ME 230 Kinematics and Dynamics

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Planar kinetics of a rigid body: Force and acceleration
Chapter 17

Chapter objectives

- Introduce the methods used to determin the mass moment of inertia of a body
- To develop the planar kinetic equations of motion for a symmetric rigid body
- To discuss applications of these equations to bodies undergoing translation, rotation about fixed axis, and general plane motion

Lecture 18

- **Planar kinetics of a rigid body: Force and acceleration**

Equations of Motion: Rotation about a Fixed Axis

Equations of Motion: General Plane Motion

- 17.4-17.5

Material covered

- Planar kinetics of a rigid body : Force and acceleration

Equations of motion

1) Rotation about a fixed axis

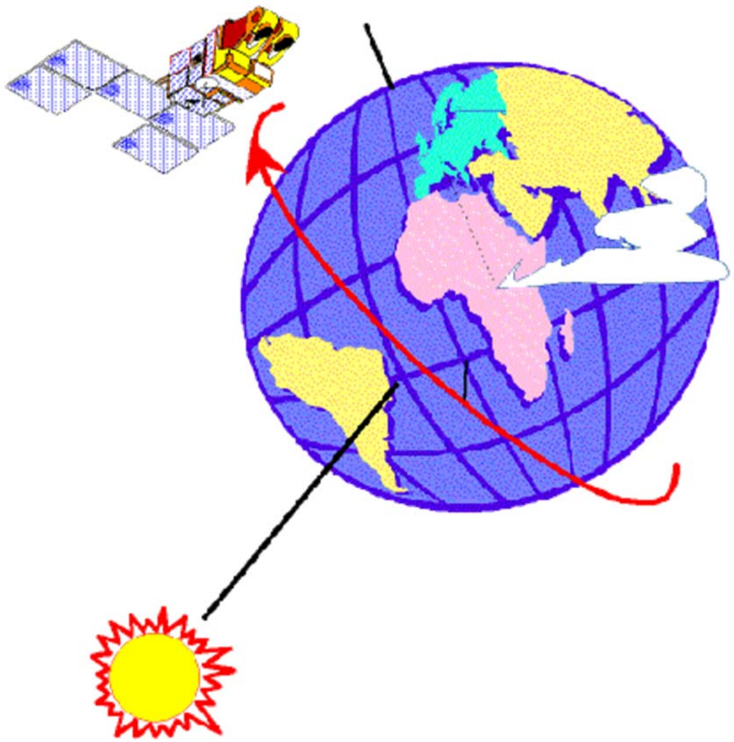
2) General plane motion

…Next lecture…Start Chapter 18

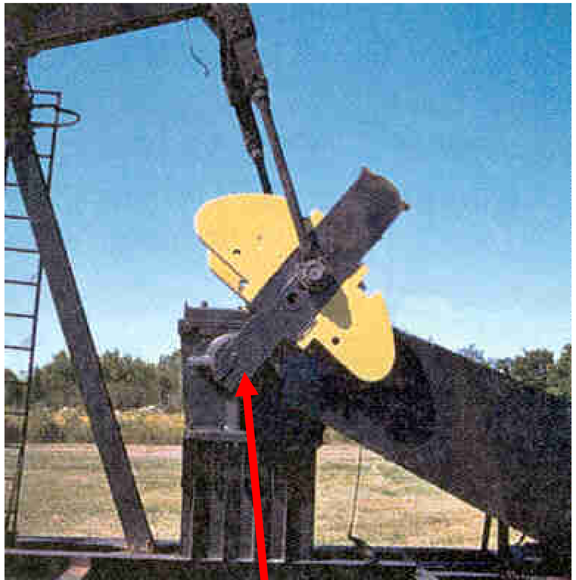
Today’s Objectives

Students should be able to:

1. Analyze the planar kinetics of a rigid body undergoing rotational motion
2. Analyze the planar kinetics of a rigid body undergoing general plane motion



Applications (17.4)



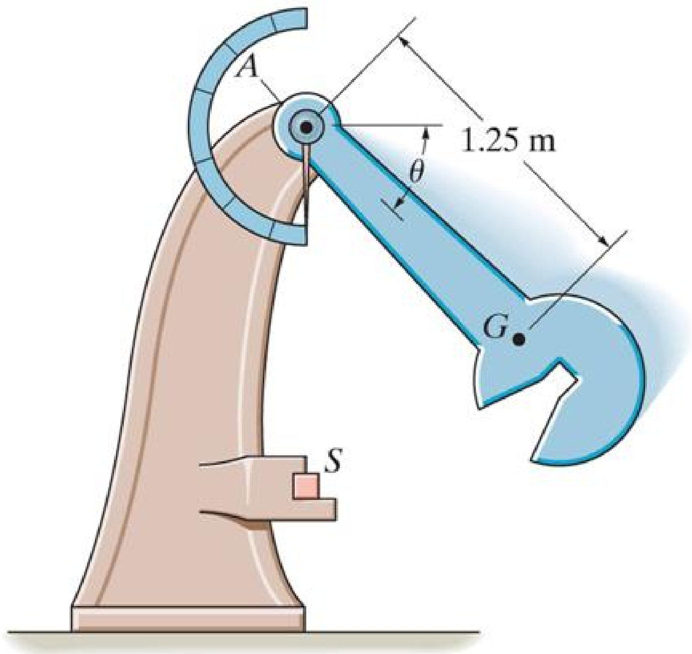
The crank on the oil-pump rig undergoes rotation about a fixed axis, caused by the driving torque M from a motor.

As the crank turns, a dynamic reaction is produced at the pin. This reaction is a function of angular velocity, angular acceleration, and the orientation of the crank.

Pin at the center of rotation.

If the motor exerts a constant torque M on the crank, does the crank turn at a constant angular velocity? Is this desirable for such a machine?

APPLICATIONS (continued)



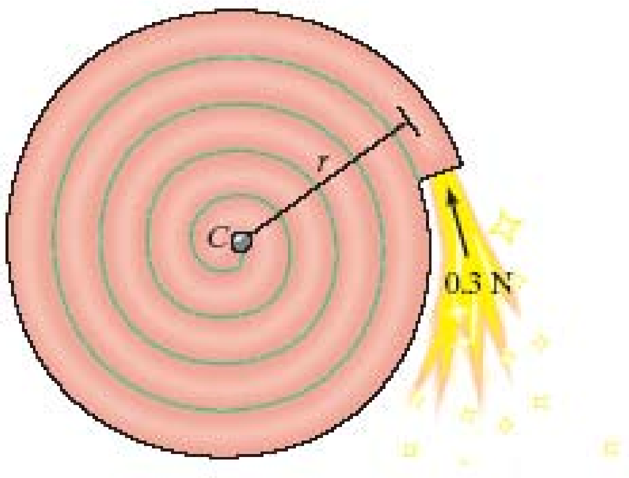
The pendulum of the Charpy impact machine is released from rest when $\theta = 0°$. Its angular velocity (ω) begins to increase.

Can we determine the angular velocity when it is in vertical position?

On which property (P) of the pendulum does the angular acceleration (α) depend?

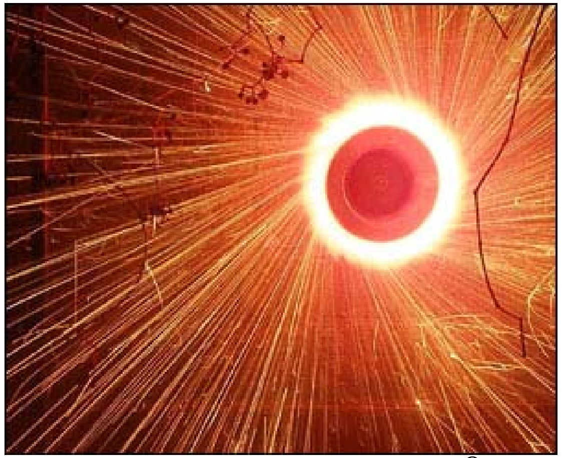
What is the relationship between P and α?

Applications (17.4) (continued)

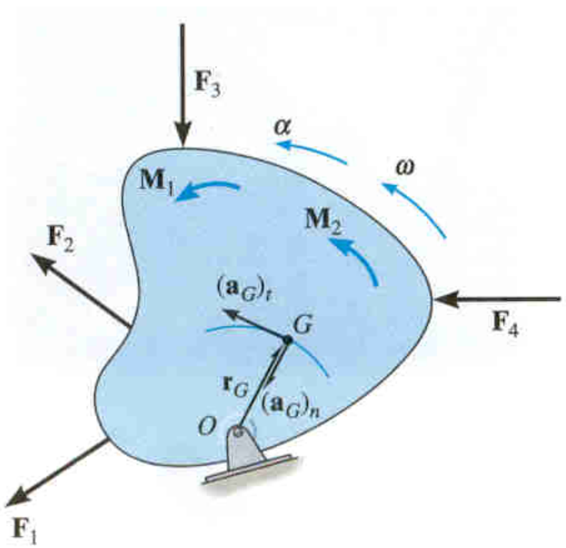


The “Catherine wheel” is a fireworks display consisting of a coiled tube of powder pinned at its center.

As the powder burns, the mass of powder decreases as the exhaust gases produce a force directed tangent to the wheel. This force tends to rotate the wheel.



Equations of motion for pure rotation (17.4)



When a rigid body rotates about a fixed axis perpendicular to the plane of the body at point O, the body's center of gravity G moves in a circular path of radius r_G. Thus, the acceleration of point G can be represented by a tangential component $(a_G)_t = r_G\,\alpha$ and a normal component $(a_G)_n = r_G\,\omega^2$.

Since the body experiences an angular acceleration, its inertia creates a moment of magnitude $I_G\alpha$ equal to the moment of the external forces about point G. Thus, the scalar equations of motion can be stated as:

$$\sum F_n = m\,(a_G)_n = m\,r_G\,\omega^2$$
$$\sum F_t = m\,(a_G)_t = m\,r_G\,\alpha$$
$$\sum M_G = I_G\,\alpha$$

Equations of motion for pure rotation (17.4) (continues)

Note that the $\sum M_G$ moment equation may be replaced by a moment summation about any arbitrary point. Summing the moment about the center of rotation O yields

$$\sum M_O = I_G \alpha + r_G\, m\, (a_G)_t = (\underline{I_G + m\,(r_G)^2}\,)\,\alpha$$

From the parallel axis theorem, $I_O = I_G + m(r_G)^2$, therefore the term in parentheses represents I_O. Consequently, we can write the three equations of motion for the body as:

$$\sum F_n = m\,(a_G)_n = m\, r_G\, \omega^2$$

$$\sum F_t = m\,(a_G)_t = m\, r_G\, \alpha$$

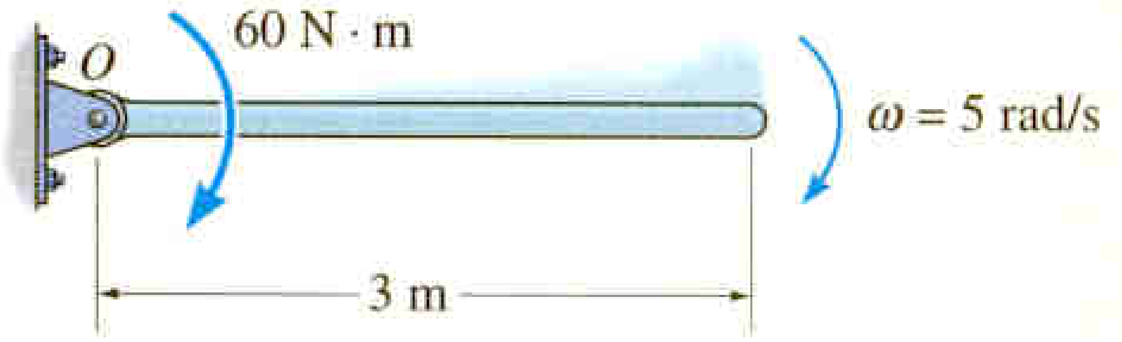
$$\sum M_O = I_O\, \alpha$$

Procedure of analysis (17.4)

Problems involving the kinetics of a rigid body rotating about a fixed axis can be solved using the following process.

1. Establish an inertial coordinate system and specify the sign and direction of $(a_G)_n$ and $(a_G)_t$.
2. Draw a free body diagram accounting for all external forces and couples. Show the resulting inertia forces and couple (typically on a separate kinetic diagram).
3. Compute the mass moment of inertia I_G or I_O.
4. Write the three equations of motion and identify the unknowns. Solve for the unknowns.
5. Use kinematics if there are more than three unknowns (since the equations of motion allow for only three unknowns).

Example (17.4)



Given: A rod with mass of 20 kg is rotating at 5 rad/s at the instant shown. A moment of 60 N·m is applied to the rod.

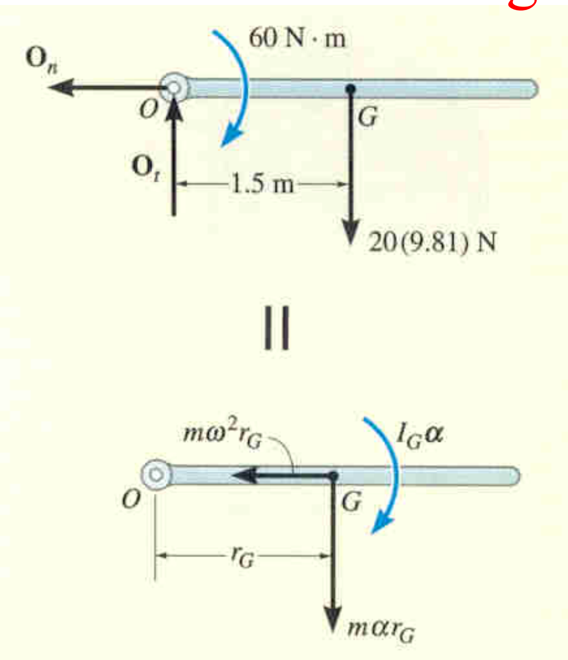
Find: The angular acceleration α and the reaction at pin O when the rod is in the horizontal position.

Plan: Since the mass center, G, moves in a circle of radius 1.5 m, it's acceleration has a normal component toward O and a tangential component acting downward and perpendicular to r_G. Apply the problem solving procedure.

Example (17.4) continues…

Solution:

FBD & Kinetic Diagram



Equations of motion:

$\overset{+}{\leftarrow} \sum F_n = ma_n = mr_G\omega^2$

$O_n = 20(1.5)(5)^2 = 750$ N

$\downarrow + \sum F_t = ma_t = mr_G\alpha$

$-O_t + 20(9.81) = 20(1.5)\alpha$

$\circlearrowright + \sum M_O = I_G\,\alpha + m\,r_G\,\alpha\,(r_G)$

Using $I_G = (ml^2)/12$ and $r_G = (0.5)(l)$, we can write:

$\sum M_O = \alpha[(ml^2/12) + (ml^2/4)] = (ml^2/3)\alpha$ where $(ml^2/3) = I_O$.

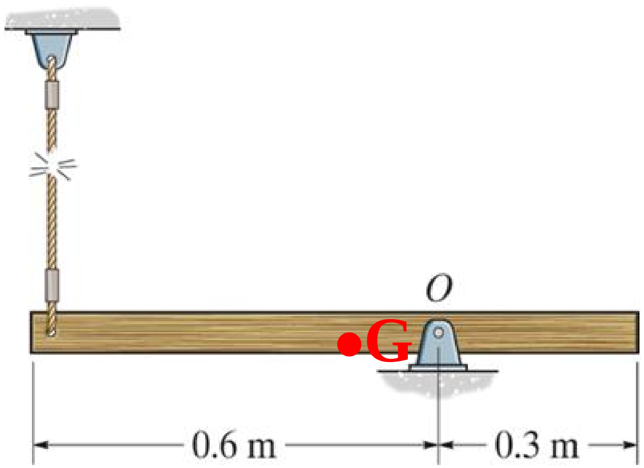
After substituting:

$60 + 20(9.81)(1.5) = 20(3^2/3)\alpha$

Solving: $\alpha = 5.9$ rad/s^2

$O_t = 19$ N

EXAMPLE



Given: The uniform slender rod has a mass of 15 kg and its mass center is at point G.

Find: The reactions at the pin O and the angular acceleration of the rod just after the cord is cut.

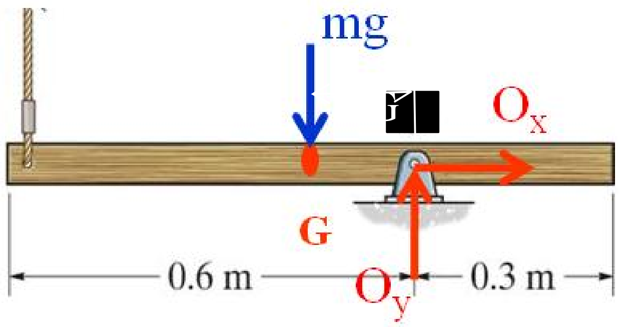
Plan: Since the mass center, G, moves in a circle of radius 0.15 m, it's acceleration has a normal component toward O and a tangential component acting downward and perpendicular to r_G.

Apply the problem solving procedure.

EXAMPLE (continued)

Solution:

FBD & Kinetic Diagram



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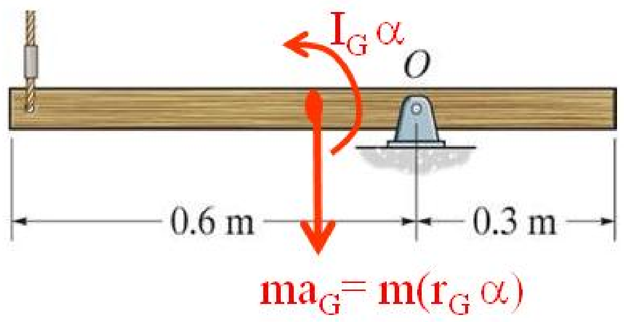


Equations of motion:

$\xrightarrow{+} \sum F_n = ma_n = mr_G\omega^2 \Rightarrow O_x = 0$ N

$+\downarrow \sum F_t = ma_t = mr_G\alpha \Rightarrow -O_y + 15(9.81) = 15(0.15)\alpha \quad (1)$

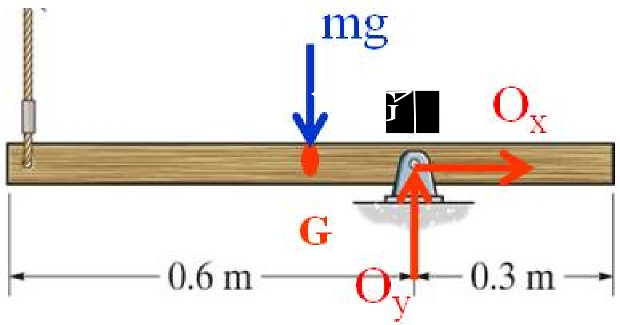
$\circlearrowright + \sum M_O = I_G\,\alpha + m\, r_G\,\alpha\,(r_G) \Rightarrow (0.15)\ 15(9.81) = I_G\,\alpha + m(r_G)^2\,\alpha$

Using $I_G = (ml^2)/12$ and $r_G = (0.15)$, we can write:

$I_G\,\alpha + m(r_G)^2\,\alpha = [(15\times0.9^2)/12 + 15(0.15)^2]\,\alpha = 1.35\,\alpha$

EXAMPLE (continued)

FBD & Kinetic Diagram



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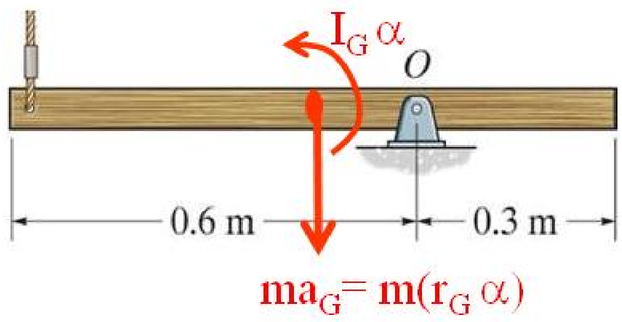


After substituting:

$22.07 = 1.35\,\alpha \quad \Rightarrow \alpha = 16.4 \text{ rad/s}^2$

From Eq (1) :

$-O_y + 15(9.81) = 15(0.15)\alpha$

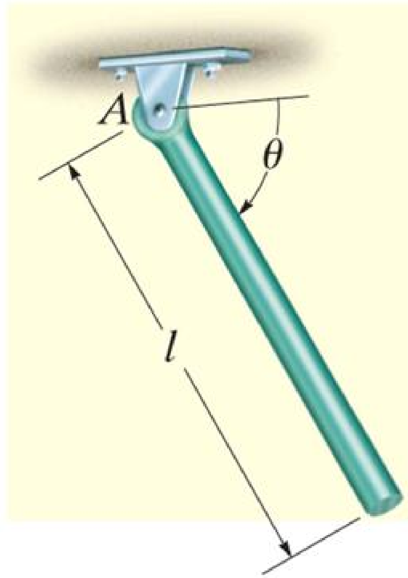
$\Rightarrow \; O_y = 15(9.81) - 15(0.15)16.4 = 110 \text{ N}$

CONCEPT QUIZ

1. If a rigid bar of length l (above) is released from rest in the horizontal position ($\theta = 0$), the magnitude of its angular acceleration is at maximum when

   A) $\theta = 0$ B) $\theta = 90°$

   C) $\theta = 180°$ D) $\theta = 0°$ and $180°$

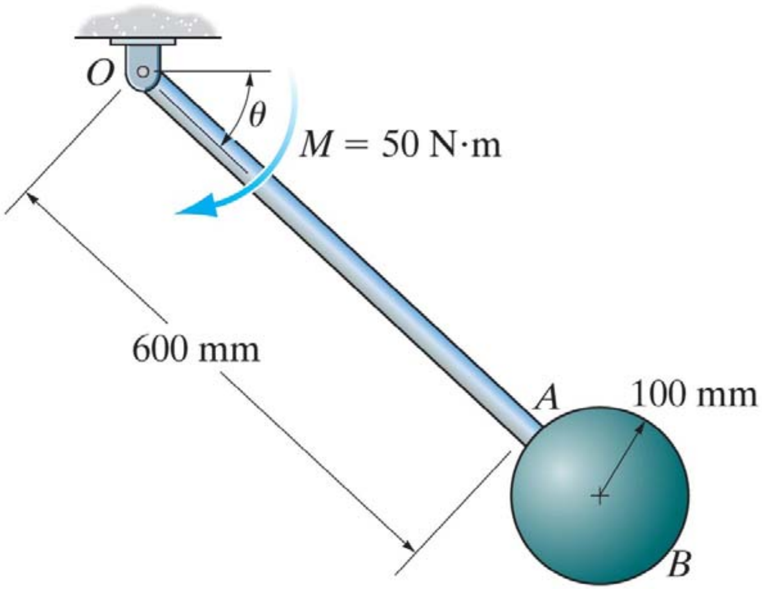


2. In the above problem, when $\theta = 90°$, the horizontal component of the reaction at pin O is __________.

   A) zero B) m g

   C) m (l/2) ω^2 D) None of the above

Example



Given: m_{sphere} = 15 kg,
m_{rod} = 10 kg.
The pendulum has an angular velocity of 3 rad/s when θ = 45° and the external moment of 50 N m.

Find: The reaction at the pin O when θ = 45°.

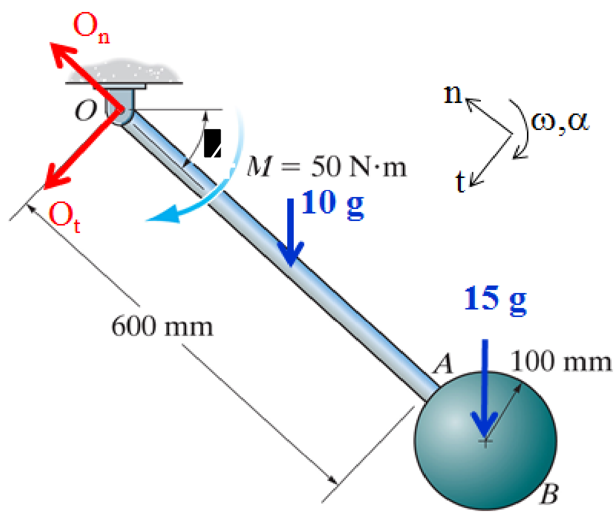
Plan:

Draw the free body diagram and kinetic diagram of the rod and sphere as one unit.

Then apply the equations of motion.

Example (continued)

Solution: FBD and kinetic diagram;



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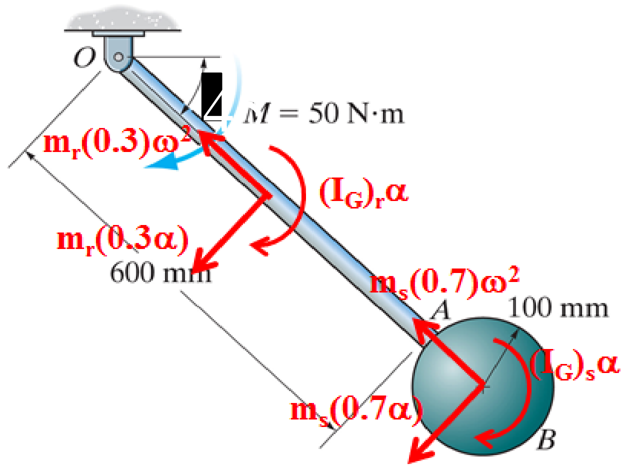
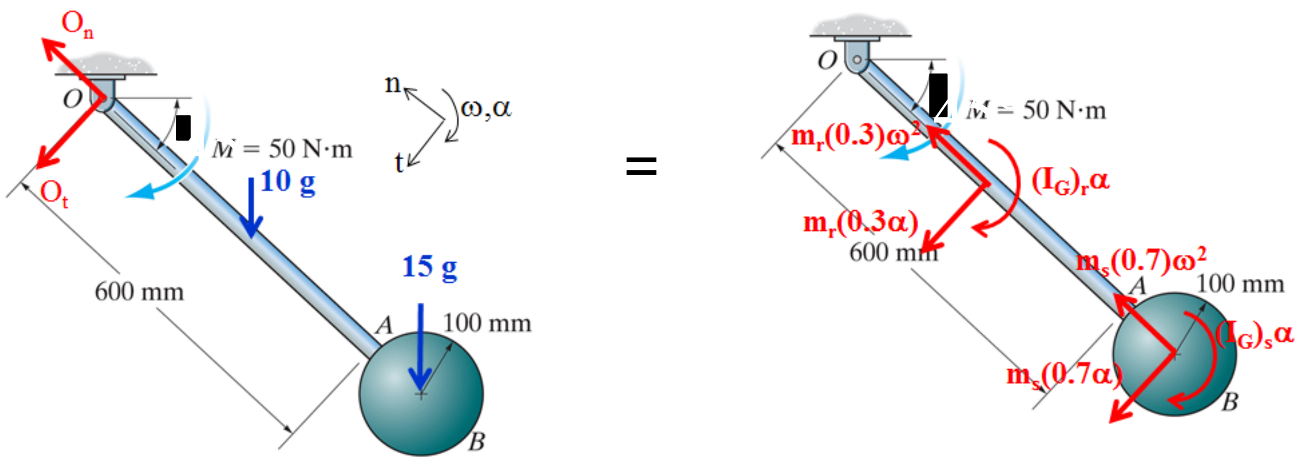


Equations of motion: $\sum F_n = m(a_G)_n$ $\quad \sum F_n = m\,(a_G)_n = m\,r_G\,\omega^2$

$$O_n - 10\,(9.81)\cos 45^\circ - 15\,(9.81)\cos 45^\circ = 10(0.3)\omega^2 + 15(0.7)\omega^2$$

Since $\omega = 3$ rad/s $\Rightarrow O_n = 295$ N

Example (continued)



$\Sigma F_t = m\,(a_G)_t = m\,r_G\,\alpha$

$\Sigma M_O = I_O\,\alpha = I_G\alpha + r_G\,m\,(a_G)_t = (I_G + m\,(r_G)^2)\,\alpha$

$\Sigma F_t = m(a_G)_t$

$O_t + 10\,(9.81)\sin 45 + 15\,(9.81)\sin 45 = 10(0.3)\alpha + 15(0.7)\alpha$

$\Rightarrow\;\; O_t = -173.4 + 13.5\,\alpha$

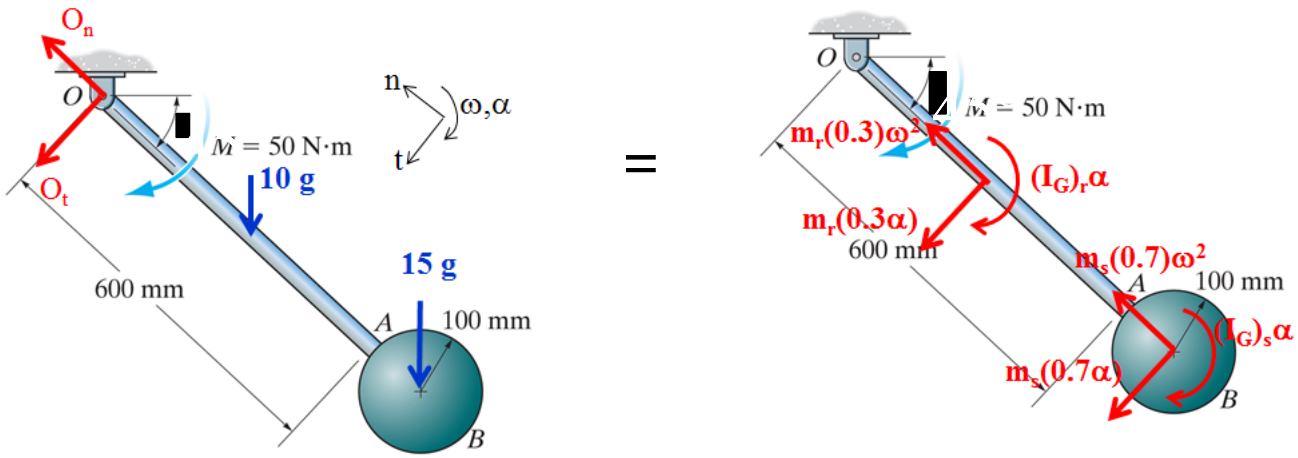
$\Sigma M_O = I_o\alpha$:

$10\,(9.81)\cos 45\,(0.3) + 15\,(9.81)\cos 45\,(0.7) + 50$

$= [(1/3)\,10\,(0.6)^2]_{rod}\,\alpha + [(2/5)\,15\,(0.1)^2 + 15\,(0.7)^2]_{sphere}\,\alpha$

$143.67 = 8.61\,\alpha \quad\Rightarrow\quad \alpha = 16.7\ \text{rad/s}^2$

Example (continued)



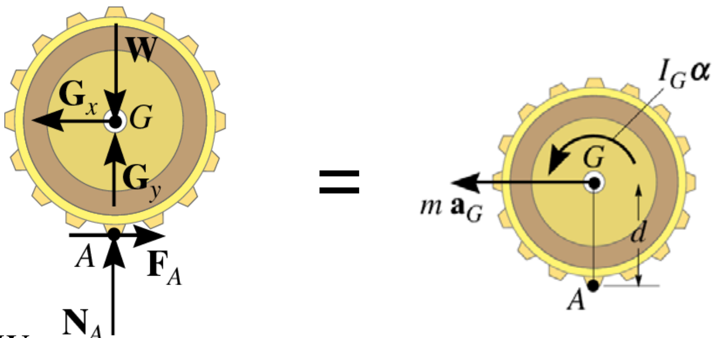
Therefore $O_t = 52.1$ N and $O_n = 295$ N

The magnitude of the reaction at O is

$$O = \sqrt{52.1^2 + 295^2} = 299 \text{ N}$$

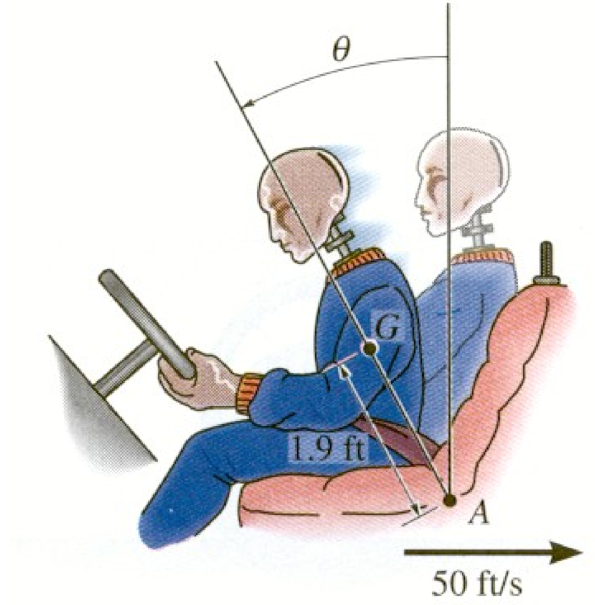
Applications (17.5)

As the soil compactor accelerates forward, the front roller experiences general plane motion (both translation and rotation).



The forces shown on the roller's FBD cause the accelerations shown on the kinetic diagram.

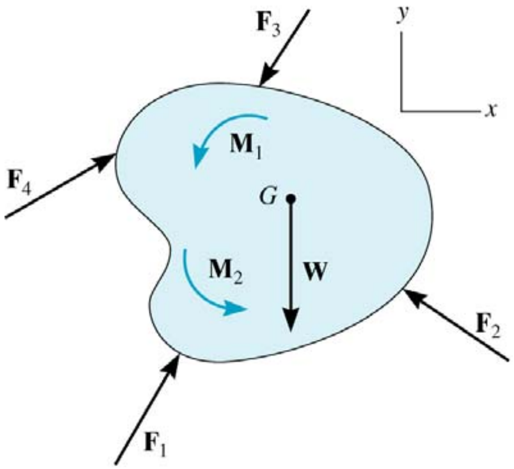
Applications (17.5) (continued)



During an impact, the center of gravity of this crash dummy will decelerate with the vehicle, but also experience another acceleration due to its rotation about point A.

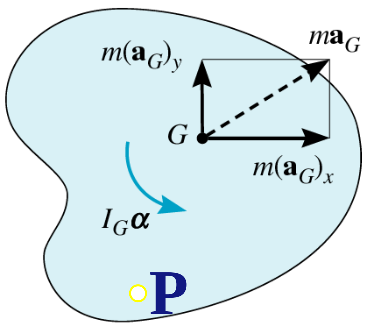
How can engineers use this information to determine the forces exerted by the seat belt on a passenger during a crash?

General plane motion (17.5)



When a rigid body is subjected to external forces and couple-moments, it can undergo both translational motion as well as rotational motion. This combination is called general plane motion.

Using an x-y inertial coordinate system, the equations of motions about the center of mass, G, may be written as

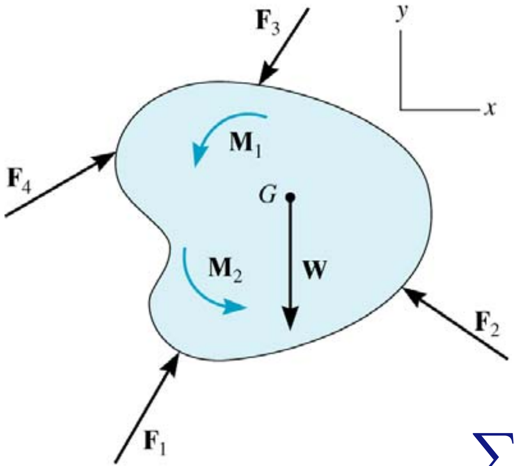


$$\sum F_x = m\,(a_G)_x$$

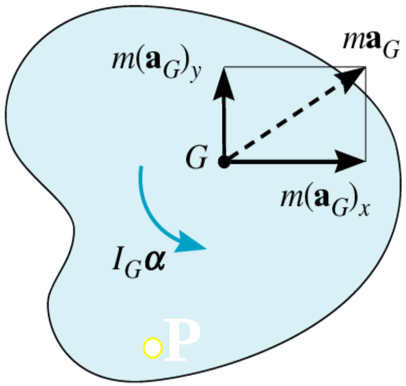
$$\sum F_y = m\,(a_G)_y$$

$$\sum M_G = I_G\,\alpha$$

General plane motion (17.5) continues…



Sometimes, it may be convenient to write the moment equation about some point P other than G. Then the equations of motion are written as follows.



$$\sum F_x = m\,(a_G)_x$$

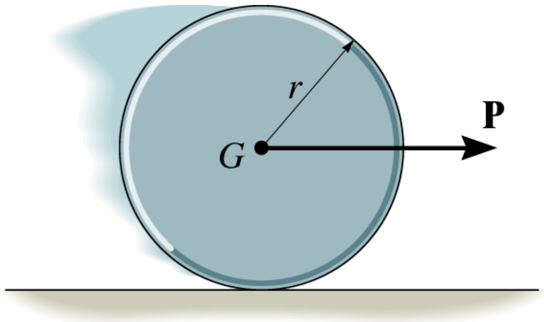
$$\sum F_y = m\,(a_G)_y$$

$$\sum M_P = \sum (M_k) = I_G\alpha + r_G\, m\,(a_G)_t = (I_G + m\,(r_G)^2)\,\alpha$$

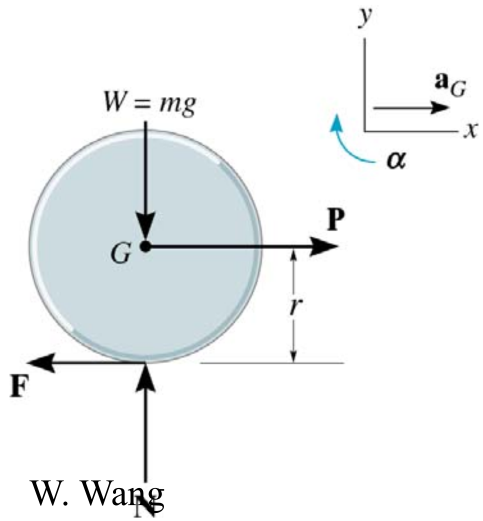
In this case, $\sum (M_k)_P$ represents the sum of the moments of $I_G\alpha$ and $m a_G$ about point P.

Frictional rolling problems

When analyzing the rolling motion of wheels, cylinders, or disks, it may not be known if the body rolls without slipping or if it slides as it rolls.



For example, consider a disk with mass m and radius r, subjected to a known force P.



The equations of motion will be

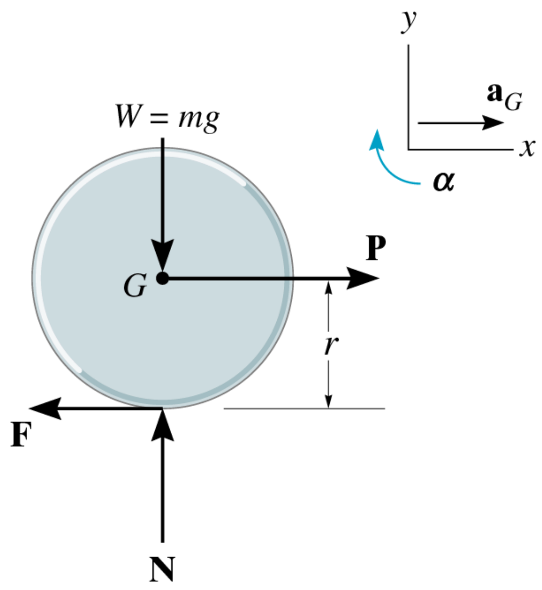
$\sum F_x = m(a_G)_x \quad => \quad P - F = ma_G$

$\sum F_y = m(a_G)_y \quad => \quad N - mg = 0$

$\sum M_G = I_G\alpha \quad => \quad F\,r = I_G\alpha$

There are 4 unknowns (F, N, α, and a_G) in these three equations.

Frictional rolling problems (continued)



Hence, we have to make an assumption to provide another equation. Then we can solve for the unknowns.

The 4^{th} equation can be obtained from the slip or non-slip condition of the disk.

Case 1:

Assume no slipping and use $a_G = \alpha\ r$ as the 4^{th} equation and DO NOT use $F_f = \mu_s N$. After solving, you will need to verify that the assumption was correct by checking if $F_f \leq \mu_s N$.

Case 2:

Assume slipping and use $F_f = \mu_k N$ as the 4^{th} equation. In this case, $a_G \neq \alpha r$.

Rolling Friction

A rolling wheel requires a certain amount of friction so that the point of contact of the wheel with the surface will not slip. The amount of traction which can be obtained for an auto tire is determined by the coefficient of static friction between the tire and the road. If the wheel is locked and sliding, the force of friction is determined by the coefficient of kinetic friction and is usually significantly less.

Assuming that a wheel is rolling without slipping, the surface friction does no work against the motion of the wheel and no energy is lost at that point. However, there is some loss of energy and some deceleration from friction for any real wheel, and this is sometimes referred to as rolling friction. It is partly **friction at the axle** and can be partly **due to flexing of the wheel** which will dissipate some energy. Figures of 0.02 to 0.06 have been reported as effective coefficients of rolling friction for automobile tires, compared to about 0.8 for the maximum static friction coefficient between the tire and the road.

Procedure of analysis (17.5)

Problems involving the kinetics of a rigid body undergoing general plane motion can be solved using the following procedure.

1. Establish the x-y inertial coordinate system. Draw both the free body diagram and kinetic diagram for the body.

2. Specify the direction and sense of the acceleration of the mass center, $\boldsymbol{a}_G$, and the angular acceleration $\boldsymbol{\alpha}$ of the body. If necessary, compute the body's mass moment of inertia I_G.

3. If the moment equation $\Sigma M_p = \Sigma(M_k)_p$ is used, use the kinetic diagram to help visualize the moments developed by the components $m(a_G)_x$, $m(a_G)_y$, and $I_G\alpha$.

4. Apply the three equations of motion.

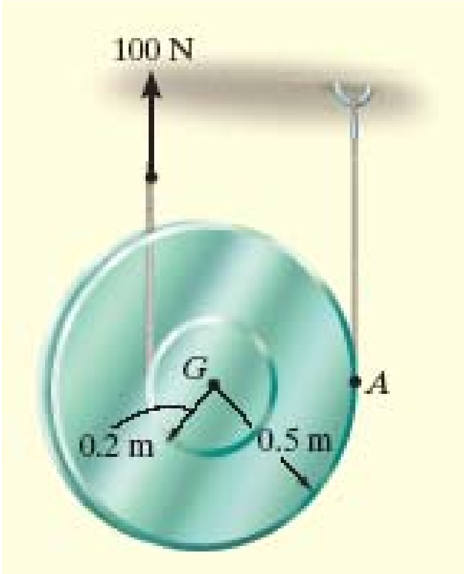
Procedure of analysis (17.5) continues…

5. Identify the unknowns. If necessary (i.e., there are four unknowns), make your slip-no slip assumption (typically no slipping, or the use of $a_G = \alpha\, r$, is assumed first).
6. Use kinematic equations as necessary to complete the solution.
7. If a slip-no slip assumption was made, check its validity!!!

Key points to consider:

1. Be consistent in assumed directions. The direction of $\boldsymbol{a}_G$ must be consistent with $\boldsymbol{\alpha}$.
2. If $F_f = \mu_k N$ is used, F_f must oppose the motion. As a test, assume no friction and observe the resulting motion. This may help visualize the correct direction of F_f.

Example (17.5)



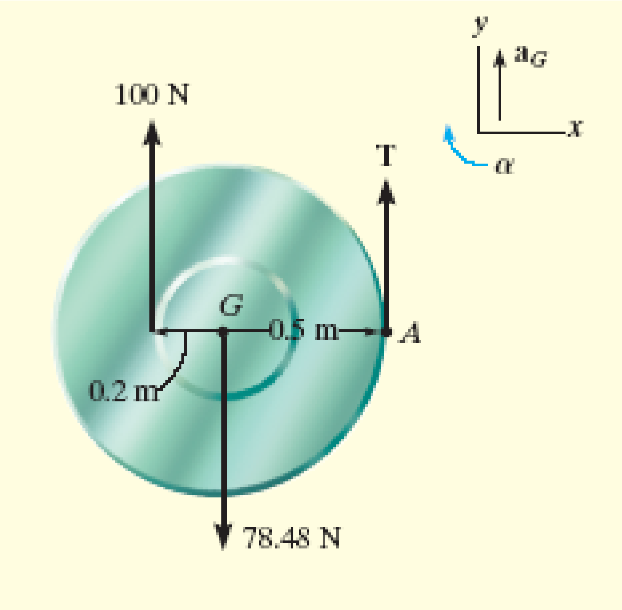
Given: A spool has a mass of 8 kg and a radius of gyration (k_G) of 0.35 m. Cords of negligible mass are wrapped around its inner hub and outer rim. There is no slipping.

Find: The angular acceleration (α) of the spool.

Plan: Focus on the spool. Follow the solution procedure (draw a FBD, etc.) and identify the unknowns.

Example (17.5) continues

Solution:



The moment of inertia of the spool is

$I_G = m\,(k_G)^2 = 8\,(0.35)^2 = 0.980\ kg{\cdot}m^{2}$

Method I

Equations of motion:

$\sum F_y = m\,(a_G)_y$

$T + 100 - 78.48 = 8\,a_G$

$\sum M_G = I_G\,\alpha$

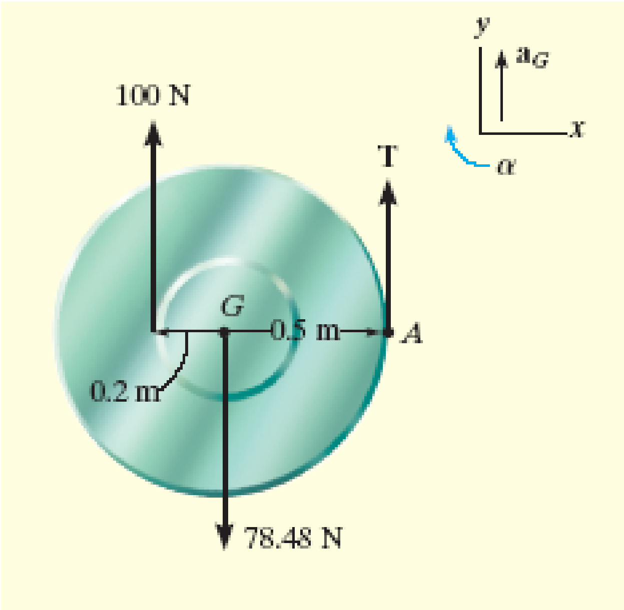
$100\,(0.2) - T(0.5) = 0.98\,\alpha$

There are three unknowns, T, a_G, α. We need one more equation to solve for 3 unknowns. Since the spool rolls on the cord at point A without slipping, $a_G = \alpha r$. So the third equation is: $a_G = 0.5\alpha$

Solving these three equations, we find:

$\alpha = 10.3\ rad/s^2$, $a_G = 5.16\ m/s^2$, $T = 19.8\ N$

Example (17.5) continues



Method II

Now, instead of using a moment equation about G, a moment equation about A will be used. This approach will eliminate the unknown cord tension (T).

$\Sigma M_A = \Sigma (M_k)_A$: 100 (0.7) - 78.48(0.5) = 0.98 α + (8 a_G)(0.5)

Using the non-slipping condition again yields $a_G = 0.5\alpha$.

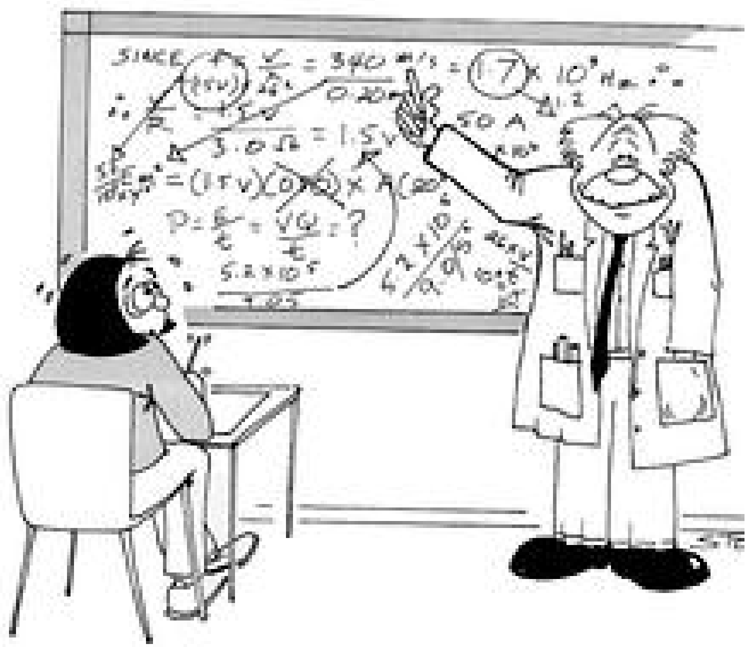
Solving these two equations, we get
$\alpha = 10.3$ rad/s^2, $a_G = 5.16$ m/s^2

Homework Assignment

Chapter17- 6, 23, 27,33, 38, 43, 53, 59, 74, 79,95, 98, 102,109

Due next Monday !!!

Exams



Chapter reviews

Chapter 12: **pages 101-105**

Chapter 13: **pages 166-167**

Chapter 14: **pages 217-219**

Chapter 15: **pages 295-297**

Chapter 16: **pages 391-393**

Chapter 17: **pages 452-453**

Chapter 18: **pages 490-493**

Chapter 19: **pages 531-533**

Book chapter reviews give you a good but brief idea about each chapter…

General exam rules

- Midterm exam will consist of 4 questions. **3** questions must be solved. The 4^{th} question will be a bonus question.
- Midterm exam counts for **25%** of the total mark
- Come on time. Since the lecture theatre will be used for another class at 1:30, there will be no extra time
- **All problems except the bonus question will be solved symbolically just like last time.**

Planar kinetics of a rigid body: Work and Energy
Chapter 18

Chapter objectives

- Develop formulations for the kinetic energy of a body, and define the various ways a force and couple do work.
- Apply the principle of work and energy to solve rigid-body planar kinetic problems that involve force, velocity and displacement
- Show how the conservation of energy can be used to solve rigid-body planar kinetic problems

Lecture 19

- **Planar kinetics of a rigid body: Work and Energy**

Kinetic energy

Work of a force

Work of a couple

Principle of work and energy

- 18.1-18.4

Material covered

- **Planar kinetics of a rigid body :Work and Energy**

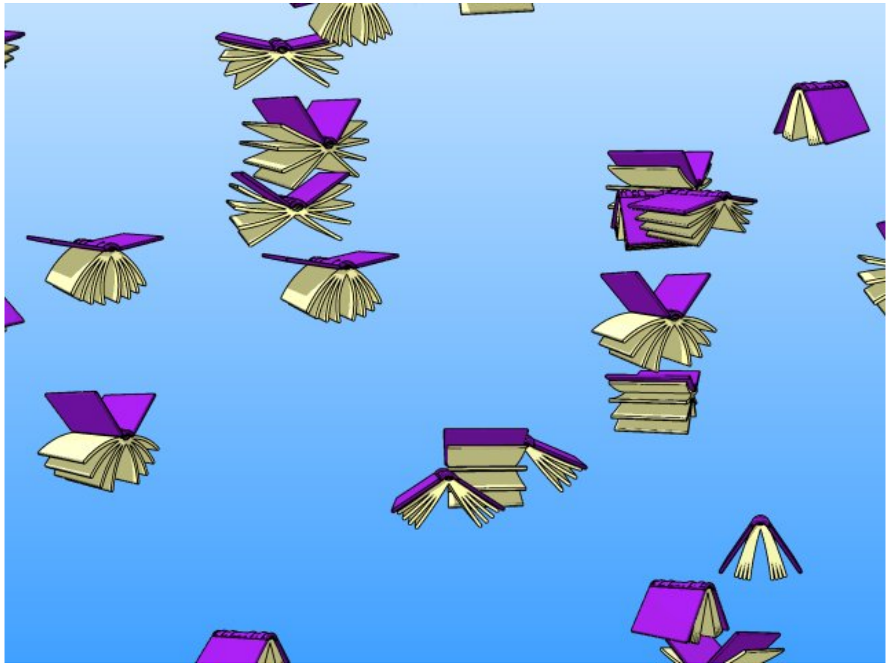
18.1: Kinetic Energy

18.2: The Work of a Force

18.3: The work of a couple

18.4: Principle of Work and Energy

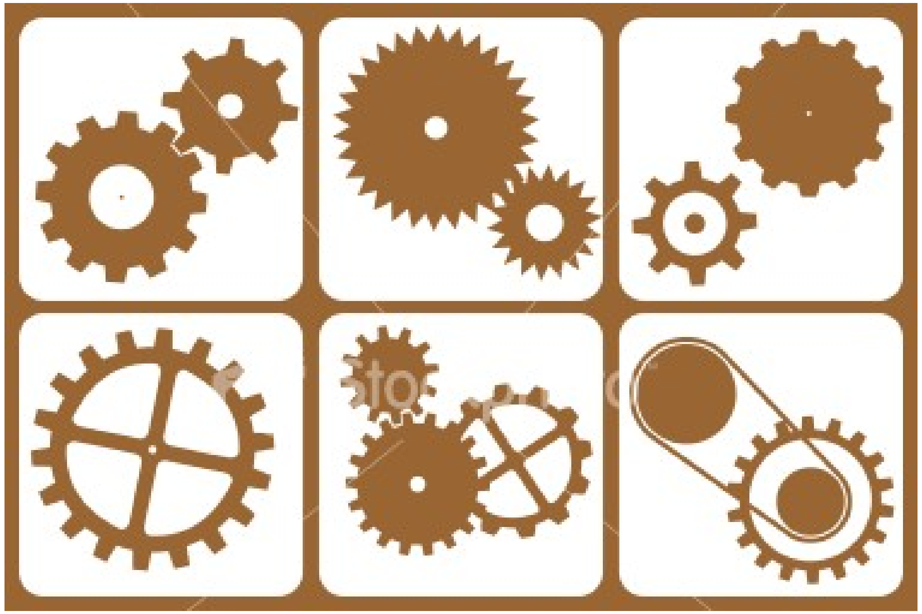
…Next lecture…18.5



Today's Objectives

Students should be able to:

1. **Define the various ways that a force and couple do work.**
2. **Apply the principle of work and energy to a rigid body**



Applications 1

The work of the torque (or moment) developed by the driving gears on the two motors on the concrete mixer is transformed into rotational kinetic energy of the mixing drum.

If the motor gear characteristics are known, could the velocity of the mixing drum be found?

Applications 2

The work done by the soil compactor's engine is transformed into translational kinetic energy of the frame and translational and rotational kinetic energy of its roller and wheels (excluding the additional kinetic energy developed by the moving parts of the engine and drive train).

Are the kinetic energies of the frame and the roller related to each other? How?

Kinetic energy (18.1)

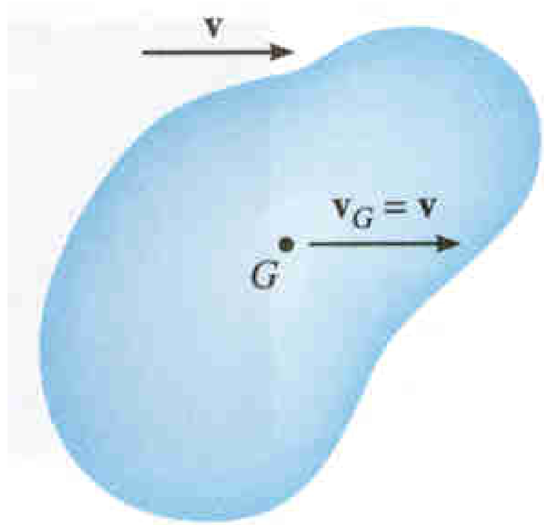
The kinetic energy of a rigid body can be expressed as the sum of its translational and rotational kinetic energies. In equation form, a body in general plane motion has kinetic energy given by

$$T = 1/2\ m\ (v_G)^2 + 1/2\ I_G\ \omega^2$$

Several simplifications can occur.

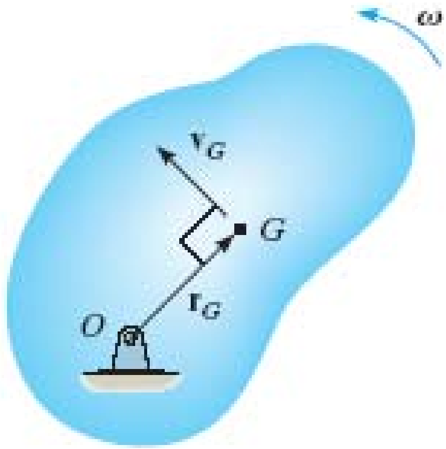
1. Pure Translation: When a rigid body is subjected to only curvilinear or rectilinear translation, the rotational kinetic energy is zero ($\omega = 0$). Therefore,

$$T = 1/2\ m\ (v_G)^2$$



Translation

Kinetic energy (18.1) continues



Rotation About a Fixed Axis

2. Pure Rotation: When a rigid body is rotating about a fixed axis passing through point O, the body has both translational and rotational kinetic energy. Thus,

$$T = 0.5m(v_G)^2 + 0.5I_G\omega^2$$

Since $v_G = r_G\omega$, we can express the kinetic energy of the body as

$$T = 0.5(I_G + m(r_G)^2)\omega^2 = 0.5I_O\omega^2$$

If the rotation occurs about the mass center, G, then what is the value of v_G?

In this case, the velocity of the mass center is equal to zero. So the kinetic energy equation reduces to

$$T = 0.5\ I_G\ \omega^2$$

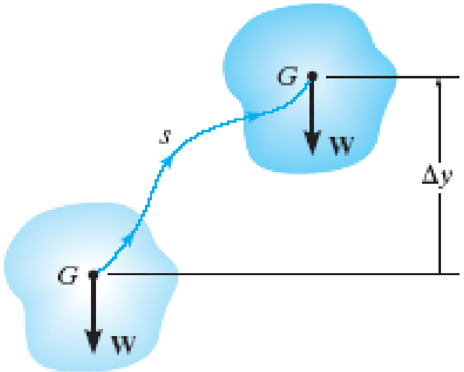
The work of a force (18.2)

Recall that the work done by a force can be written as

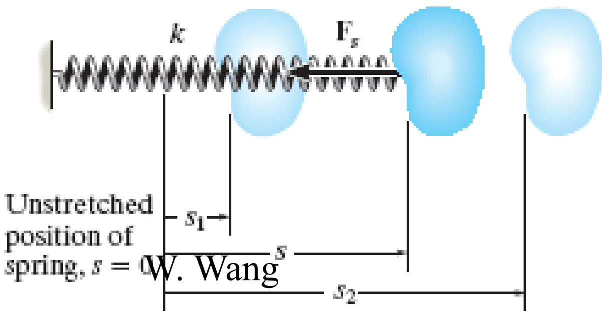
$$U_F = \int \boldsymbol{F} \bullet d\boldsymbol{r} = \int_s (F \cos \theta)\, ds.$$

When the force is constant, this equation reduces to

$U_{Fc} = (F_c \cos\theta)s$ where $F_c\cos\theta$ represents the component of the force acting in the direction of displacement s.



Work of a weight: As before, the work can be expressed as $U_w = -W\Delta y$. Remember, if the force and movement are in the same direction, the work is positive.

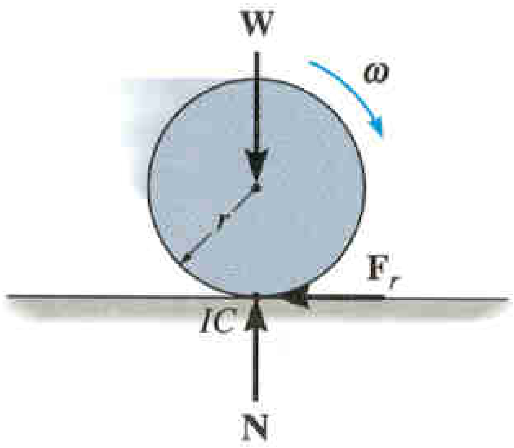


Work of a spring force: For a linear spring, the work is

$$U_s = -0.5k[(s_2)^2 - (s_1)^2]$$

Forces that do no work

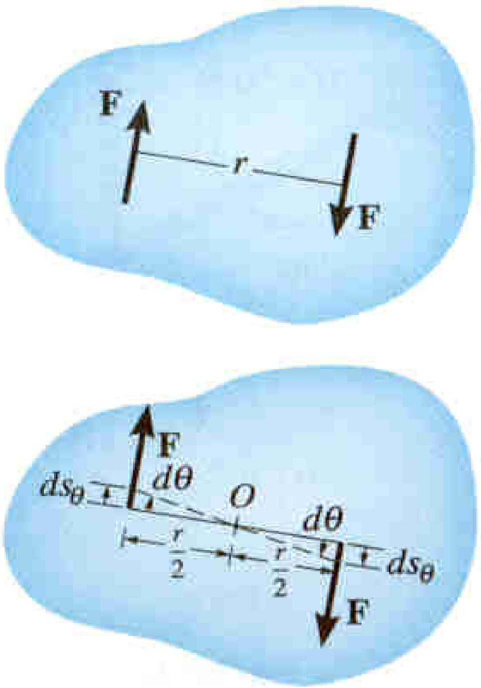
There are some external forces that do no work. For instance, reactions at fixed supports do no work because the displacement at their point of application is zero.



Normal forces and friction forces acting on bodies as they roll without slipping over a rough surface also do no work since there is no instantaneous displacement of the point in contact with ground (it is an instant center, IC).

Internal forces do no work because they always act in equal and opposite pairs. Thus, the sum of their work is zero.

The work of a couple (18.3)



Rotation

When a body subjected to a couple experiences general plane motion, the two couple forces do work only when the body undergoes rotation.

If the body rotates through an angular displacement $d\theta$, the work of the couple moment, M, is

$$U_M = \int_{\theta_1}^{\theta_2} M \, d\theta$$

If the couple moment, M, is constant, then

$$U_M = M\,(\theta_2 - \theta_1)$$

Here the work is positive, provided M and $(\theta_2 - \theta_1)$ are in the same direction.

Principle of work and energy (18.4)

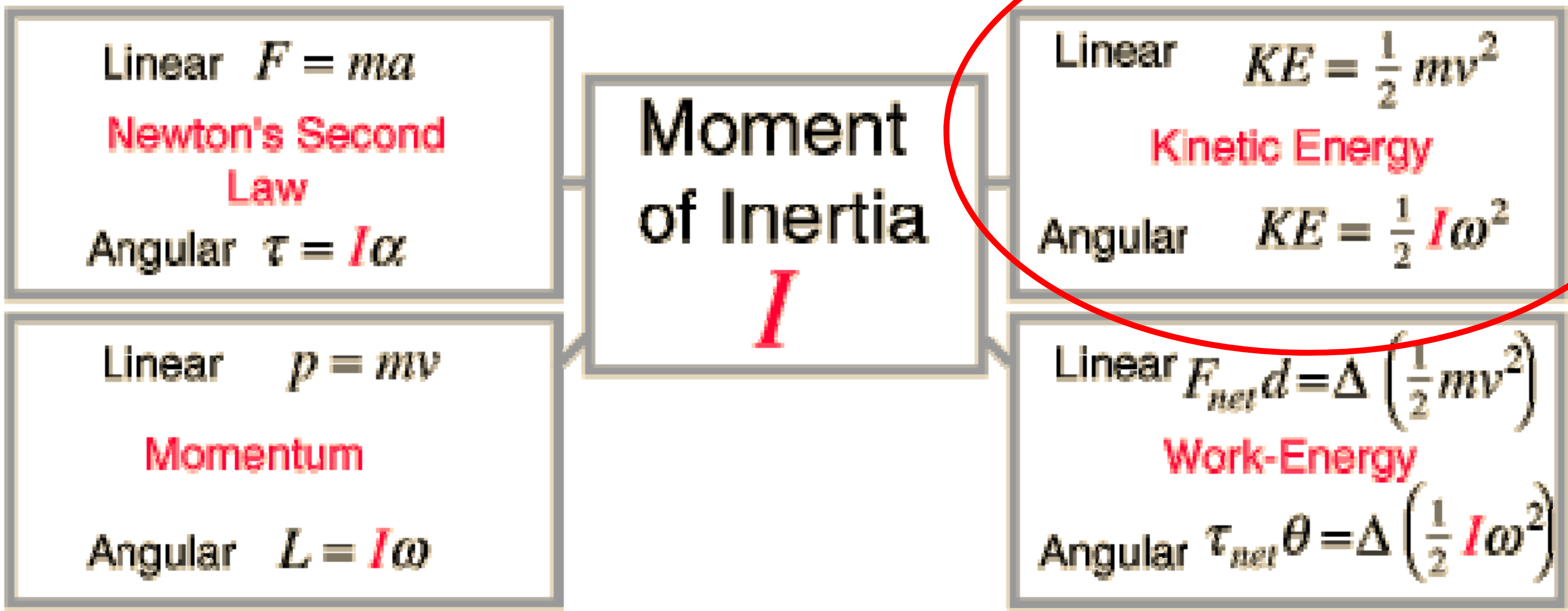
Recall the statement of the principle of work and energy used earlier:

$$T_1 + \Sigma U_{1\text{-}2} = T_2$$

In the case of general plane motion, this equation states that the sum of the initial kinetic energy (both translational and rotational) and the work done by all external forces and couple moments equals the body's final kinetic energy (translational and rotational).

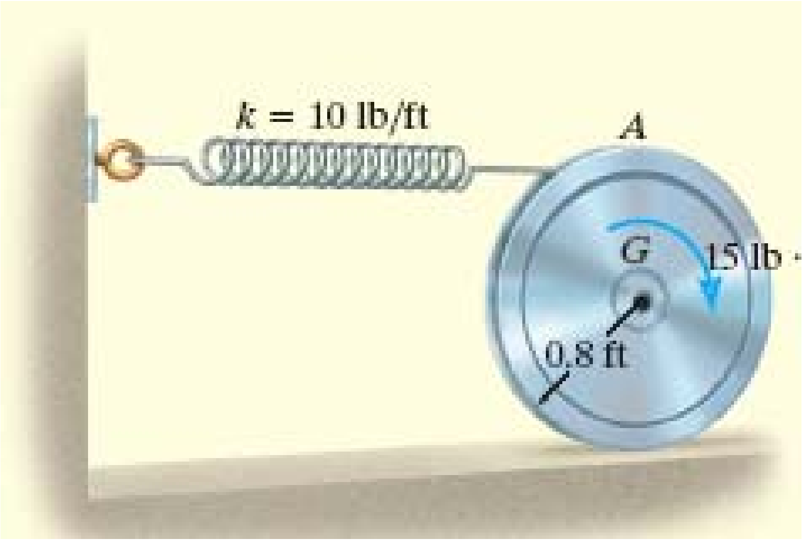
This equation is a scalar equation. It can be applied to a system of rigid bodies by summing contributions from all bodies.

Moment of Inertia



Moment of inertia is the name given to rotational inertia, the rotational analog of mass for linear motion. For a point mass the moment of inertia is just the mass times the square of perpendicular distance to the rotation axis, $I = mr^2$.

Example



Given: The disk weighs 40 lb and has a radius of gyration (k_G) of 0.6 ft. A 15 ft·lb moment is applied and the spring has a spring constant of 10 lb/ft.

Find: The angular velocity of the wheel when point G moves 0.5 ft. The wheel starts from rest and rolls without slipping. The spring is initially un-stretched.

Plan: Use the principle of work and energy since distance is the primary parameter. Draw a free body diagram of the disk and calculate the work of the external forces.

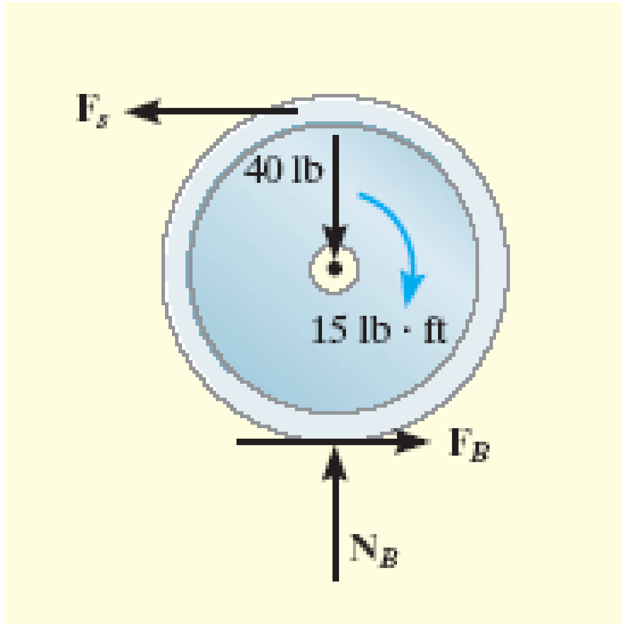
Example continues

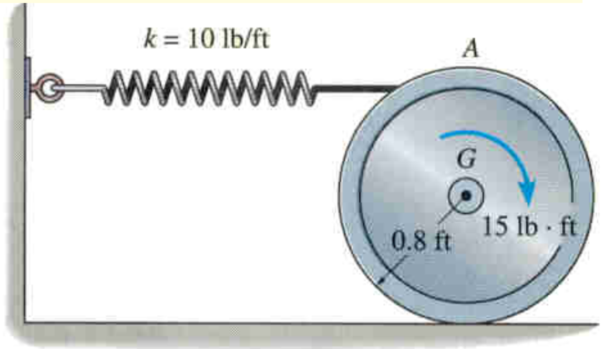
Solution:

Free body diagram of the disk:

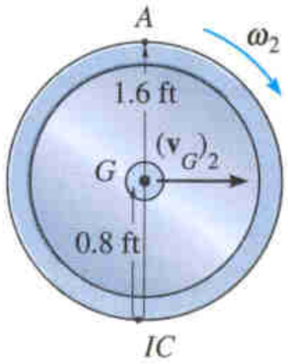
Since the body rolls without slipping on a horizontal surface, only the spring force and couple moment M do work.

Since the spring is attached to the top of the wheel, it will stretch twice the amount of displacement of G, or 1 ft.





Example continues



Work: $U_{1\text{-}2} = -0.5k[(s_2)^2 - (s_1)^2] + M(\theta_2 - \theta_1)$

$U_{1\text{-}2} = -0.5(10)(1^2 - 0) + 15(0.5/0.8) = 4.375\ \text{ft}\cdot\text{lb}$

Kinematic relation: $v_G = r\,\omega = 0.8\omega$

Kinetic energy: $T_1 = 0$

$T_2 = 0.5m\,(v_G)^2 + 0.5\,I_G\,\omega^2$

$T_2 = 0.5(40/32.2)(0.8\omega)^2 + 0.5(40/32.2)(0.6)^2\omega^2$

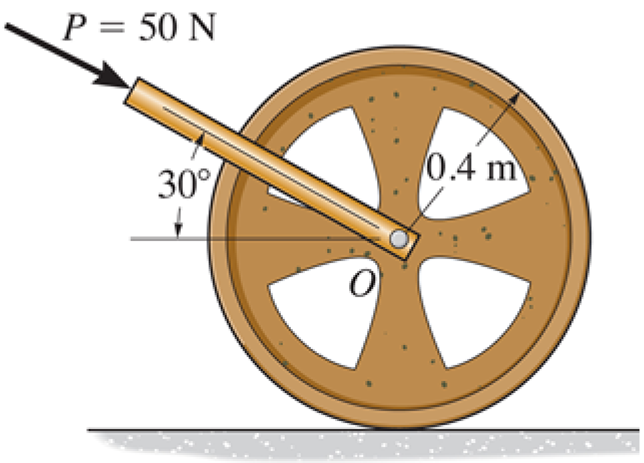
$T_2 = 0.621\,\omega^2$

Work and energy: $T_1 + U_{1\text{-}2} = T_2$

$0 + 4.375 = 0.621\,\omega^2$

$\omega = 2.65\ \text{rad/s}$

EXAMPLE



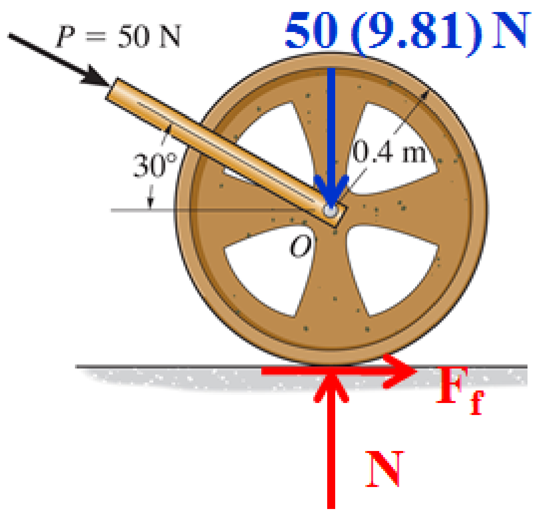
Given: The 50 kg wheel is subjected to a force of 50 N. The radius of gyration of the wheel about its mass center O is $k_O = 0.3$ m.

Find: The angular velocity of the wheel after it has rotated 10 revolutions. The wheel starts from rest and rolls without slipping.

Plan: Use the principle of work and energy to solve the problem since distance is the primary parameter. Draw a free body diagram of the wheel and calculate the work of the external forces.

EXAMPLE (continued)

Solution: Free body diagram of the wheel:



Since the wheel rolls without slipping on a horizontal surface, only the force P's horizontal component does work.

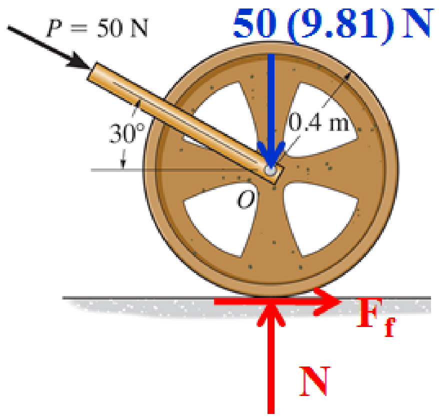
Why don't forces P_y, F_f and N do any work?

EXAMPLE (continued)

Work:

$$U_{1\text{-}2} = \int \boldsymbol{F} \bullet d\boldsymbol{r} = \int (P \cos 30)\, ds.$$

$$U_{1\text{-}2} = (50 \cos 30^{o})\, 10(\underline{\pi\ 0.8}) = 1088\ \text{N}\cdot\text{m}$$



Kinematic relation: $v_O = r\,\omega = 0.4\,\omega$

$2\pi r$

Kinetic energy:

$$T_1 = 0$$

$$T_2 = 0.5\, m\, (v_O)^2 + 0.5\, I_O\, \omega^2$$

$$= 0.5(50)\,(0.4\,\omega)^2 + 0.5\,(50)\,(0.3)^2\,\omega^2$$

$$T_2 = 6.25\,\omega^2$$

Work and energy: $T_1 + U_{1\text{-}2} = T_2$

$$0 + 1088 = 6.25\,\omega^2$$

$$\omega = 13.2\ \text{rad/s}$$

CONCEPT QUIZ

1. If a rigid body rotates about its center of gravity, its translational kinetic energy is ___________ at all times.

   A) constant

   B) equal to its rotational kinetic energy

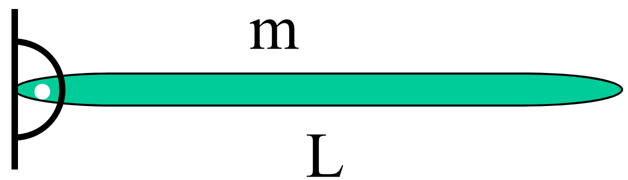
   C) zero

   D) Cannot be determined

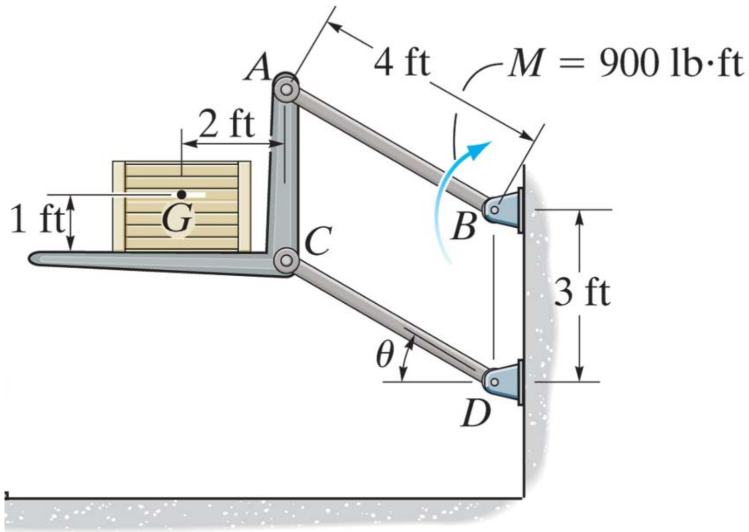
2. A rigid bar of mass m and length L is released from rest in the horizontal position. What is the rod's angular velocity when it has rotated through 90°?

   A) $\sqrt{g/3L}$ B) $\sqrt{3g/L}$

   C) $\sqrt{12g/L}$ D) $\sqrt{g/L}$



Example



Given: The combined weight of the load and the platform is 200 lb, with the center of gravity located at G. A couple moment is applied to link AB. The system is at rest when $\theta = 0$. Neglect the weight of the links.

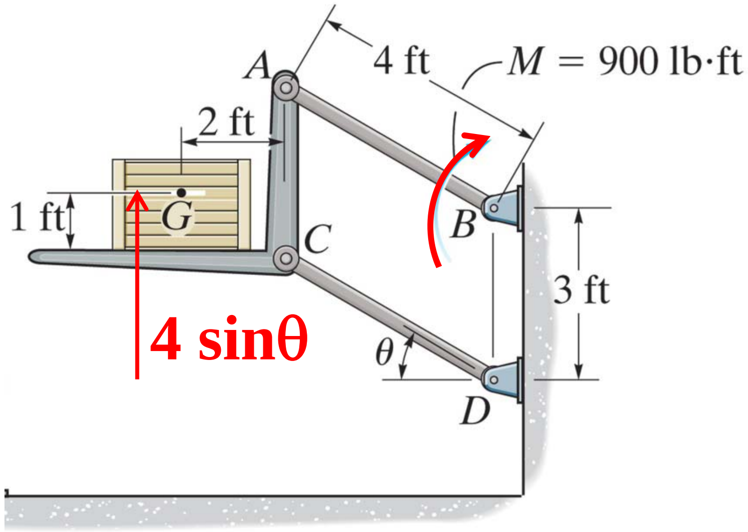
Find: The angular velocity of links AB & CD at $\theta = 60^{o}$.

Plan: Since the problem involves distance, the principle of work and energy is an efficient solution method.

Example (continued)

Solution: Work done by the external loads

Calculate the vertical distance the mass center moves.



$\Delta y = 4 \sin \theta$ where $\theta = 60°$

Then, determine the work due to the weight.

$U_w = -W \Delta y = - W (4 \sin \theta)$

$U_w = - 200 (4 \sin 60^o)$

$= - 692.8$ ft·lb

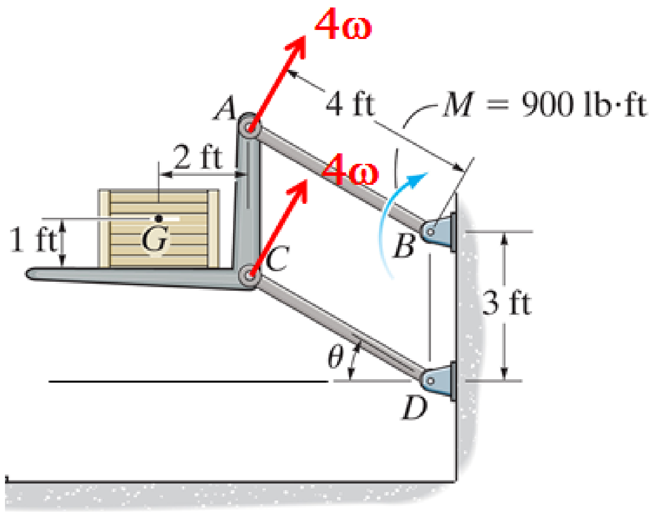
Work due to the moment (careful of θ's units!):

$$U_M = \mathbf{M \cdot \theta} = 900 (60°) (\pi/180) = 942.5 \text{ ft·lb}$$

Therefore, $U_{1\text{-}2} = - 692.8 + 942.5 = 249.7$ ft·lb

Example (continued)

As links AB and CD rotate, the platform will be subjected to only curvilinear translational motion with a speed of 4ω.



Kinetic energy:

$T_1 = 0$

$T_2 = 0.5\ m\ (v_G)^2$

$= 0.5\ (200/32.2)\ (4\omega)^2$

$= 49.69\ \omega^2\ \text{ft·lb}$

Now apply the principle of work and energy equation:

$T_1 + U_{1\text{-}2} = T_2$

$0\ + 249.7 = 49.69\ \omega^2$

$\omega = 2.24\ \text{rad/s}$

Homework Assignment

Chapter18- 17, 37, 43, 47
Due next Wednesday !!!

Homework Assignment

Chapter17- 6, 23, 27,33, 38, 43, 53, 59, 74, 79,95, 98, 102,109

Due next Monday !!!

Chapter reviews

Chapter 12: **pages 101-105**

Chapter 13: **pages 166-167**

Chapter 14: **pages 217-219**

Chapter 15: **pages 295-297**

Chapter 16: **pages 391-393**

Chapter 17: **pages 452-453**

Chapter 18: **pages 490-493**

Chapter 19: **pages 531-533**

Book chapter reviews give you a good but brief idea about each chapter…

General exam rules

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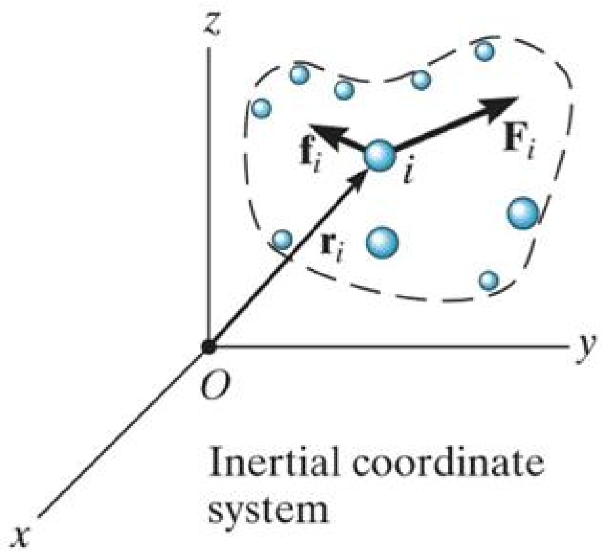
Chapter 15: make sure you review the conservation of momentum and conservation of energy.

Principle of linear impulse and momentum for a system of particles
(Section 15.2)



For the system of particles shown, the internal forces f_i between particles always occur in pairs with equal magnitude and opposite directions. Thus the internal impulses sum to zero.

The linear impulse and momentum equation for this system only includes the impulse of external forces.

$$\sum m_i(v_i)_1 + \sum \int_{t_1}^{t_2} F_i \, dt = \sum m_i(v_i)_2$$

Conservation of linear momentum for a system of particles (Section 15.3)

When the sum of external impulses acting on a system of objects is zero, the linear impulse-momentum equation simplifies to

$$\sum m_i(\boldsymbol{v}_i)_1 = \sum m_i(\boldsymbol{v}_i)_2$$

This important equation is referred to as the conservation of linear momentum. Conservation of linear momentum is often applied when particles collide or interact. When particles impact, only impulsive forces cause a change of linear momentum.

The sledgehammer applies an impulsive force to the stake. The weight of the stake can be considered negligible, or non-impulsive, as compared to the force of the sledgehammer. Also, provided the stake is driven into soft ground with little resistance, the impulse of the ground's reaction on the stake can also be considered negligible or non-impulsive.

Central impact (continued)

In most problems, the initial velocities of the particles, $(v_A)_1$ and $(v_B)_1$, are known, and it is necessary to determine the final velocities, $(v_A)_2$ and $(v_B)_2$. So the first equation used is the conservation of linear momentum, applied along the line of impact.

$$(m_A v_A)_1 + (m_B v_B)_1 = (m_A v_A)_2 + (m_B v_B)_2$$

This provides one equation, but there are usually two unknowns, $(v_A)_2$ and $(v_B)_2$. So another equation is needed. The principle of impulse and momentum is used to develop this equation, which involves the coefficient of restitution, or *e*.

Central impact (continued)

The coefficient of restitution, e, is the ratio of the particles' relative separation velocity after impact, $(v_B)_2 - (v_A)_2$, to the particles' relative approach velocity before impact, $(v_A)_1 - (v_B)_1$. The coefficient of restitution is also an indicator of the energy lost during the impact.

The equation defining the coefficient of restitution, e, is

$$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

If a value for e is specified, this relation provides the second equation necessary to solve for $(v_A)_2$ and $(v_B)_2$.

Theory: Moment and Angular Momentum

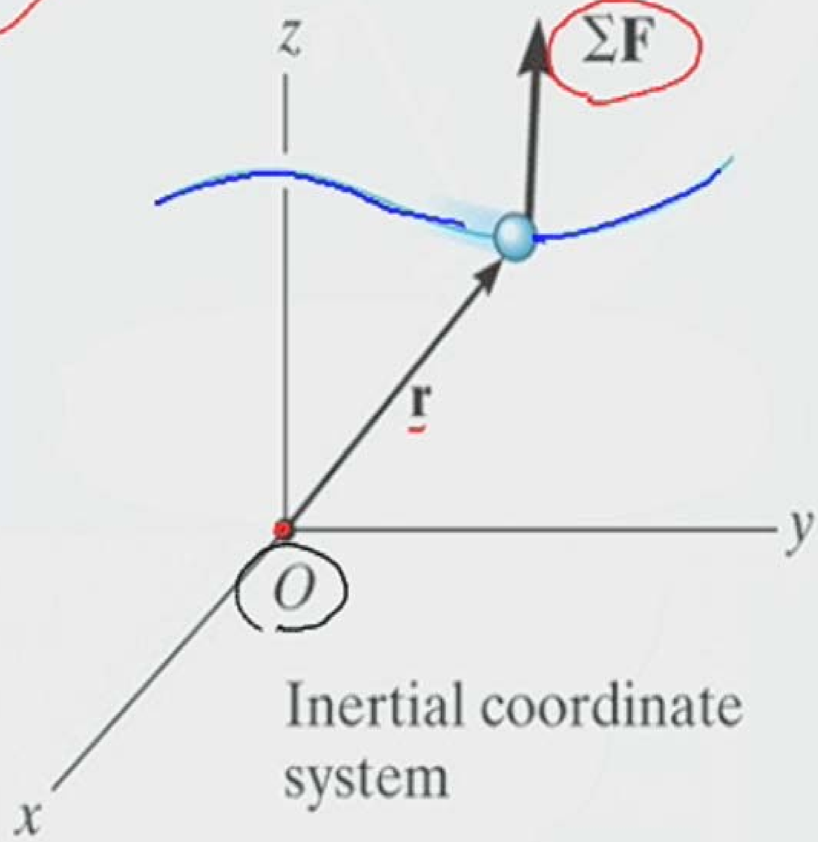
$$\Sigma \bar{F} = m\bar{a} \Rightarrow \Sigma \bar{F} = m\dot{\bar{v}}$$

$$\bar{H}_o = \bar{r} \times (m\bar{v})$$

$$\bar{r} \times (\Sigma \bar{F}) = \bar{r} \times (m\dot{\bar{v}})$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underset{\Sigma}{\bar{M}_o} = \underset{\Sigma}{\dot{H}_o}$$



W. Wang

Relationship between moment of a force and angular momentum (15.6)

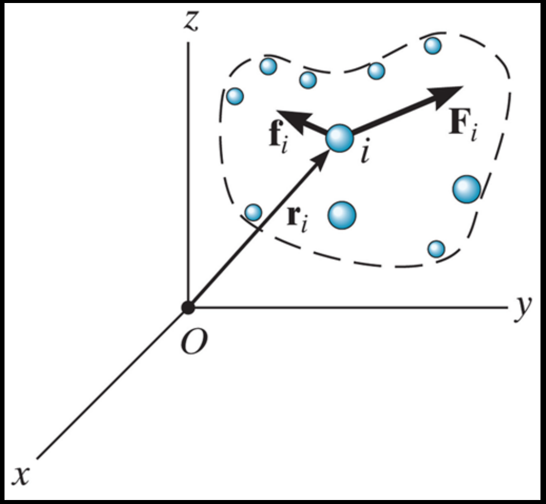
The resultant force acting on the particle is equal to the time rate of change of the particle's linear momentum. Showing the time derivative using the familiar "dot" notation results in the equation

$$\sum \boldsymbol{F} = \dot{\boldsymbol{L}} = m\dot{\boldsymbol{v}}$$

We can prove that the resultant moment acting on the particle about point O is equal to the time rate of change of the particle's angular momentum about point O or

$$\sum \boldsymbol{M}_o = \boldsymbol{r} \times \boldsymbol{F} = \dot{\boldsymbol{H}}_o$$

MOMENT AND ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES



$$\sum \boldsymbol{M}_o = r \times \boldsymbol{F} = \dot{\boldsymbol{H}}_o$$

The same form of the equation can be derived for the system of particles.

Then, the moments of these forces for the particles can be written as $\sum(r_i \times \boldsymbol{F}_i) + \sum(r_i \times \boldsymbol{f}_i) = \sum(\dot{\boldsymbol{H}}_i)_o$

The second term is zero since the internal forces occur in equal but opposite collinear pairs. Thus,

$$\sum \boldsymbol{M}_o = \sum(r_i \times \boldsymbol{F}_i) = \sum(\dot{\boldsymbol{H}}_i)_o$$

Angular impulse and momentum principles (Section 15.7)

Considering the relationship between moment and time rate of change of angular momentum

$$\sum \boldsymbol{M}_o = \dot{\boldsymbol{H}}_o = d\boldsymbol{H}_o/dt$$

By integrating between the time interval t_1 to t_2

$$\sum \int_{t_1}^{t_2} \boldsymbol{M}_o \, dt = (\boldsymbol{H}_o)_2 - (\boldsymbol{H}_o)_1 \quad \text{or} \quad (\boldsymbol{H}_o)_1 + \sum \int_{t_1}^{t_2} \boldsymbol{M}_o \, dt = (\boldsymbol{H}_o)_2$$

This equation is referred to as the principle of angular impulse and momentum. The second term on the left side, $\sum\int M_o \, dt$, is called the angular impulse. In cases of 2D motion, it can be applied as a scalar equation using components about the z-axis.

Chapter 16: Planar kinematics of a rigid body

Theory: Translation (16.2)

- all points on a rigid body subject to either rectilinear or curvilinear translation (i.e., no rotation) move with the same velocity and acceleration

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

$$\bar{r}_B = \bar{r}_A + \bar{r}_{B/A} \text{ relative position}$$

$$\frac{d}{dt}$$

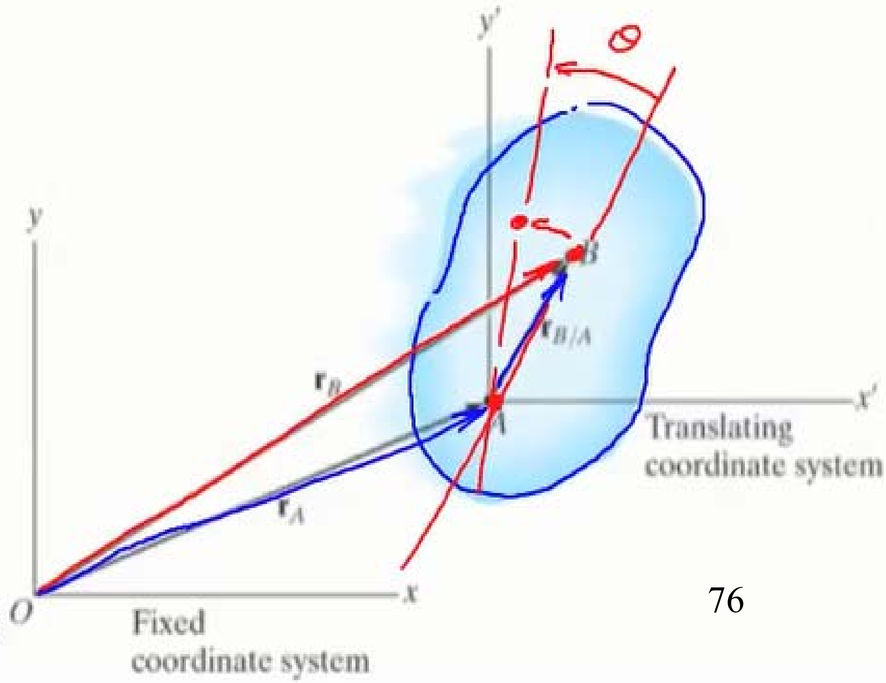
$$\bar{V}_B = \bar{V}_A + \bar{V}_{B/A}$$

$$\frac{d}{dt}$$

$$\bar{a}_B = \bar{a}_A + \bar{a}_{B/A}$$

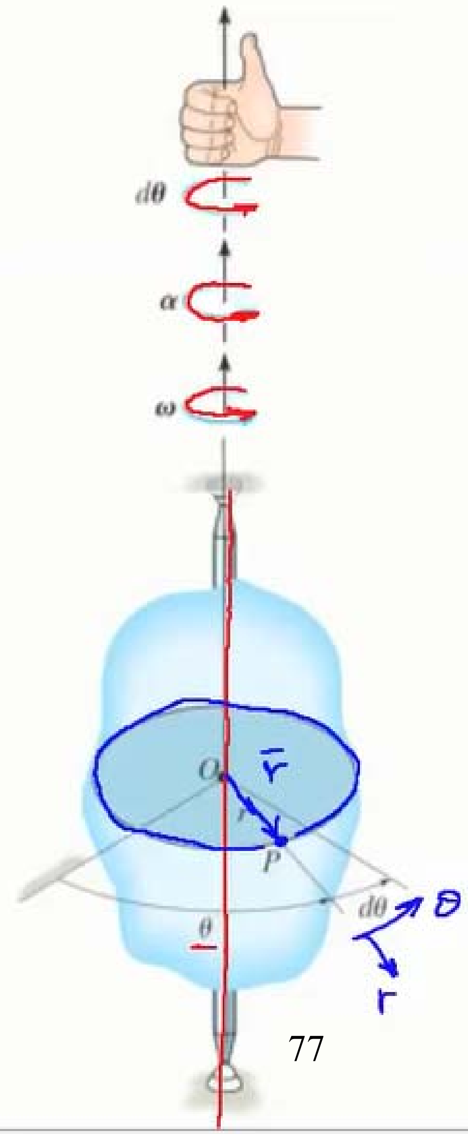
(i) what is changing?
- distances (A & B)
- orientations

(ii) what coord. system do I use to express these changes?



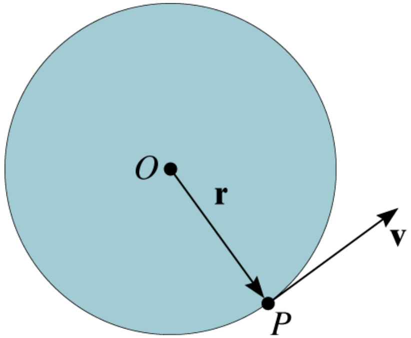
Concept: Rotation about a Fixed Axis (16.3)

- if a body rotates about a fixed axis, then all points on that body follow a circular path
- we define several kinematic quantities:
  - angular position θ
  - angular displacement $d\theta, \Delta\theta$
  - angular velocity ω
  - angular acceleration α

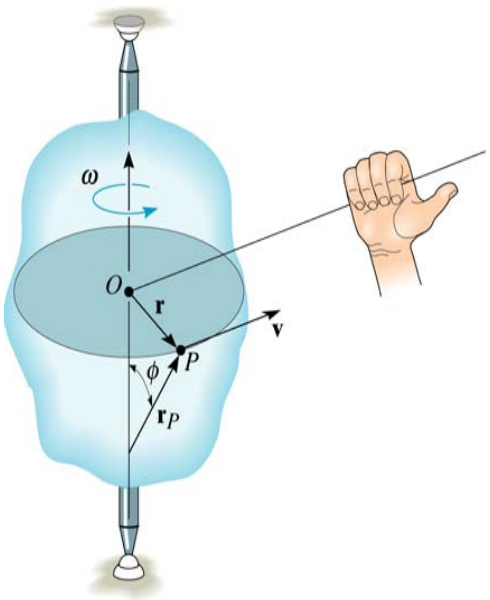


$$\alpha = \dot{\omega} = \ddot{\theta}$$

Rigid body rotation – Velocity of point P



The magnitude of the velocity of P is equal to ωr (the text provides the derivation). The velocity's direction is tangent to the circular path of P.

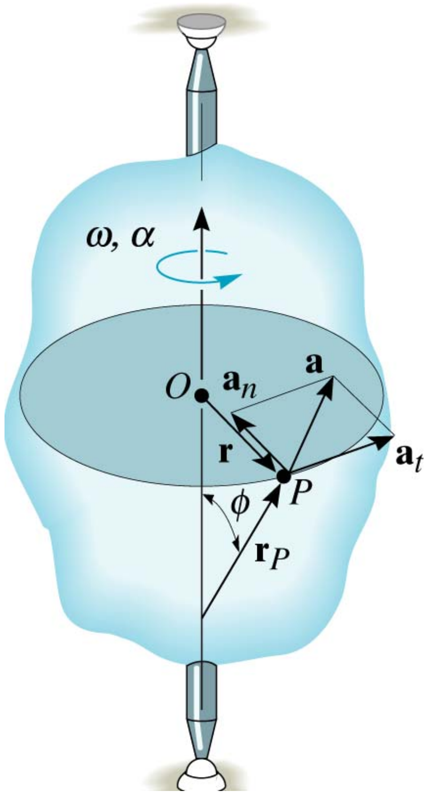


In the vector formulation, the magnitude and direction of $\boldsymbol{v}$ can be determined from the cross product of $\boldsymbol{\omega}$ and $\boldsymbol{r}_p$. Here $\boldsymbol{r}_p$ is a vector from any point on the axis of rotation to P.

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}_p = \boldsymbol{\omega} \times \boldsymbol{r}$$

The direction of $\boldsymbol{v}$ is determined by the right-hand rule.

Rigid body rotation – Acceleration of point P (continued)



Using the vector formulation, the acceleration of P can also be defined by differentiating the velocity. (we derived it earlier in week 2)

$$\boldsymbol{a} = d\boldsymbol{v}/dt = d\boldsymbol{\omega}/dt \text{ x } \boldsymbol{r}_P + \boldsymbol{\omega} \text{ x } d\boldsymbol{r}_P/dt$$

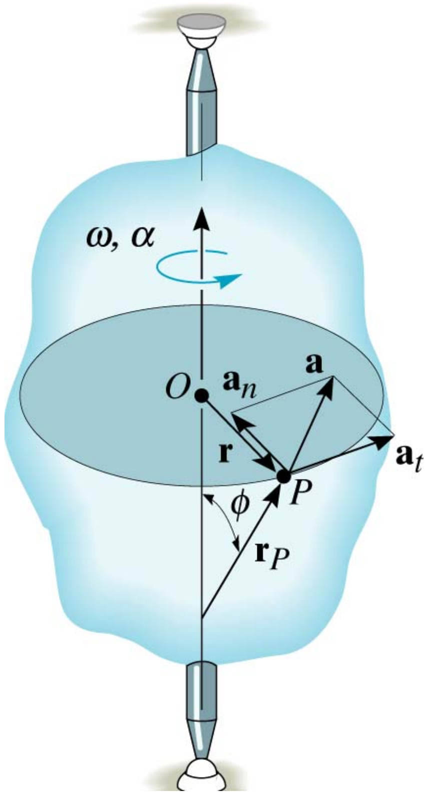
$$= \boldsymbol{\alpha} \text{ x } \boldsymbol{r}_P + \boldsymbol{\omega} \text{ x } (\boldsymbol{\omega} \text{ x } \boldsymbol{r}_P)$$

It can be shown that this equation reduces to

$$\boldsymbol{a} = \boldsymbol{\alpha} \text{ x } \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_t + \boldsymbol{a}_n$$

The magnitude of the acceleration vector is $a = \sqrt{(a_t)^2 + (a_n)^2}$

Rigid body rotation – Acceleration of point P (continued)



Using the vector formulation, the acceleration of P can also be defined by differentiating the velocity. (we derived it earlier in week 2)

$$\boldsymbol{a} = d\boldsymbol{v}/dt = d\boldsymbol{\omega}/dt \times \boldsymbol{r}_P + \boldsymbol{\omega} \times d\boldsymbol{r}_P/dt$$

$$= \boldsymbol{\alpha} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_P)$$

It can be shown that this equation reduces to

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_t + \boldsymbol{a}_n$$

The magnitude of the acceleration vector is $a = \sqrt{(a_t)^2 + (a_n)^2}$

Acceleration in the n-t coordinate system II

The tangential component of acceleration is constant, $a_t = (a_t)_c$.

In this case,

$$s = s_o + v_o t + (1/2)(a_t)_c t^2$$

Rigid body rotation:

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_t + \boldsymbol{a}_n$$

$$v = v_o + (a_t)_c t$$

$$v^2 = (v_o)^2 + 2(a_t)_c(s - s_o)$$

$$\boldsymbol{a} = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n$$

As before, s_o and v_o are the initial position and velocity of the particle at $t = 0$

Then accleration in polar coordinates:

$$a = \dot{v} = \ddot{r}u_r + \dot{r}\dot{u}_r + \dot{r}\dot{\theta}u_\theta + r\ddot{\theta}u_\theta + r\dot{\theta}\dot{u}_\theta$$

$$= \ddot{r}u_r + \dot{r}\dot{\theta}u_\theta + \dot{r}\dot{\theta}u_\theta + r\ddot{\theta}u_\theta - r\dot{\theta}\dot{\theta}u_r$$

$$= (\ddot{r}u_r - r\dot{\theta}\dot{\theta}u_r) + (2\dot{r}\dot{\theta}u_\theta + r\ddot{\theta}u_\theta)$$

$$= (\ddot{r} - r\dot{\theta}\dot{\theta})\,\boldsymbol{u}_r + (\cancel{2\dot{r}\dot{\theta}} + r\ddot{\theta})\,\boldsymbol{u}_\theta$$

Rigid body rotation:

$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_t + \boldsymbol{a}_n$

Theory: Relative Motion Analysis (16.5)

- another way to characterize this general = translation + rotation motion is to use a moving coordinate frame
- consider a rigid body (bar) AB, whose motion is "general"
- we attach a moving (translating!) reference frame to point A, and look at the rotation of B around the origin of this moving frame
- the moving reference frame does not rotate

translating, NOT rotating

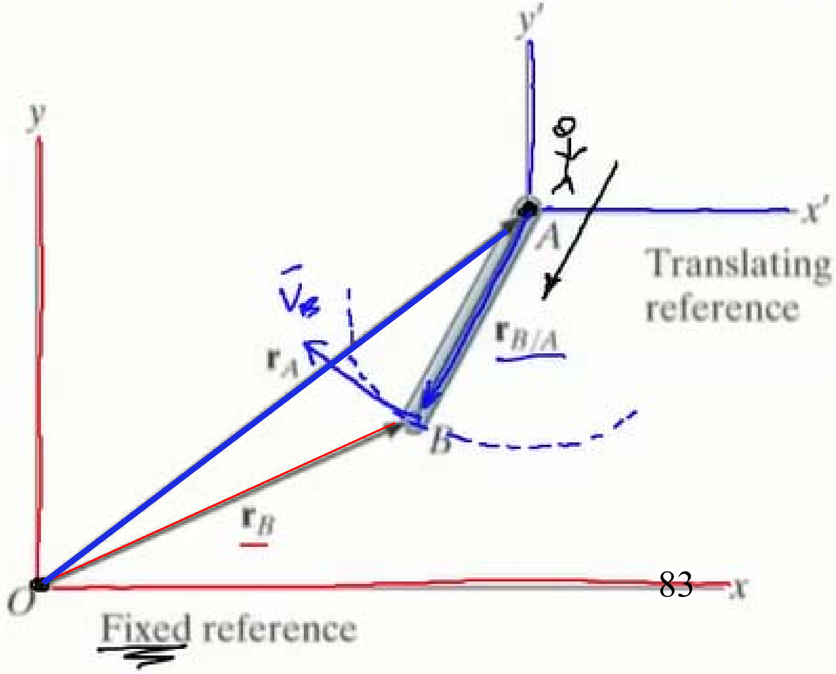
$$\bar{r}_B = \bar{r}_A + \bar{r}_{B/A}$$

$$\bar{v}_B = \bar{v}_A + \bar{v}_{B/A} \Rightarrow \text{translating frame}$$

$\Downarrow$

pure rotation

$$\bar{\omega}_{AB} \times \bar{r}_{B/A}$$



Theory: Position at Two Times

① pure translation, $d\bar{r}_A \Rightarrow$

② rotating B with respect to A

"pure" rotation, $d\theta$

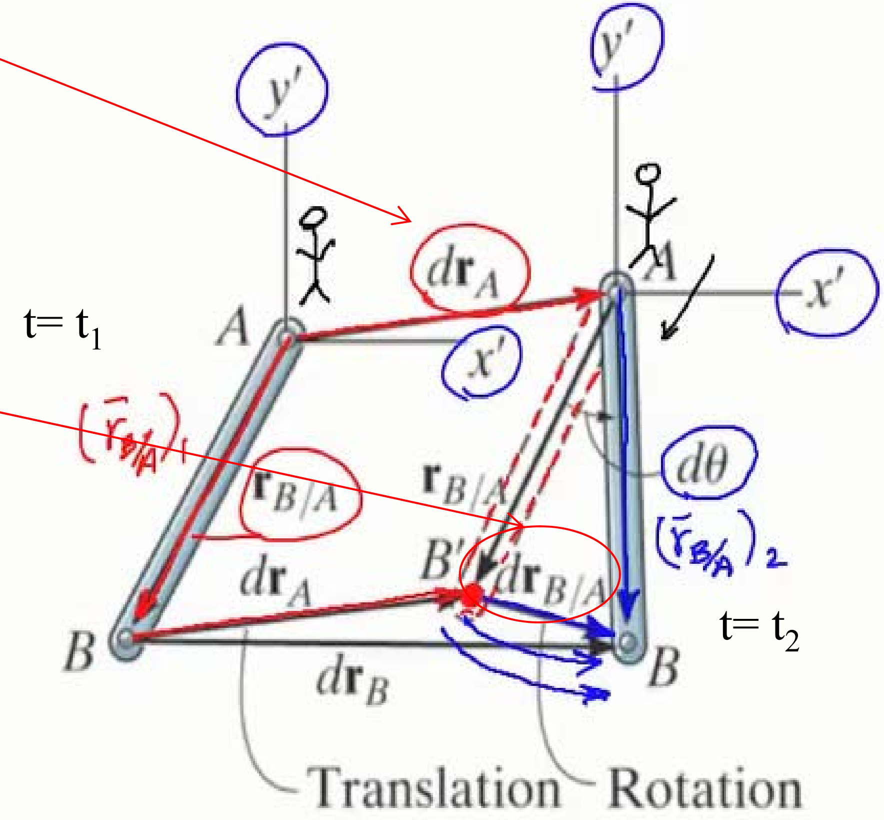
$$d\bar{r}_{B/A} \Rightarrow \bar{\omega}\times\bar{r}$$

general motion: ① + ②

$$\bar{V}_{B/A} = \bar{\omega}_{AB} \times \bar{r}_{B/A}$$

"circular motion"

↳ rigid body



Theory: Relative Motion, Acceleration (16.7)

- acceleration analysis for general motion also follows the "relative" form, by differentiation of the relative velocity equation:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

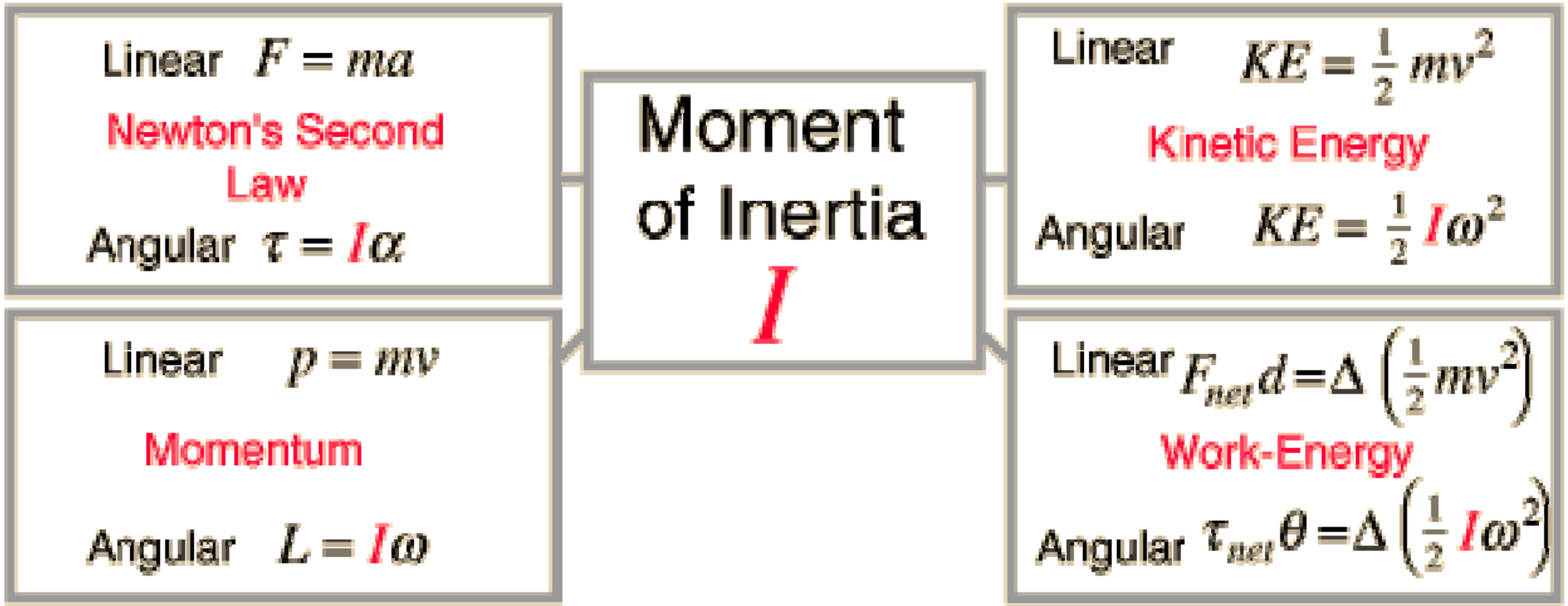
$$\underbrace{\bar{a}_B = \bar{a}_A}_{\text{absolute!}} + \underbrace{\bar{a}_{B/A}}_{\text{relative}}$$

$$\bar{a}_{B/A} = \underbrace{\bar{\alpha}_{AB} \times \bar{r}_{B/A}}_{\text{tang.}} + \underbrace{\bar{\omega}_{AB} \times (\vec{\omega}_{AB} \times \bar{r}_{B/A})}_{\text{normal}}$$

Chapter 17:

Planar kinetics of a rigid body: Force and acceleration

Moment of Inertia



Moment of inertia is the name given to rotational inertia, the rotational analog of mass for linear motion. For a point mass the moment of inertia is just the mass times the square of perpendicular distance to the rotation axis, $I = mr^2$.

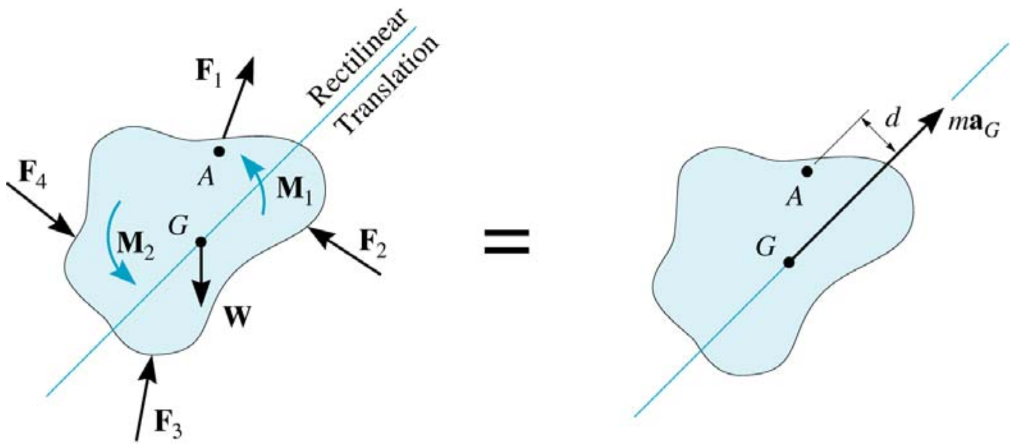
Equations of motion: Translation (17.3)

When a rigid body undergoes only translation, all the particles of the body have the same acceleration so $\boldsymbol{a}_G = \boldsymbol{a}$ and $\alpha = 0$. The equations of motion become:

$$\Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

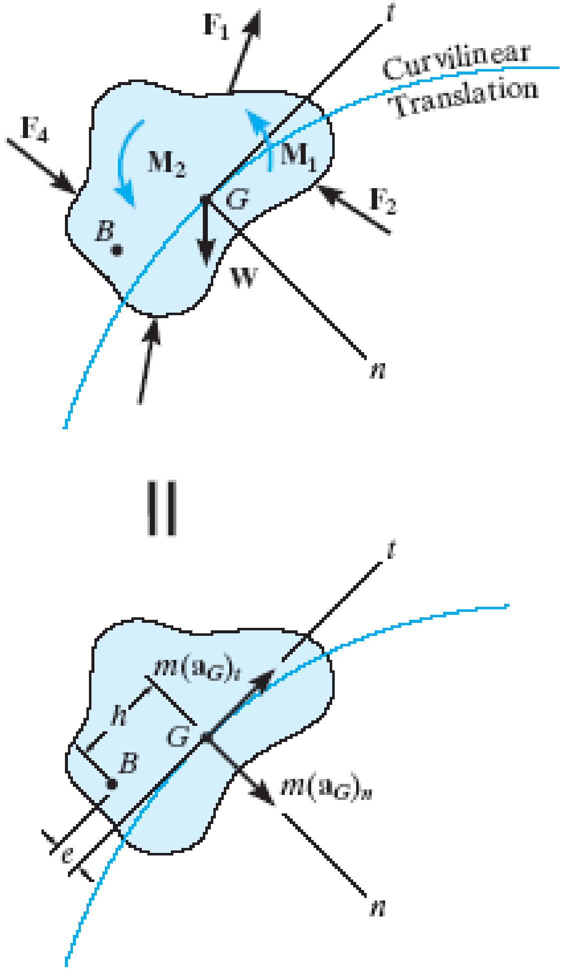
$$\Sigma M_G = 0$$



Note that, if it makes the problem easier, the moment equation can be applied about other points instead of the mass center. In this case,

$$\Sigma M_A = \boldsymbol{r}_G \times m\boldsymbol{a}_G = (m\, a_G)\, d\, .$$

Equations of motion: Translation (17.3) (continues)



When a rigid body is subjected to curvilinear translation, it is best to use an n-t coordinate system. Then apply the equations of motion, as written below, for n-t coordinates.

$$\Sigma F_n = m(a_G)_n$$

$$\Sigma F_t = m(a_G)_t$$

$$\Sigma M_G = 0 \quad \text{or}$$

$$\Sigma M_B = \boldsymbol{r}_G \times m\boldsymbol{a}_G = e[m(a_G)_t] - h[m(a_G)_n]$$

Equations of motion for pure rotation (17.4) (continues)

Note that the $\sum M_G$ moment equation may be replaced by a moment summation about any arbitrary point. Summing the moment about the center of rotation O yields

$$\sum M_O = I_G \alpha + r_G\, m\, (a_G)_t = \underline{(I_G + m\,(r_G)^2)}\, \alpha$$

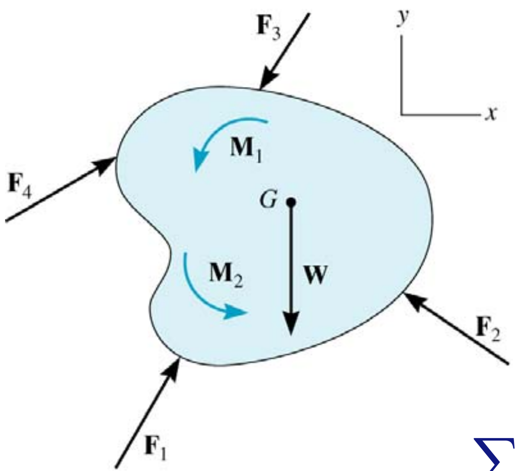
From the parallel axis theorem, $I_O = I_G + m(r_G)^2$, therefore the term in parentheses represents I_O. Consequently, we can write the three equations of motion for the body as:

$$\sum F_n = m\,(a_G)_n = m\, r_G\, \omega^2$$

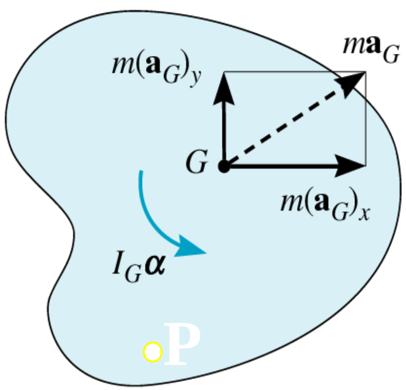
$$\sum F_t = m\,(a_G)_t = m\, r_G\, \alpha$$

$$\sum M_O = I_O\, \alpha$$

General plane motion (17.5) continues…



Sometimes, it may be convenient to write the moment equation about some point P other than G. Then the equations of motion are written as follows.



$$\sum F_x = m\,(a_G)_x$$

$$\sum F_y = m\,(a_G)_y$$

$$\sum M_P = \sum (M_k) = I_G\alpha + r_G\, m\,(a_G)_t = (I_G + m\,(r_G)^2)\,\alpha$$

In this case, $\sum (M_k)_P$ represents the sum of the moments of $I_G\alpha$ and $m\boldsymbol{a}_G$ about point P.

Remember…

…next time is Exam time