Some Transformations of the Behre Equation of Tree Form

Note by Donald Bruce

Abstract. The described equations permit flexible transformations of Behre's hyperbolic equations from one top diameter to another. They also make it possible to determine coefficients applicable to any top diameter from measurements taken to some other merchantable top. ***Forest Sci. 18:164–166.***

Additional key words. Tree form, research techniques, tree volume, taper equation.

THIS IS A slight revision of an unpublished paper dated 1953; it explains some of the methods used in application of Behre's hyperbola to prepare the Mason, Bruce, and Girard volume tables. There is a continuing demand for these transformation equations by those who want to estimate volume to a merchantable top other than those in published tables. These equations are especially useful when volumes are calculated by electronic computers.

Many equations have been suggested for describing tree form. One of the most successful was proposed by C. E. Behre.[1] It is that of a hyperbola and may be written

$$D = L/(AL + B) \qquad (1)$$

where D is the tree diameter at any point, expressed in percentage of the basal diameter, and L is the length to the point of diameter measurement, measured downwards from the tip and expressed in percentage of the total length, while A and B are constants. Furthermore,

$$A + B = 1 \qquad (2)$$

so that the equation contains virtually a single parameter.

The significance of the constant can be seen in Figure 1. The curve is one branch of the hyperbola. The coordinate axes are $O\,L$ and $O\,D$ with O the origin. The broken lines are the asymptotes to the hyperbola. The horizontal asymptote is $1/A$ above the horizontal axis, and the vertical asymptote is B/A to the left of the vertical axis.

The useful range of A values is between 0 and 1. If $A = 1$ (and $B = 0$), the equation becomes $D = 1$. At the other extreme, if $A = 0$ (and $B = 1$), the equation becomes $D = L$. In these limiting cases, the hyperbola degenerates into straight lines which correspond with cylindrical and conical trees, respectively. Negative values of A would apply to trees of concave outline. As A increases from 0 towards 1, the hyperbola represents trees of progressively fuller form.

A transformation of the equation will simplify fitting it to actual tree measurements If $1 - A$ is substituted for B, the following form can be obtained by transposition

$$A = (D - L)/[D\,(1 - L)] \qquad (3)$$

Values for A can now be calculated directly for every pair of values of D and L, and they can be combined into a single value by averaging. If the measurements are taken at the ends of logs, each log will then have equal weight.

For truncated trees, the equation in the form described above presents difficulties. If the distance from the top log to the extreme tip is unknown, values of L as defined above are not available.

Let us consider first the case where the limit of merchantability is equal to some fixed fraction of basal diameter such as 0.5. The diameter axis is regraduated so that 0.5 relative diameter replaces 0, and 1.0 relative diameter is unchanged (see Fig. 1). The equation of the curve now can be written:

$$D = 0.5 + L_{.5}/(A_{.5}L_{.5} + B_{.5}) \qquad (4)$$

In this form $L_{.5}$ must be defined as the length to the point of diameter measurement, measured downward from the point of truncation, and expressed in percentage of length from point of truncation to base. (The subscripts are

[1] Behre, C. Edward. Form-class taper tables and volume tables and their application. J Agr Res 35:673–744, illus. 1927.

The author, deceased 1966, was a professor of mensuration at the University of California and later a partner of Mason, Bruce and Girard, a western forest consulting and management firm in Portland, Oregon. This paper was revised for publication by David Bruce, a project leader at the Pacific Northwest Forest and Range Exp. Sta, USDA Forest Service, Portland. Manuscript received Sept. 2, 1971.

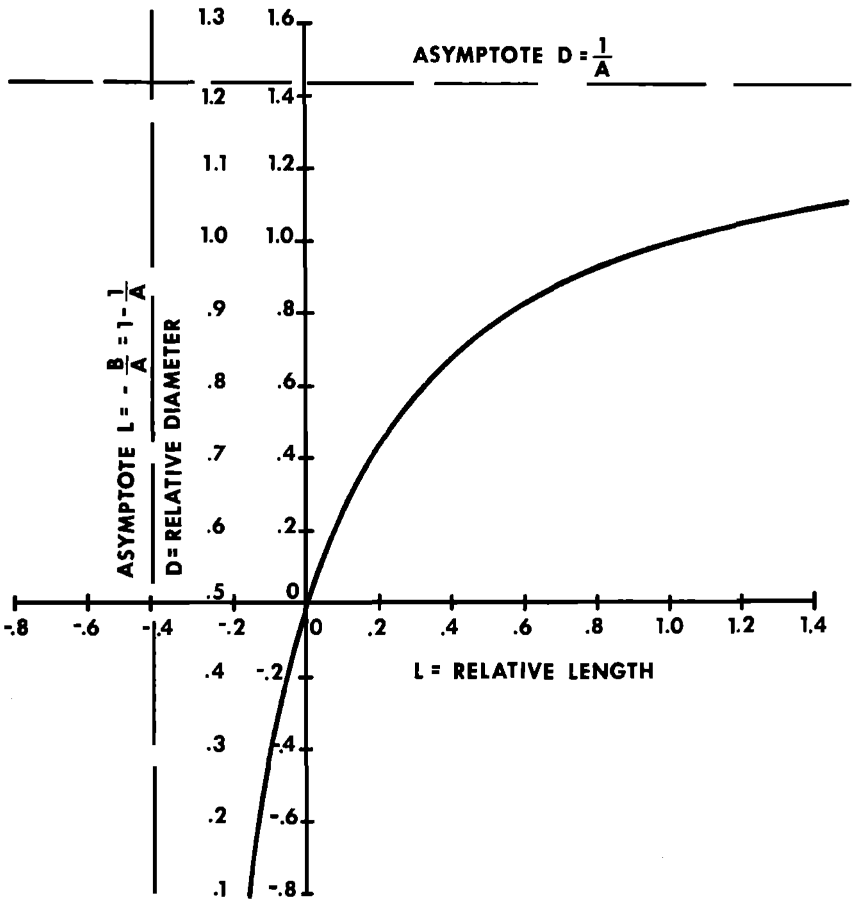

FIGURE 1. Behre's hyperbola showing regraduation of diameter axis. Original diameter graduations are to right, regraduations to left.

used to indicate relative diameter at point from which length is measured.)

The relation $A + B = 1$ is now no longer true. Since the curve still passes through the point (1, 1), we may substitute 1 for both D and L and thus find

$$1 - 0.5 = 1/(A_{.5} + B_{.5})$$

and
$$A_{.5} + B_{.5} = 2 \tag{5}$$

For convenience in fitting tree measurements, eq. 4 may be transposed into a form which corresponds to that of eq. 3 after substituting $2 - A_{.5}$ for $B_{.5}$ as follows:

$$A_{.5} = \frac{2D - L_{.5} - 1}{(1 - L_{.5})\ (D - 0.5)} \tag{6}$$

The last three equations may be generalized for use where some relative top cutting limit other than $D = 0.5$ is used. Let $T =$ the merchantable top diameter in percentage of basal diameter. Then

$$D = T + [L_T/(A_T L_T + B_T)] \tag{7}$$

and
$$A_T + B_T = 1/(1 - T) \tag{8}$$

Eq. 7 can now be transposed into

$$A_T = \frac{(D - T)/(1 - T) - L_T}{(1 - L_T)\ (D - T)} \tag{9}$$

It is useful to be able to convert equations derived from truncated data into the corresponding equations based on the entire tree to the tip. This can be done by extending the truncated curve to the point where $D = 0$, then shifting the axes of coordinates to that point with an appropriate change in scale. After some rather tedious algebra this gives

$$A_0 = A_T/(1 + A_T T) \tag{10}$$

or $$A_T = A_0/(1 - A_0 T) \qquad (11)$$

where the A_0 is the constant in eq. 1 to 3 and A_T is the corresponding constant in eq. 7 to 9. (The subscript 0 indicates zero diameter or the tip of tree.)

To illustrate the use of this equation, let us transform the following which applies to trees truncated at $D = 0.5$:

$$D = 0.5 + [L_{.5}/(0.9\, L_{.5} + 1.1)]$$

In this, $A_T = 0.9$ and $T = 0.5$. Substituting in eq. 10 we have

$$A_0 = 0.9/[1 + (0.9)(0.5)] = 0.62$$

From eq. 3, $B_1 = 1 - 0.62 = 0.38$. The corresponding equation for the complete tree is then

$$D = L_0/[(0.62\, L_0 + 0.38)]$$

The two equations will give identical values of D if L is appropriately measured in each case.

Again if we need to alter T from 0.5 to 0.4, the first step is to derive the equation of the complete tree as above and the second, to convert it into the desired form by using eq. 11 as follows:

$$A_T = 0.62/[1 - 0.62\,(0.4)]$$

whence $$A_T = 0.83$$

Furthermore, from eq. 8,

$$0.83 + B_T = 1/(1 - 0.4) = 1.67$$

whence $$B_T = 0.84$$

The desired equation is thus

$$D - 0.4 = L_{.4}/(0.83\, L_{.4} + 0.84)$$

where L is measured from the truncation point, $D = 0.4$.

Thus we have available a set of transformation equations. These permit flexible transformations of hyperbolic equations from any top diameter to any other. Also tree measurements taken to variable top diameters can be used in ascertaining average values of A and B applicable to some other top diameter.

A possible procedure is as follows:

For each tree—

(a) Calculate T,
(b) Using eq. 9, calculate all possible values of A_T and average them, and
(c) Using eq. 10, calculate the corresponding value of A_0 for the entire tree.

For all trees—

(d) Average all values of A_0,
(e) Using eq. 11, convert average A_0 to corresponding A_T for whatever value of T is desired,
(f) Using eq. 8, calculate corresponding values of B_T, and
(g) Write the equation as in eq. 7.

The same procedure can be used where heights to a fixed top cutting limit are demanded. The first four steps are as above, but since the value of T corresponding to a fixed top will be different for each diameter class, it will be necessary to repeat steps (e) to (g) for each such class.

This equation, like most others, does not fit well that part of the tree close to the ground where butt swell produces a concave outline. Where volumes are measured in board feet by some log rule which makes either no allowance for taper or a standard allowance for it, the form of the butt log below its scaling diameter is immaterial. In this case, it is convenient to take as basal diameter the diameter at the top of the butt log, normally 16 ft long. Much better fitting equations will result.

All the foregoing formulas may still be used with a simple change in the definition of L In eq. 1, for example, L is now the length to the point of diameter measurement, measured downwards from the tip and expressed in percentage of the total length, from the tip to the top of the butt 16-ft log. Corresponding changes in eq. 4 and 7 are self-evident.