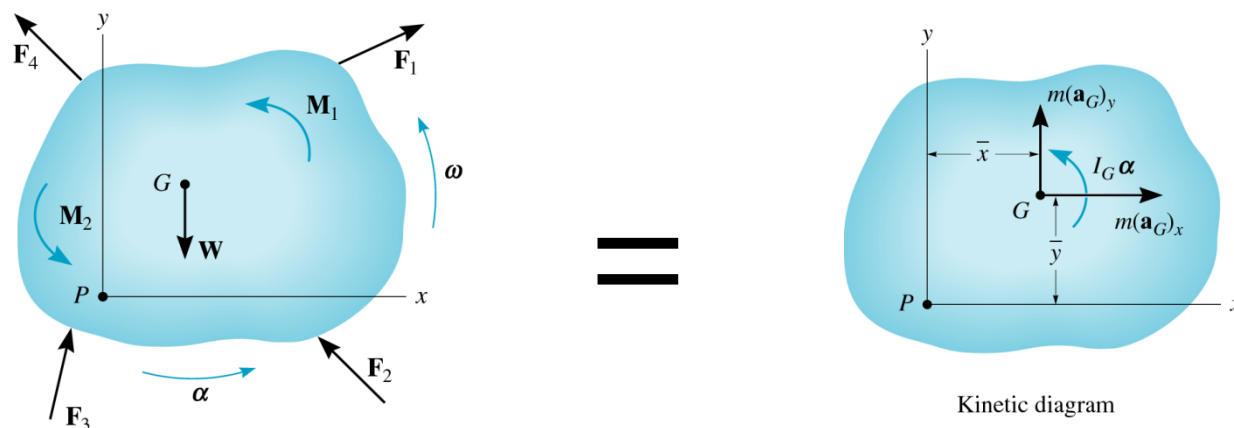


Planar kinetic equations of motion (17.2) (continued)

- If a body undergoes **translational motion**, the **equation of motion** is $\Sigma \mathbf{F} = m \mathbf{a}_G$. This can also be written in scalar form as

$$\Sigma F_x = m(a_G)_x \quad \text{and} \quad \Sigma F_y = m(a_G)_y$$

- In words: the sum of all the external forces acting on the body is equal to the body's mass times the acceleration of its mass center.



Equation of rotational motion (17.2)

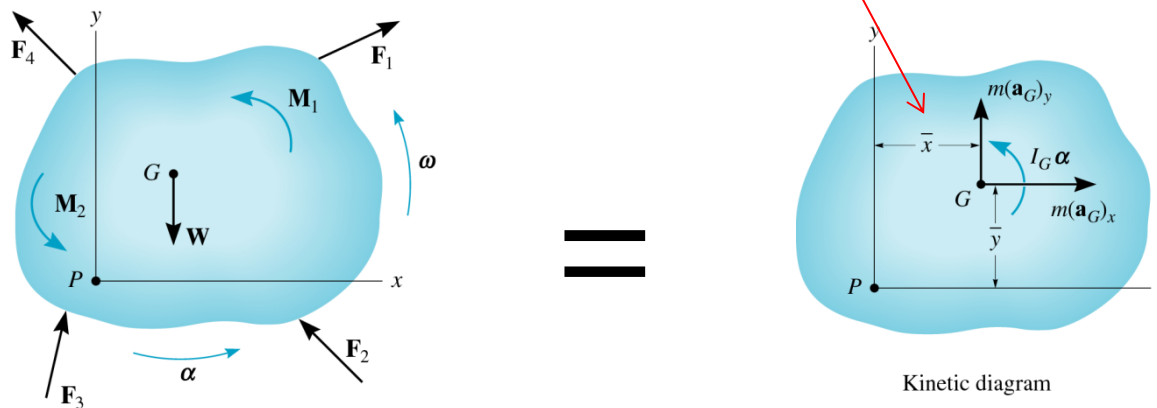
We need to determine the effects caused by the moments of the external force system. The moment about point P can be written as

$$\Sigma (\mathbf{r}_i \times \mathbf{F}_i) + \Sigma \mathbf{M}_i = \mathbf{r}_G \times m\mathbf{a}_G + I_G \boldsymbol{\alpha}$$

$$\Sigma M_p = \Sigma (M_k)_p$$

Wont have this equation if it's particle

where ΣM_p is the resultant moment about P due to all the external forces. The term $\Sigma (\mathbf{M}_k)_p$ is called the **kinetic moment** about point P.



Equation of rotational motion (17.2) (continues)

If point P coincides with the mass center G, this equation reduces to the **scalar equation** of $\Sigma M_G = I_G \alpha$.

In words: the resultant (summation) moment about the mass center due to all the external forces is equal to the moment of inertia about G times the angular acceleration of the body.

Thus, **three** independent **scalar** equations of motion may be used to describe the general planar motion of a rigid body. These equations are:

$$\Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

$$\text{and } \Sigma M_G = I_G \alpha \quad \text{or} \quad \Sigma M_p = \Sigma (M_k)_p$$

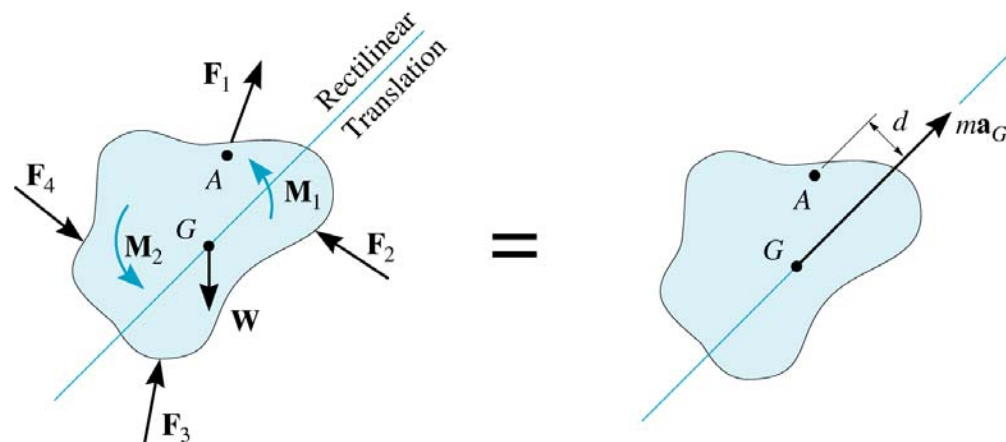
Equations of motion: Translation (17.3)

When a rigid body undergoes **only translation**, all the particles of the body have the same acceleration so $\mathbf{a}_G = \mathbf{a}$ and $\boldsymbol{\alpha} = \mathbf{0}$. The equations of motion become:

$$\Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

$$\Sigma M_G = 0$$



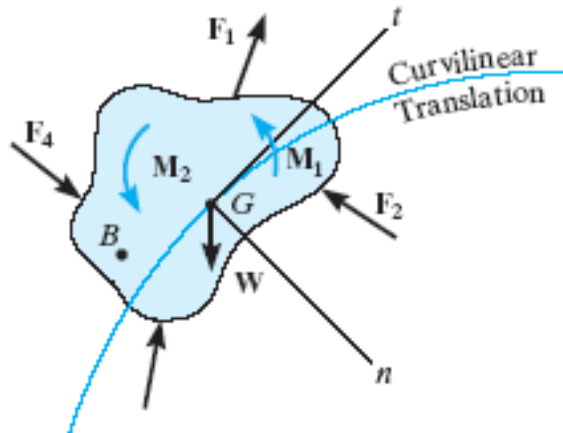
Note that, if it makes the problem easier, the moment equation can be applied about other points instead of the mass center. In this case,

$$\Sigma M_A = \mathbf{r}_G \times m\mathbf{a}_G = (m a_G) d .$$

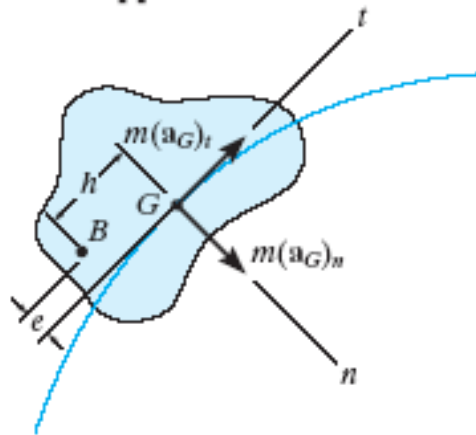
Equations of motion: Translation (17.3)

(continues)

When a rigid body is subjected to **curvilinear translation**, it is best to use an **n-t coordinate system**. Then apply the equations of motion, as written below, for n-t coordinates.



||



$$\Sigma F_n = m(a_G)_n$$

$$\Sigma F_t = m(a_G)_t$$

$$\Sigma M_G = 0 \quad \text{or}$$

$$\Sigma M_B = \mathbf{r}_G \times m\mathbf{a}_G = e[m(a_G)_t] - h[m(a_G)_n]$$

Procedure of analysis

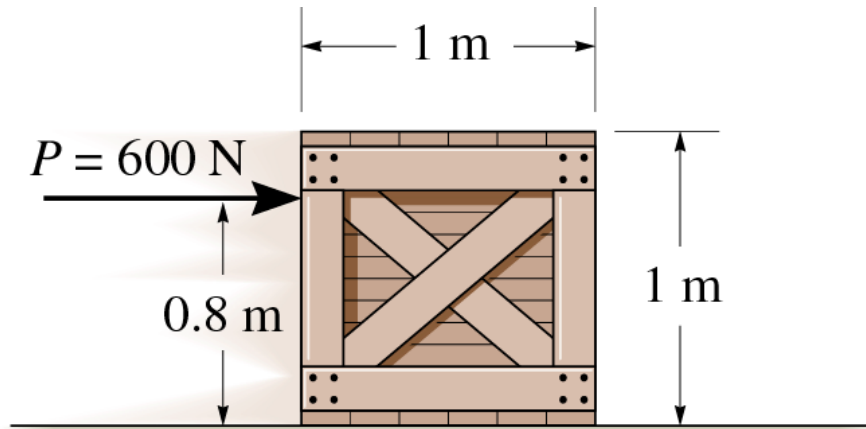
Problems involving kinetics of a rigid body in only translation should be solved using the following procedure:

1. Establish an (x-y) or (n-t) inertial coordinate system and specify the sense and direction of acceleration of the mass center, \mathbf{a}_G .
2. Draw a FBD and kinetic diagram showing all external forces, couples and the **inertia forces and couples**.
3. Identify the unknowns.
4. Apply the **three equations of motion**:

$$\left. \begin{array}{l} \Sigma F_x = m(a_G)_x \quad \Sigma F_y = m(a_G)_y \\ \Sigma M_G = 0 \quad \text{or} \quad \Sigma M_P = \Sigma (M_k)_P \end{array} \right| \begin{array}{l} \Sigma F_n = m(a_G)_n \quad \Sigma F_t = m(a_G)_t \\ \Sigma M_G = 0 \quad \text{or} \quad \Sigma M_P = \Sigma (M_k)_P \end{array}$$

5. Remember, friction forces always act on the body opposing the motion of the body.

Example



Given: A 50 kg crate rests on a horizontal surface for which the kinetic friction coefficient $\mu_k = 0.2$.

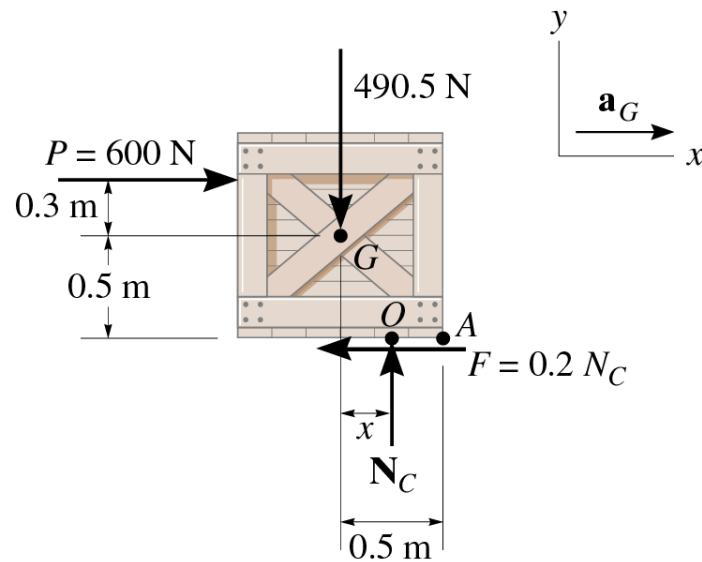
Find: The acceleration of the crate if $P = 600 \text{ N}$.

Plan: Follow the procedure for analysis.

Note that the load P can cause the crate either to slide or to tip over. Let's assume that the crate slides. We will check this assumption later.

Example continues

Solution:



The coordinate system and **FBD** are as shown. The weight of $(50)(9.81)$ N is applied at the center of mass and the normal force N_c acts at O. Point O is some distance x from the crate's center line. The unknowns are N_c , x , and a_G .

Applying the **equations of motion**:

$$\Sigma F_x = m(a_G)_x: 600 - 0.2 N_c = 50 a_G$$

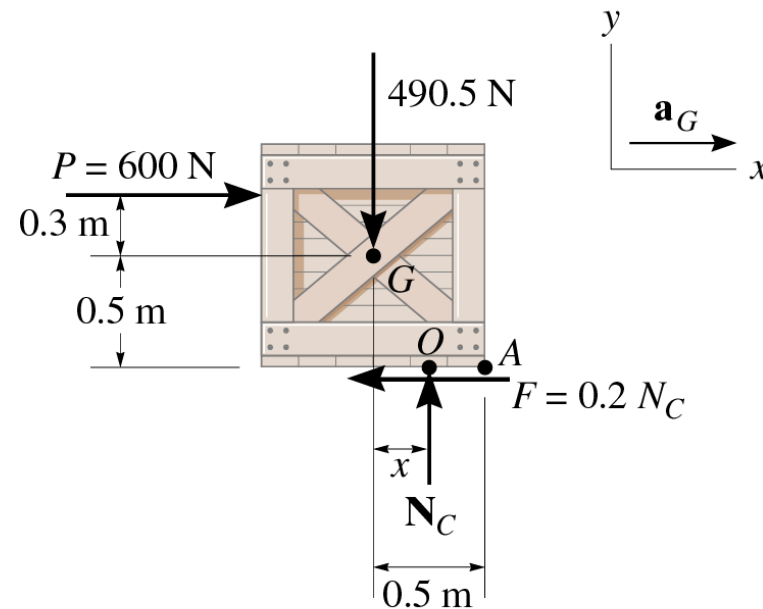
$$N_c = 490 \text{ N}$$

$$\Sigma F_y = m(a_G)_y: N_c - 490.5 = 0$$

$$\rightarrow x = 0.467 \text{ m}$$

$$\Sigma M_G = 0: -600(0.3) + N_c(x) - 0.2 N_c (0.5) = 0 \quad a_G = 10.0 \text{ m/s}^2$$

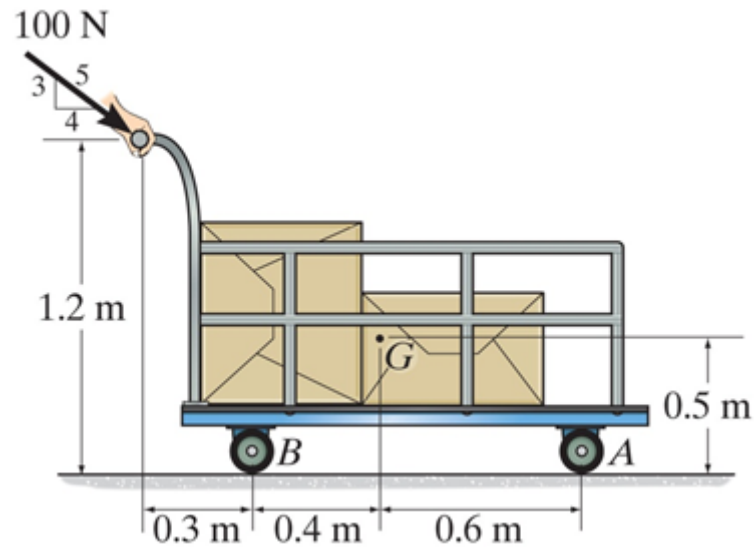
Example continues



Since $x = 0.467 \text{ m} < 0.5 \text{ m}$, the crate slides as originally assumed.

If x was greater than 0.5 m , the problem would have to be reworked with the assumption that tipping occurred.

EXAMPLE



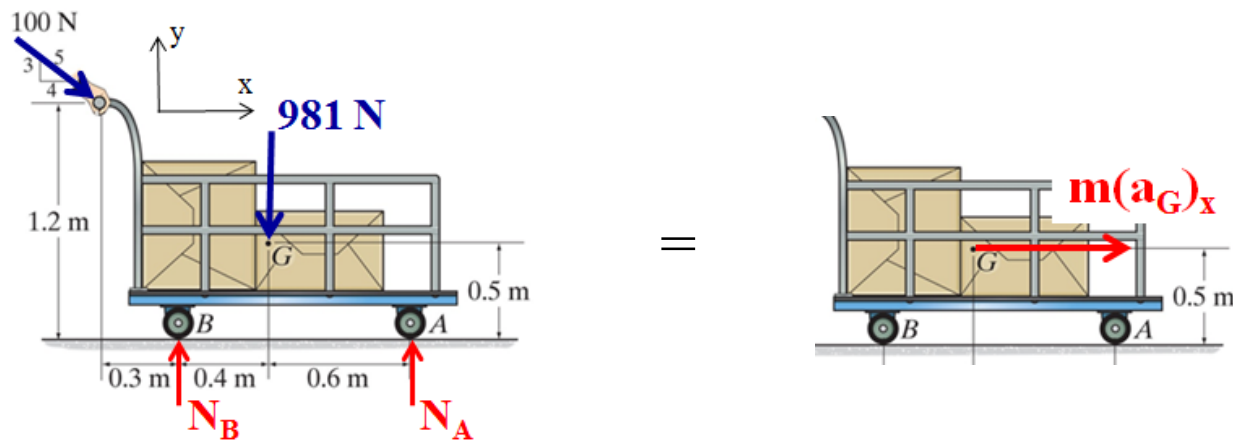
Given: The cart and its load have a total mass of 100 kg and center of mass at G . A force of $P = 100\text{ N}$ is applied to the handle. Neglect the mass of the wheels.

Find: The normal reactions at each of the two wheels at A and B.

Plan: Follow the procedure for analysis.

EXAMPLE (continued)

Solution: The cart will move along a rectilinear path.
Draw the **FBD** and **kinetic diagram**.

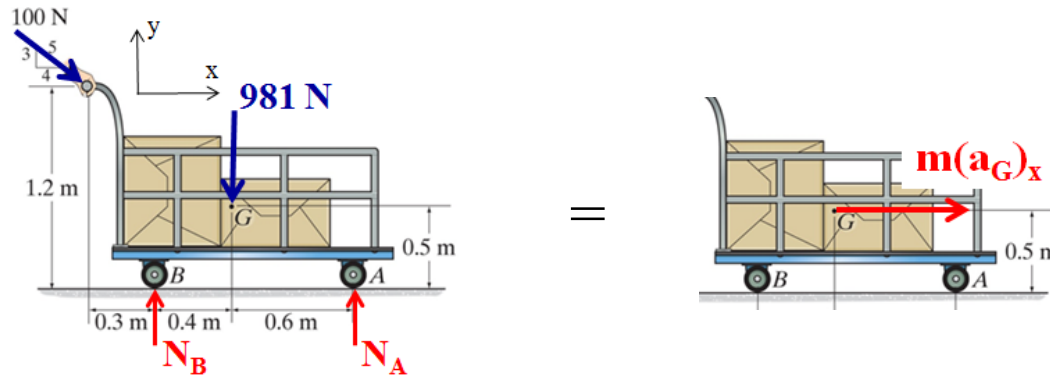


Apply the **equation of motion in the x-direction** first:

$$\begin{aligned} \overset{+}{\rightarrow} \Sigma F_x &= m(a_G)_x \\ 100 (4/5) &= 100 a_G \\ a_G &= 0.8 \text{ m/s}^2 \end{aligned}$$

EXAMPLE (continued)

Then apply the equation of motion in the y-direction and sum moments about G.



$$+\uparrow \Sigma F_y = 0 \Rightarrow N_A + N_B - 981 - 100(3/5) = 0$$

$$N_A + N_B = 1041 \text{ N} \quad (1)$$

$$+\circlearrowleft \Sigma M_G = 0$$

$$\Rightarrow N_A(0.6) - N_B(0.4) + 100(3/5)(0.7) - 100(4/5)(1.2 - 0.5) = 0$$

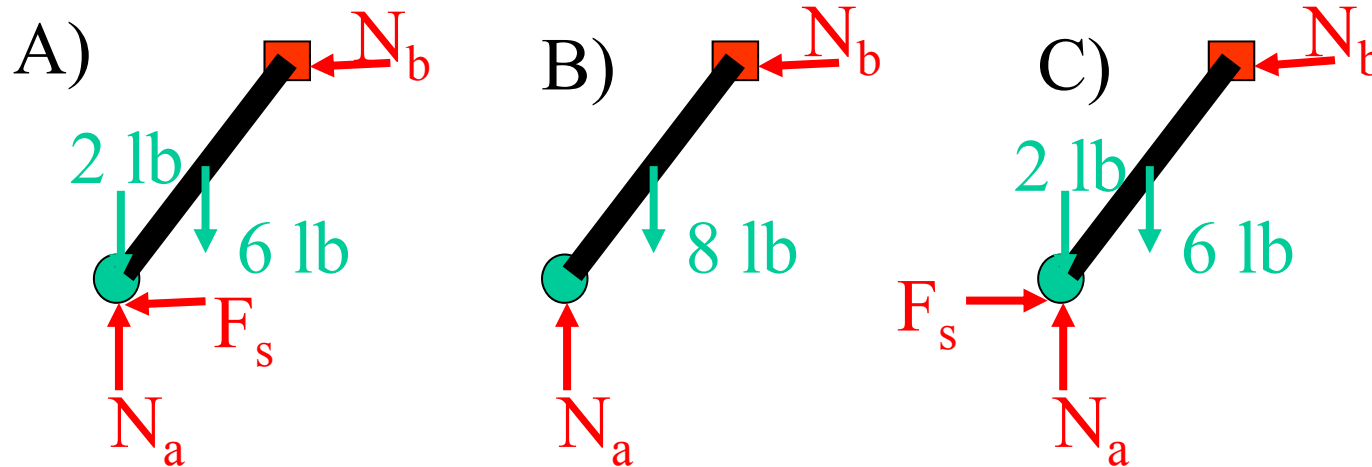
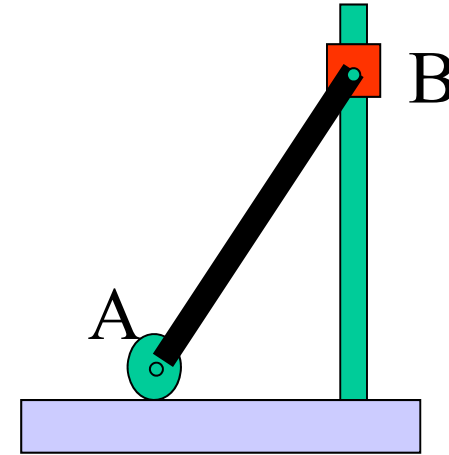
$$0.6 N_A - 0.4 N_B = 14 \text{ N m} \quad (2)$$

Using Equations (1) and (2), solve for the reactions, N_A and N_B

$$N_A = 430 \text{ N} \text{ and } N_B = 611 \text{ N}$$

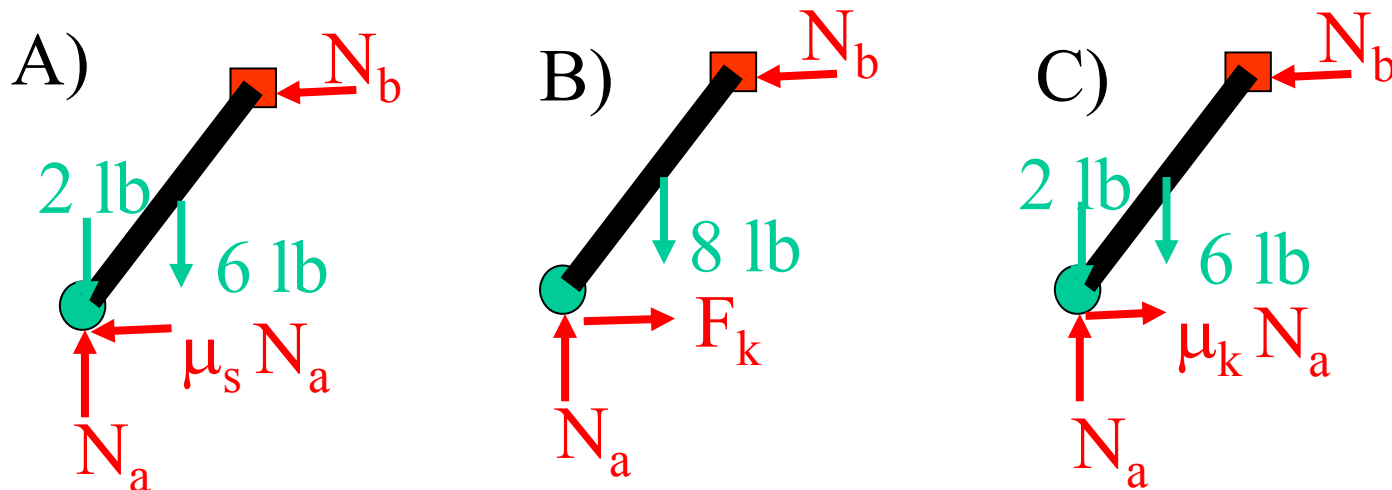
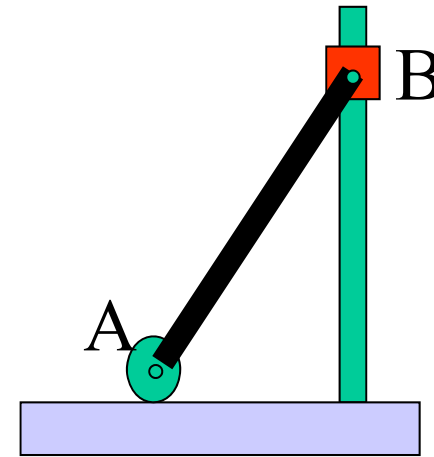
CONCEPT QUIZ

1. A 2 lb disk is attached to a uniform 6 lb rod AB with a frictionless collar at B. If the disk rolls **without** slipping, select the correct FBD.

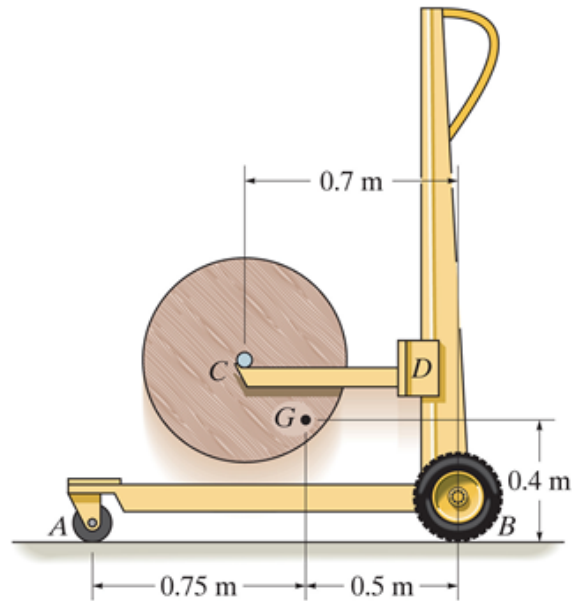


CONCEPT QUIZ

2. A 2 lb disk is attached to a uniform 6 lb rod AB with a frictionless collar at B. If the disk rolls **with** slipping, select the correct FBD.



Example



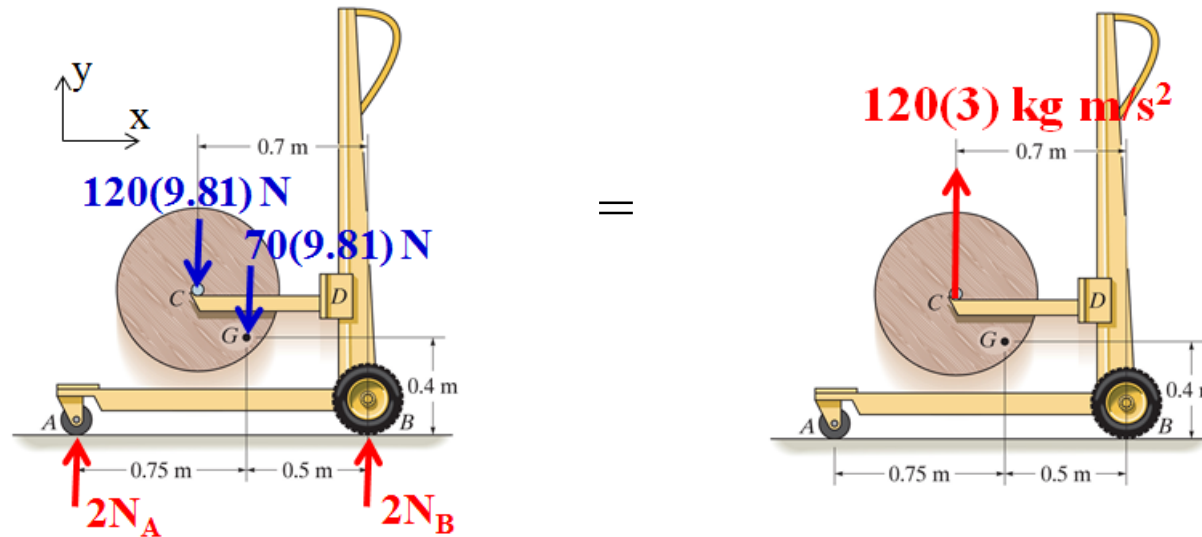
Given: The lift truck has a mass of 70 kg and mass center at G. It lifts the 120-kg spool with an acceleration of 3 m/s^2 . The spool's mass center is at C. You can neglect the mass of the movable arm CD.

Find: The normal reactions at each of the four wheels.

Plan: Follow the procedure for analysis.

Example (continued)

Solution: Draw FBD and kinetic diagram.



Applying the equations of motion:

$$\curvearrowleft + \sum M_B = \sum (M_k)_B$$

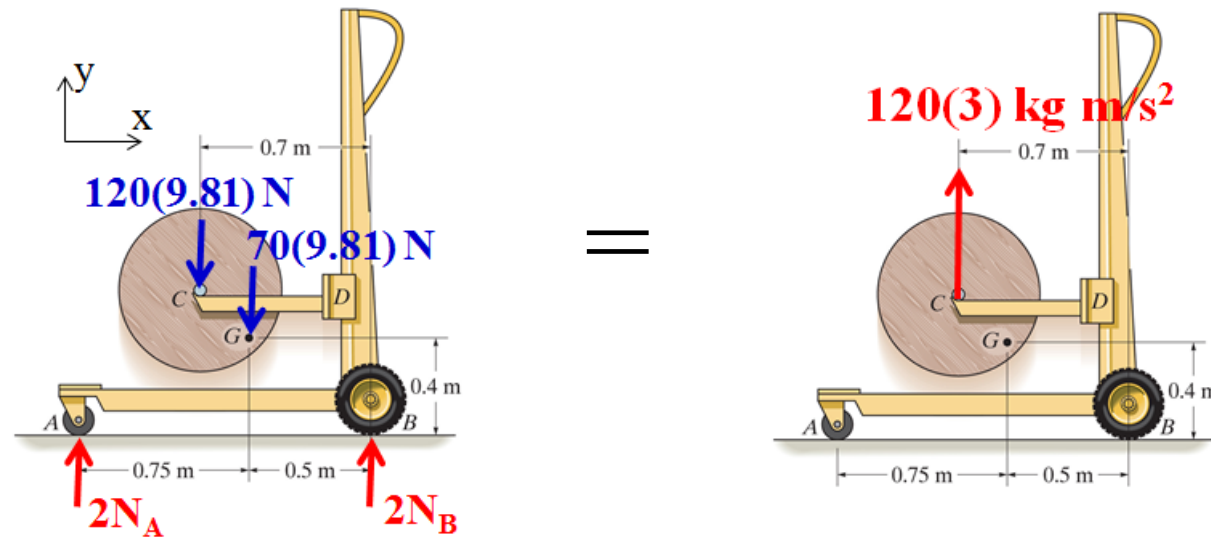
Summation of moment about B

$$\sum (\mathbf{r}_i \times \mathbf{F}_i) + \sum \mathbf{M}_i = \mathbf{r}_G \times m\mathbf{a}_G + I_G \alpha$$

$$\Rightarrow 70(9.81)(0.5) + 120(9.81)(0.7) - 2N_A(1.25) = -120(3)(0.7)$$

$$N_A = 568 \text{ N}$$

Example (continued)



$$+\uparrow \Sigma F_y = m(a_G)_y$$

$$\Rightarrow 2 N_A + 2 N_B - 120(9.81) - 70 (9.81) = 120 (3)$$

$$2 N_A + 2 N_B = 2224 \text{ N}$$

Since $N_A = 568 \text{ N}$, $N_B = 544 \text{ N}$

Homework Assignment

Chapter 17- 6, 23, 27, 33, 38, 43, 53, 59, 74, 79, 95,
98, 102, 109

Due next Wednesday !!!



Planar kinetics of a rigid body: Force and acceleration

Chapter 17

Chapter objectives

- Introduce the methods used to determine the mass moment of inertia of a body
- To develop the planar kinetic equations of motion for a symmetric rigid body
- To discuss applications of these equations to bodies undergoing translation, rotation about fixed axis, and general plane motion



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Lecture 18

- **Planar kinetics of a rigid body: Force and acceleration**

Equations of Motion: Rotation about a Fixed Axis

Equations of Motion: General Plane Motion

- 17.4-17.5



Material covered

- **Planar kinetics of a rigid body : Force and acceleration**

Equations of motion

- 1) **Rotation about a fixed axis**
- 2) **General plane motion**

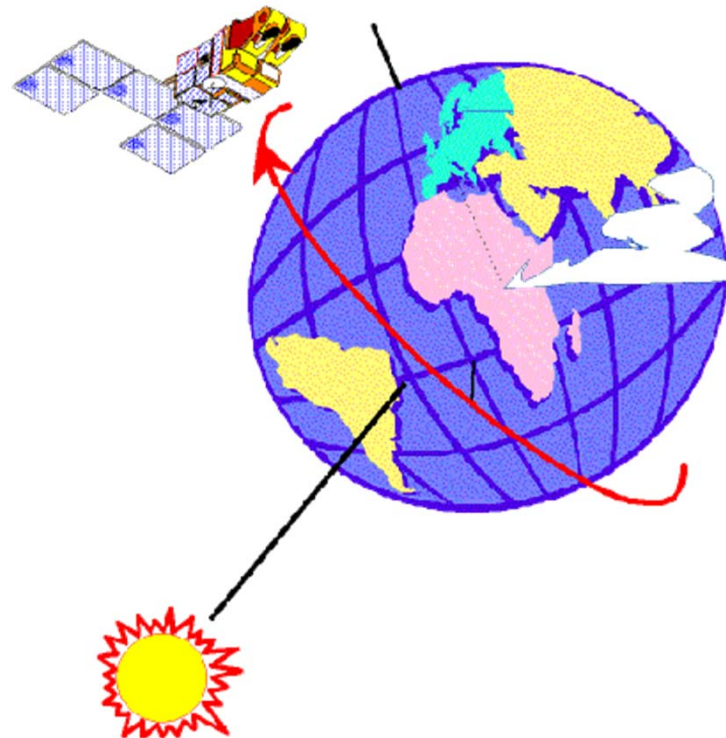
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Chapter 18



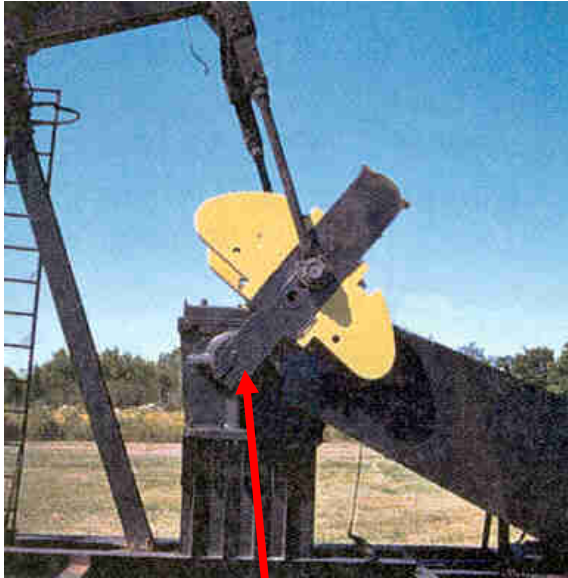
Today's Objectives

Students should be able to:

1. Analyze the planar kinetics of a rigid body undergoing rotational motion
2. Analyze the planar kinetics of a rigid body undergoing general plane motion



Applications (17.4)



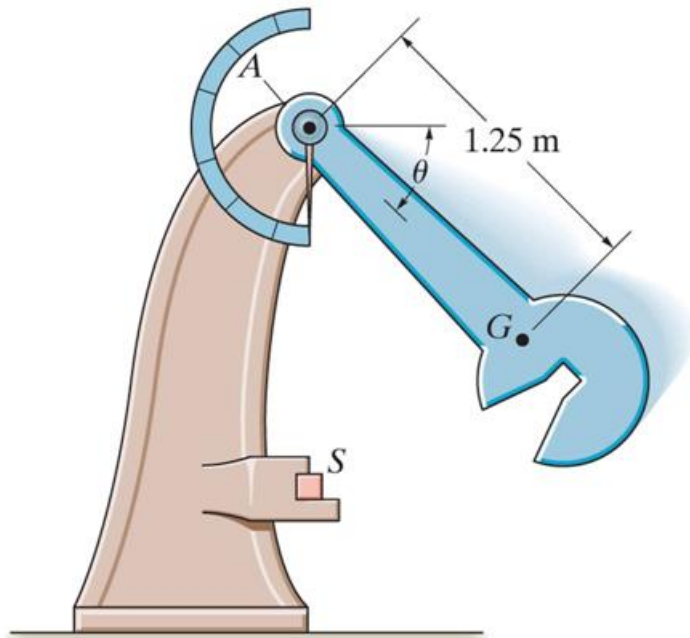
Pin at the center of rotation.

The crank on the oil-pump rig undergoes rotation about a fixed axis, caused by the driving torque M from a motor.

As the crank turns, a dynamic reaction is produced at the pin. This reaction is a function of angular velocity, angular acceleration, and the orientation of the crank.

If the motor exerts a constant torque M on the crank, does the crank turn at a constant angular velocity? Is this desirable for such a machine?

APPLICATIONS (continued)



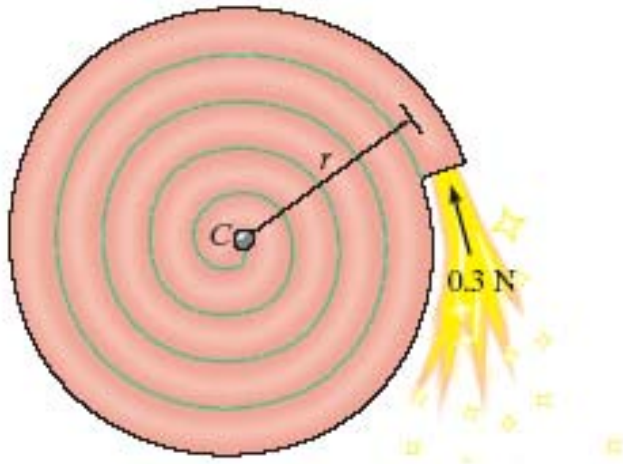
The pendulum of the Charpy impact machine is released from rest when $\theta = 0^\circ$. Its angular velocity (ω) begins to increase.

Can we determine the angular velocity when it is in vertical position?

On which property (P) of the pendulum does the angular acceleration (α) depend?

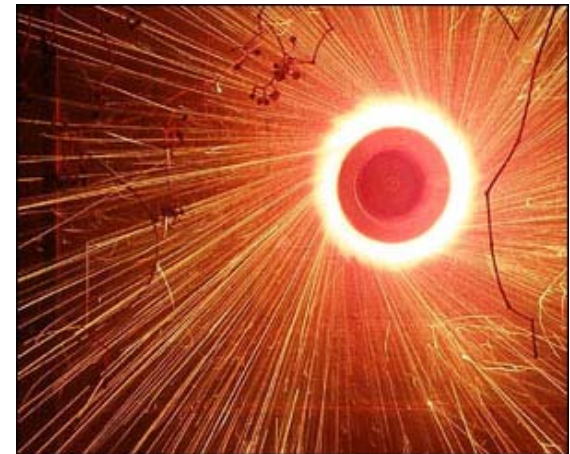
What is the relationship between P and α ?

Applications (17.4) (continued)

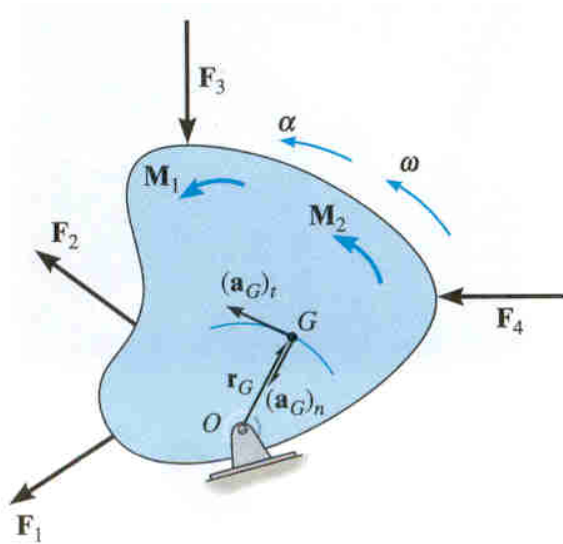


The “Catherine wheel” is a fireworks display consisting of a coiled tube of powder pinned at its center.

As the powder burns, the mass of powder decreases as the exhaust gases produce a force directed tangent to the wheel. This force tends to rotate the wheel.



Equations of motion for pure rotation (17.4)



When a rigid body rotates about a fixed axis perpendicular to the plane of the body at point O, the body's center of gravity G moves in a circular path of radius r_G . Thus, the **acceleration of point G** can be represented by a **tangential** component $(a_G)_t = r_G \alpha$ and a **normal** component $(a_G)_n = r_G \omega^2$.

Since the body experiences an angular acceleration, its inertia creates a moment of magnitude $I_G \alpha$ equal to the moment of the external forces about point G. Thus, the **scalar equations of motion** can be stated as:

$$\sum F_n = m (a_G)_n = m r_G \omega^2$$

$$\sum F_t = m (a_G)_t = m r_G \alpha$$

$$\sum M_G = I_G \alpha$$

Equations of motion for pure rotation (17.4)

(continues)

Note that the $\sum M_G$ moment equation may be replaced by a moment summation about any arbitrary point. Summing the moment about the center of rotation O yields

$$\sum M_O = I_G \alpha + r_G m (a_G)_t = \underline{(I_G + m (r_G)^2)} \alpha$$

From the parallel axis theorem, $I_O = I_G + m(r_G)^2$, therefore the term in parentheses represents I_O . Consequently, we can write the **three equations of motion** for the body as:

$$\sum F_n = m (a_G)_n = m r_G \omega^2$$

$$\sum F_t = m (a_G)_t = m r_G \alpha$$

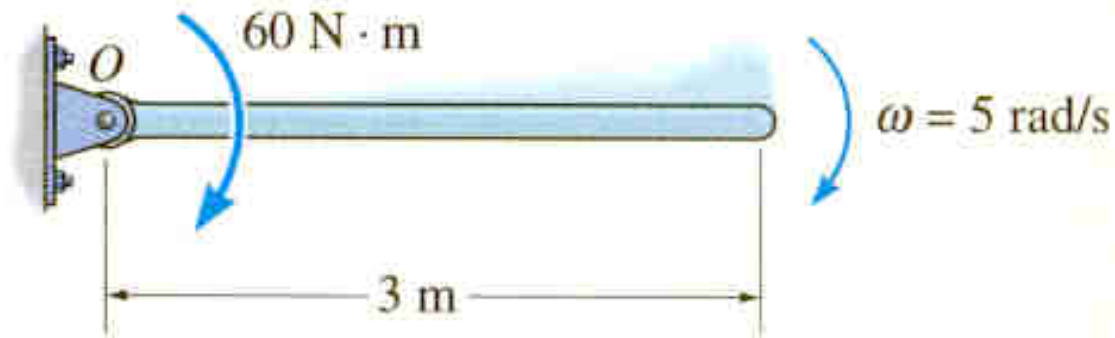
$$\sum M_O = I_O \alpha$$

Procedure of analysis (17.4)

Problems involving the kinetics of a rigid body rotating about a fixed axis can be solved using the following process.

1. Establish an inertial coordinate system and specify the sign and direction of $(a_G)_n$ and $(a_G)_t$.
2. Draw a free body diagram accounting for all external forces and couples. Show the resulting **inertia forces and couple** (typically on a separate kinetic diagram).
3. Compute the mass moment of inertia I_G or I_O .
4. Write the **three equations of motion** and identify the unknowns. Solve for the unknowns.
5. **Use kinematics if there are more than three unknowns** (since the equations of motion allow for only three unknowns).

Example (17.4)



Given: A rod with mass of 20 kg is rotating at 5 rad/s at the instant shown. A moment of 60 N·m is applied to the rod.

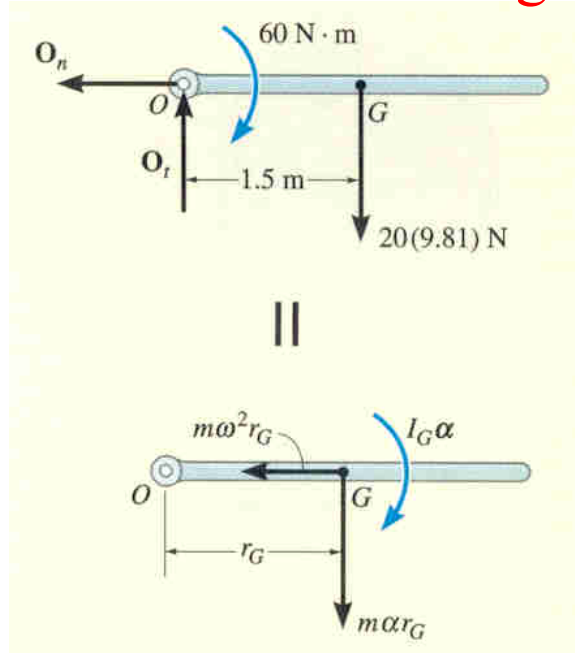
Find: The angular acceleration α and the reaction at pin O when the rod is in the horizontal position.

Plan: Since the mass center, G, moves in a circle of radius 1.5 m, its acceleration has a normal component toward O and a tangential component acting downward and perpendicular to r_G . Apply the problem solving procedure.

Example (17.4) continues...

Solution:

FBD & Kinetic Diagram



Equations of motion:

$$\leftarrow + \sum F_n = ma_n = mr_G \omega^2$$

$$O_n = 20(1.5)(5)^2 = 750 \text{ N}$$

$$\downarrow + \sum F_t = ma_t = mr_G \alpha$$

$$-O_t + 20(9.81) = 20(1.5)\alpha$$

$$\curvearrowright + \sum M_O = I_G \alpha + m r_G \alpha (r_G)$$

Using $I_G = (ml^2)/12$ and $r_G = (0.5)(l)$, we can write:

$$\sum M_O = \alpha[(ml^2/12) + (ml^2/4)] = (ml^2/3)\alpha \quad \text{where } (ml^2/3) = I_O.$$

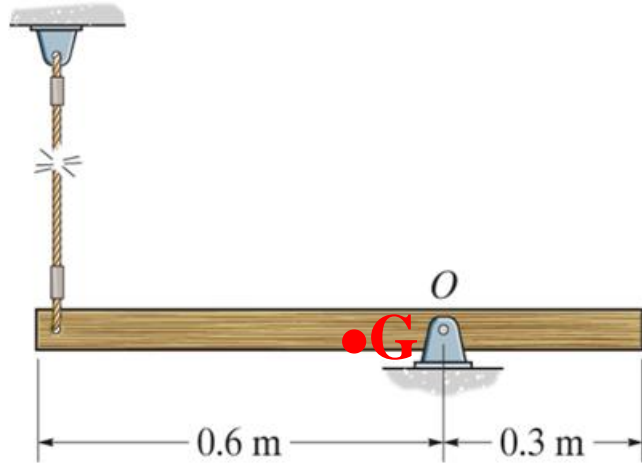
After substituting:

$$60 + 20(9.81)(1.5) = 20(3^2/3)\alpha$$

Solving: $\alpha = 5.9 \text{ rad/s}^2$

$$O_t = 19 \text{ N}$$

EXAMPLE



Given: The uniform slender rod has a mass of 15 kg and its mass center is at point G.

Find: The reactions at the pin O and the angular acceleration of the rod just after the cord is cut.

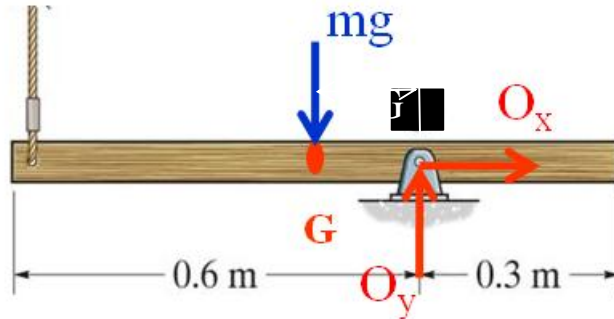
Plan: Since the mass center, G, moves in a circle of radius 0.15 m, its acceleration has a normal component toward O and a tangential component acting downward and perpendicular to r_G .

Apply the problem solving procedure.

EXAMPLE (continued)

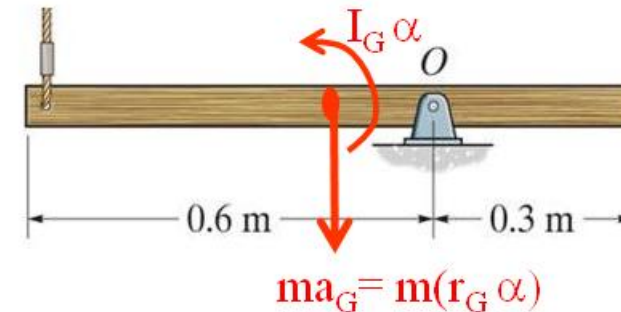
Solution:

FBD



&

Kinetic Diagram



Equations of motion:

$$+\rightarrow \sum F_n = ma_n = mr_G \omega^2 \quad \Rightarrow O_x = 0 \text{ N}$$

$$+\downarrow \sum F_t = ma_t = mr_G \alpha \quad \Rightarrow -O_y + 15(9.81) = 15(0.15)\alpha \quad (1)$$

$$\curvearrowright \sum M_O = I_G \alpha + m r_G \alpha (r_G) \Rightarrow (0.15) 15(9.81) = I_G \alpha + m(r_G)^2 \alpha$$

Using $I_G = (ml^2)/12$ and $r_G = (0.15)$, we can write:

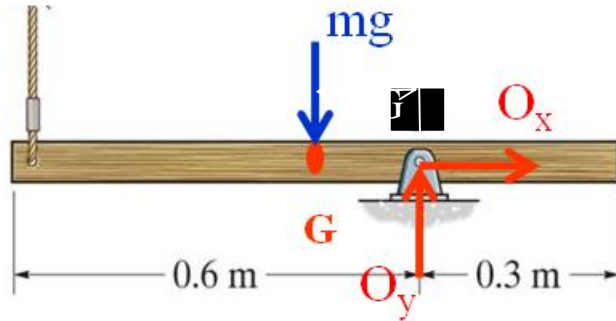
$$I_G \alpha + m(r_G)^2 \alpha = [(15 \times 0.9^2)/12 + 15(0.15)^2] \alpha = 1.35 \alpha$$

W. Wang



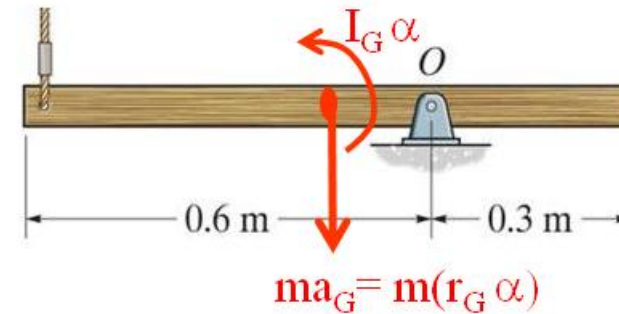
EXAMPLE (continued)

FBD



&

Kinetic Diagram



After substituting:

$$22.07 = 1.35 \alpha \Rightarrow \alpha = 16.4 \text{ rad/s}^2$$

From Eq (1) :

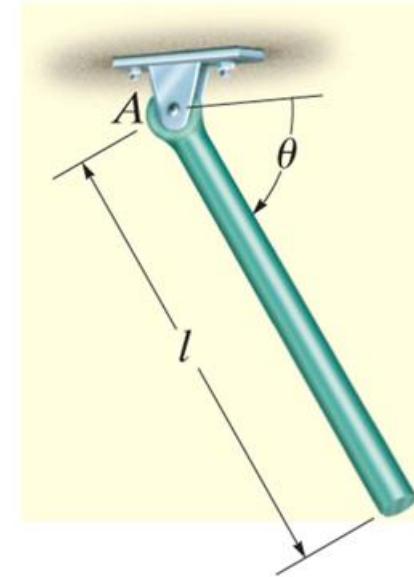
$$-O_y + 15(9.81) = 15(0.15)\alpha$$

$$\Rightarrow O_y = 15(9.81) - 15(0.15)16.4 = 110 \text{ N}$$

CONCEPT QUIZ

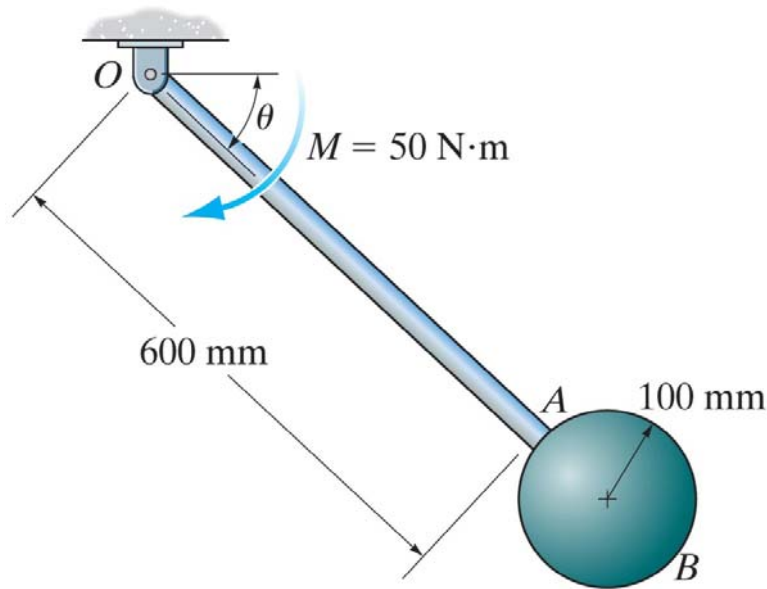
1. If a rigid bar of length l (above) is released from rest in the horizontal position ($\theta = 0$), the magnitude of its angular acceleration is at maximum when

- A) $\theta = 0$ B) $\theta = 90^\circ$
C) $\theta = 180^\circ$ D) $\theta = 0^\circ$ and 180°



2. In the above problem, when $\theta = 90^\circ$, the horizontal component of the reaction at pin O is _____.
- A) zero B) $m g$
C) $m (l/2) \omega^2$ D) None of the above

Example



Given: $m_{\text{sphere}} = 15 \text{ kg}$,
 $m_{\text{rod}} = 10 \text{ kg}$.

The pendulum has an angular velocity of 3 rad/s when $\theta = 45^\circ$ and the external moment of 50 N m .

Find: The reaction at the pin O when $\theta = 45^\circ$.

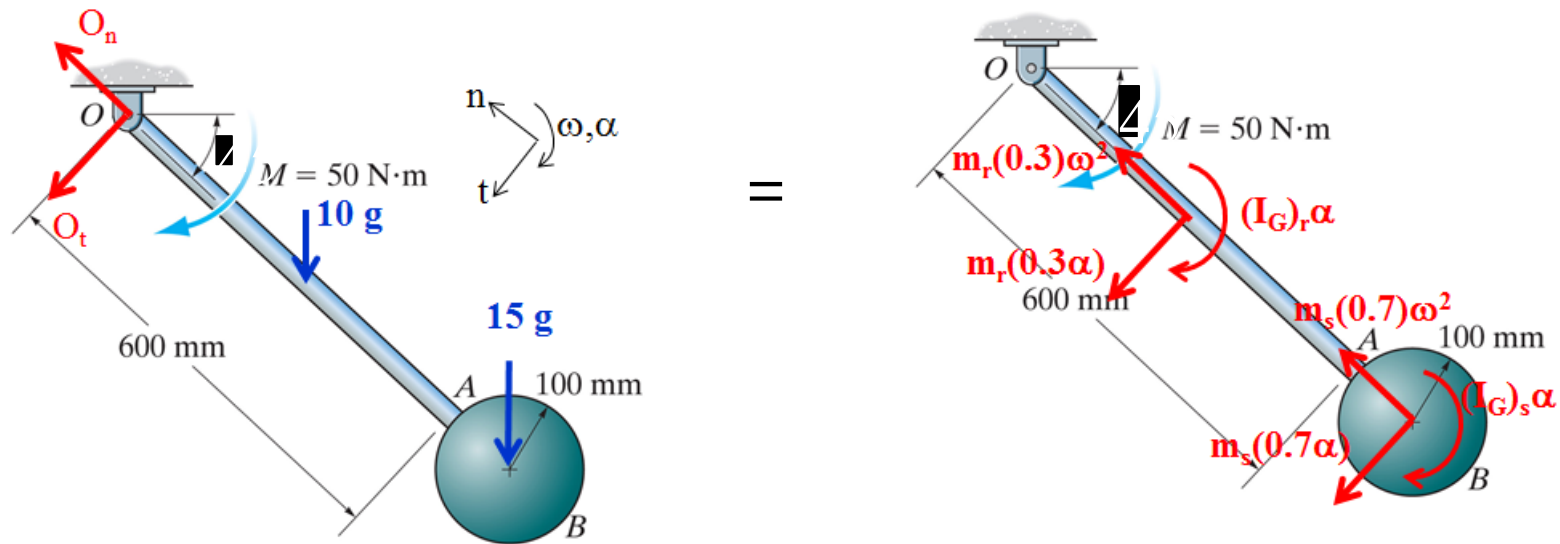
Plan:

Draw the free body diagram and kinetic diagram of the rod and sphere as one unit.

Then apply the equations of motion.

Example (continued)

Solution: FBD and kinetic diagram;

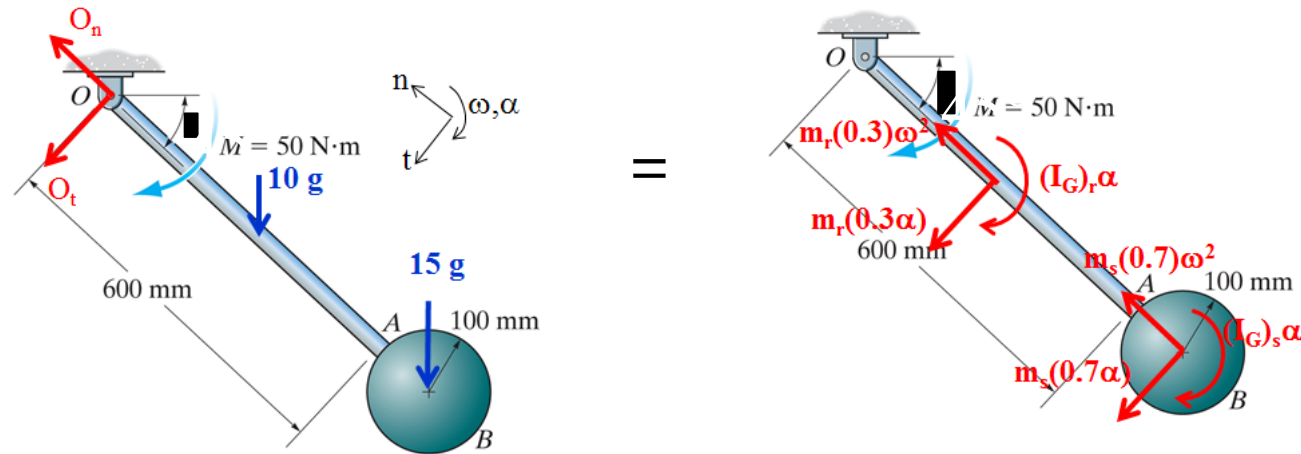


Equations of motion: $\sum F_n = m(a_G)_n$

$$O_n - 10 (9.81) \cos 45 - 15 (9.81) \cos 45 = 10(0.3)\omega^2 + 15(0.7)\omega^2$$

Since $\omega = 3 \text{ rad/s} \Rightarrow O_n = 295 \text{ N}$

Example (continued)



$$\sum F_t = m(a_G)_t$$

$$O_t + 10 (9.81) \sin 45 + 15 (9.81) \sin 45 = 10(0.3)\alpha + 15(0.7)\alpha$$

$$\Rightarrow O_t = -173.4 + 13.5 \alpha$$

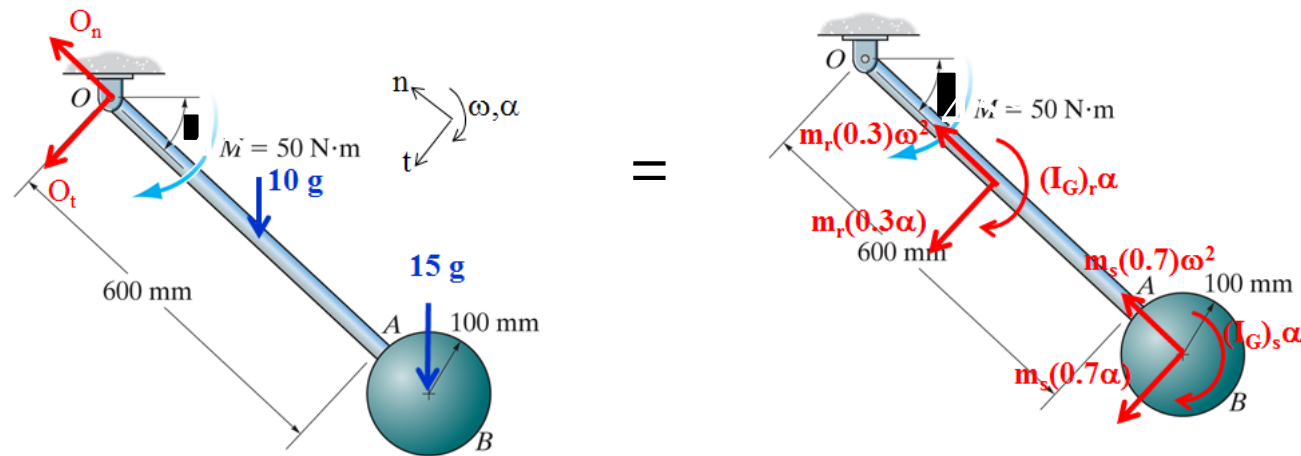
$$\sum M_O = I_O \alpha:$$

$$10 (9.81) \cos 45 (0.3) + 15 (9.81) \cos 45 (0.7) + 50$$

$$= [(1/3) 10 (0.6)^2]_{\text{rod}} \alpha + [(2/5) 15 (0.1)^2 + 15 (0.7)^2]_{\text{sphere}} \alpha$$

$$143.67 = 8.61 \alpha \Rightarrow \alpha = 16.7 \text{ rad/s}^2$$

Example (continued)



Therefore $O_t = 52.1 \text{ N}$ and $O_n = 295 \text{ N}$

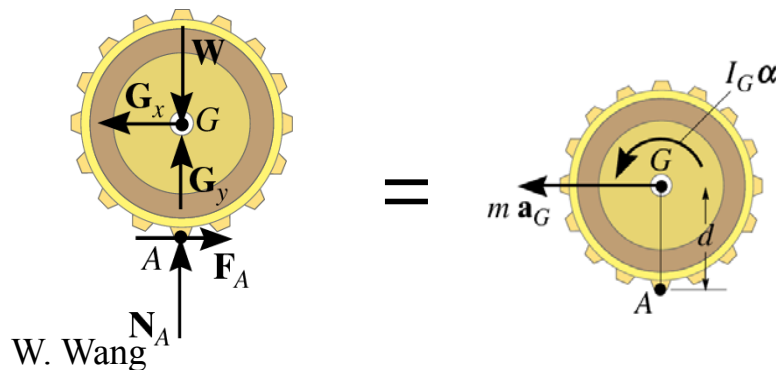
The magnitude of the reaction at O is

$$O = \sqrt{52.1^2 + 295^2} = 299 \text{ N}$$

Applications (17.5)

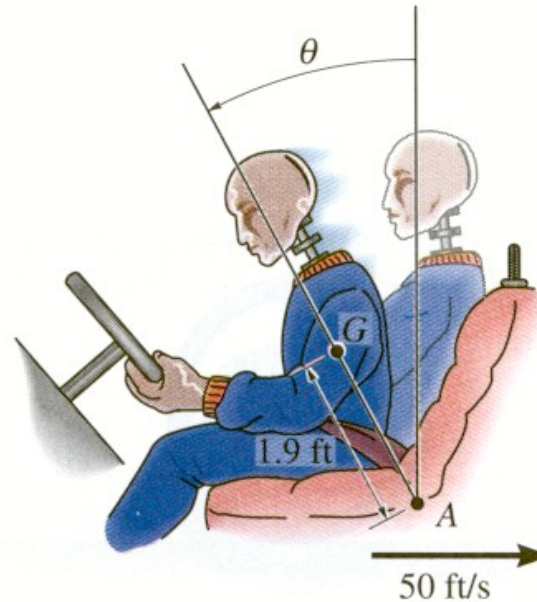


As the soil compactor accelerates forward, the front roller experiences general plane motion (both translation and rotation).



The forces shown on the roller's FBD cause the accelerations shown on the kinetic diagram.

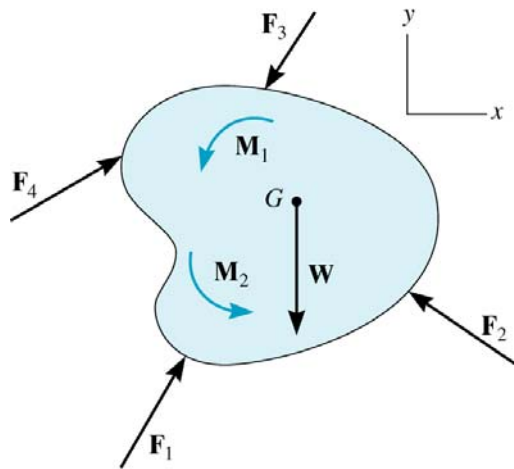
Applications (17.5) (continued)



During an impact, the center of gravity of this crash dummy will decelerate with the vehicle, but also experience another acceleration due to its rotation about point A .

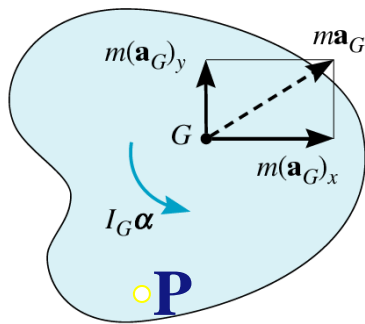
How can engineers use this information to determine the forces exerted by the seat belt on a passenger during a crash?

General plane motion (17.5)



When a rigid body is subjected to external forces and couple-moments, it can undergo both translational motion as well as rotational motion. This combination is called **general plane motion**.

Using an x-y inertial coordinate system, the equations of motions about the center of mass, G , may be written as

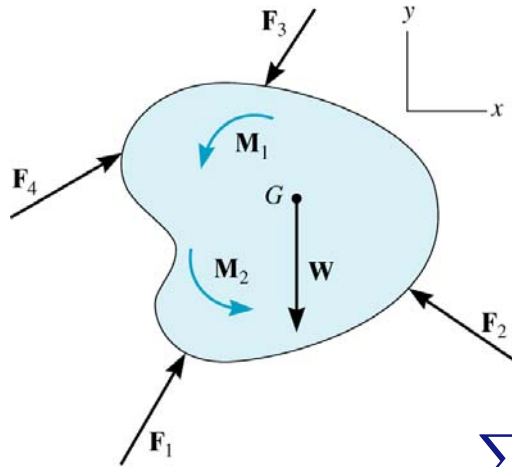


$$\sum F_x = m (a_G)_x$$

$$\sum F_y = m (a_G)_y$$

$$\sum M_G = I_G \alpha$$

General plane motion (17.5) continues...

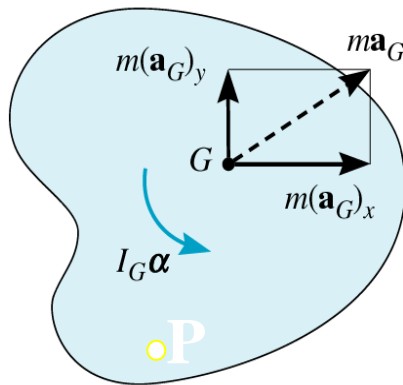


Sometimes, it may be convenient to write the moment equation about some point P other than G. Then the equations of motion are written as follows.

$$\sum F_x = m (a_G)_x$$

$$\sum F_y = m (a_G)_y$$

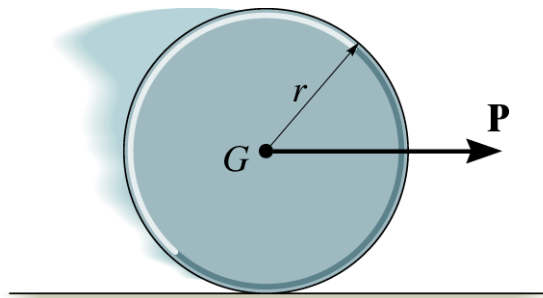
$$\sum M_P = \sum (M_k) = I_G \alpha + r_G m (a_G)_t = (I_G + m (r_G)^2) \alpha$$



In this case, $\sum (M_k)_P$ represents the sum of the moments of $I_G \alpha$ and $m \mathbf{a}_G$ about point P.

Frictional rolling problems

When analyzing the rolling motion of wheels, cylinders, or disks, it may not be known if the body rolls without slipping or if it slides as it rolls.



For example, consider a disk with mass m and radius r , subjected to a known force P .

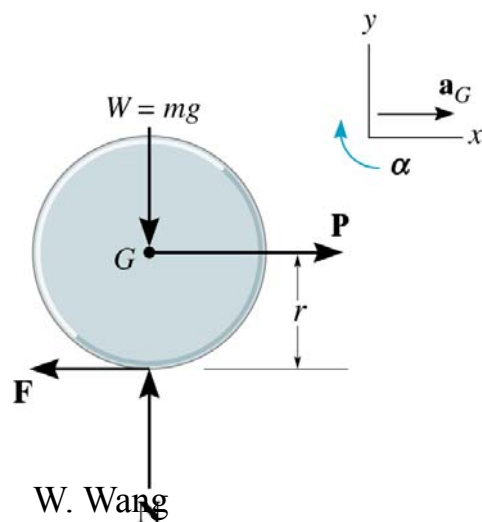
The equations of motion will be

$$\sum F_x = m(a_G)_x \Rightarrow P - F = ma_G$$

$$\sum F_y = m(a_G)_y \Rightarrow N - mg = 0$$

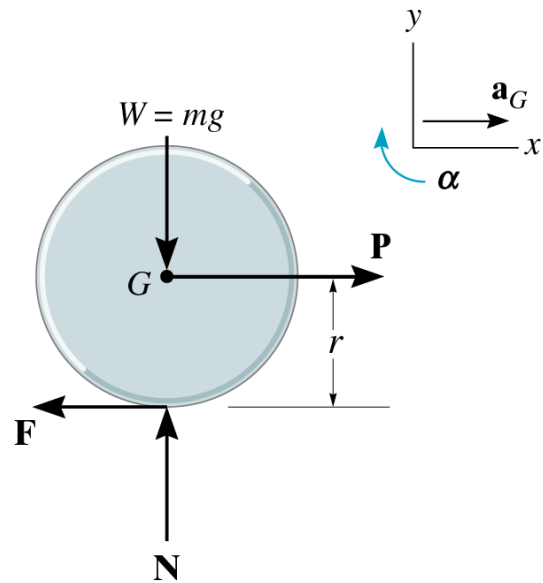
$$\sum M_G = I_G \alpha \Rightarrow Fr = I_G \alpha$$

There are 4 unknowns (F , N , α , and a_G) in these three equations.



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Frictional rolling problems (continued)



Hence, we have to make an assumption to provide another equation. Then we can solve for the unknowns.

The 4th equation can be obtained from the slip or non-slip condition of the disk.

Case 1:

Assume **no slipping** and use $a_G = \alpha r$ as the 4th equation and **DO NOT** use $F_f = \mu_s N$. After solving, you will need to verify that the assumption was correct by checking if $F_f \leq \mu_s N$.

Case 2:

Assume **slipping** and use $F_f = \mu_k N$ as the 4th equation. In this case, $a_G \neq \alpha r$.

Procedure of analysis (17.5)

Problems involving the kinetics of a rigid body undergoing general plane motion can be solved using the following procedure.

1. **Establish** the x-y inertial coordinate system. Draw both the free body diagram and kinetic diagram for the body.
2. **Specify** the direction and sense of the acceleration of the mass center, \mathbf{a}_G , and the angular acceleration α of the body. If necessary, compute the body's mass moment of inertia I_G .
3. If the moment equation $\Sigma M_p = \Sigma (M_k)_p$ is used, use the kinetic diagram to help visualize the moments developed by the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\alpha$.
4. Apply the **three equations of motion**.

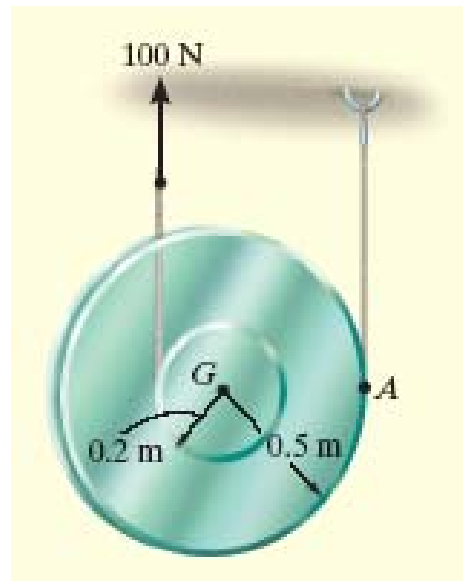
Procedure of analysis (17.5) continues...

5. **Identify the unknowns.** If necessary (i.e., there are four unknowns), make your slip-no slip assumption (typically no slipping, or the use of $a_G = \alpha r$, is assumed first).
6. Use kinematic equations as necessary to complete the solution.
7. If a slip-no slip assumption was made, **check its validity!!!**

Key points to consider:

1. Be **consistent** in assumed directions. The direction of a_G must be consistent with α .
2. If $F_f = \mu_k N$ is used, **F_f must oppose the motion**. As a test, assume no friction and observe the resulting motion. This may help visualize the correct direction of F_f .

Example (17.5)



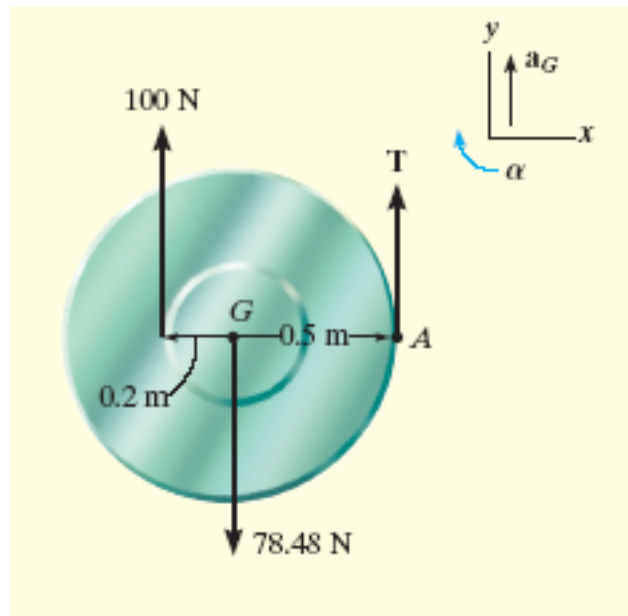
Given: A spool has a mass of 8 kg and a radius of gyration (k_G) of 0.35 m. Cords of negligible mass are wrapped around its inner hub and outer rim. There is no slipping.

Find: The angular acceleration (α) of the spool.

Plan: Focus on the spool. Follow the solution procedure (draw a FBD, etc.) and identify the unknowns.

Example (17.5) continues

Solution:



The moment of inertia of the spool is
 $I_G = m (k_G)^2 = 8 (0.35)^2 = 0.980 \text{ kg} \cdot \text{m}^2$

Method I

Equations of motion:

$$\sum F_y = m (a_G)_y$$

$$T + 100 - 78.48 = 8 a_G$$

$$\sum M_G = I_G \alpha$$

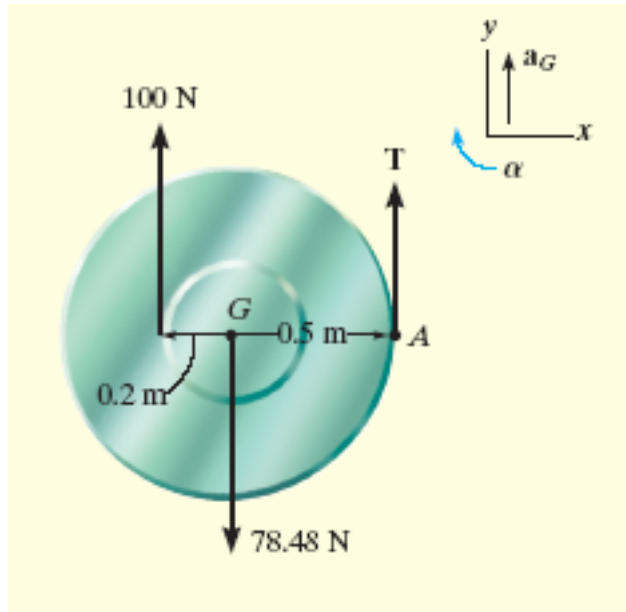
$$100 (0.2) - T(0.5) = 0.98 \alpha$$

There are **three unknowns**, T , a_G , α . We need one more equation to solve for 3 unknowns. Since the spool rolls on the cord at point A without slipping, $a_G = \alpha r$. So the third equation is: $a_G = 0.5\alpha$

Solving these **three** equations, we find:

$$\alpha = 10.3 \text{ rad/s}^2, a_G = 5.16 \text{ m/s}^2, T = 19.8 \text{ N}$$

Example (17.5) continues



Method II

Now, instead of using a moment equation about G, a moment equation **about A** will be used. This approach will eliminate the unknown cord tension (T).

$$\Sigma M_A = \Sigma (M_k)_A: 100 (0.7) - 78.48(0.5) = 0.98 \alpha + (8 a_G)(0.5)$$

Using the **non-slipping condition** again yields $a_G = 0.5\alpha$.

Solving these two equations, we get

$$\alpha = 10.3 \text{ rad/s}^2, a_G = 5.16 \text{ m/s}^2$$

Homework Assignment

Chapter 17- 6, 23, 27, 33, 38, 43, 53, 59, 74, 79, 95,
98, 102, 109

Due Wednesday !!!