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NOTES ON THE MASS DIAGRAM AS APPLIED TO EARTHWORK DISTRIBUTION

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I. FUNDAMENTALS

1. Definition

A mass diagram or mass curve, plotted in connection with a route survey involving excavations and embankments, is a curve having for abscissas the distance along the survey line and for ordinates the algebraic sums of earthwork quantities, from the beginning to each ordinate, considering cut volumes positive and fill volumes negative. At the beginning of the curve the ordinate is zero, and ordinates are calculated continuously from that initial station, which may be selected at any point past which there will be no transportation of excavated material, such as a tunnel, large bridge, or long low fill made entirely from material excavated from side ditches. The ordinate at the starting point may be assumed as 10,000 cu yds or other selected value.

The mass diagram is used to determine proper distribution of excavated material, to determine amount and location of waste and borrow, and as a basis for an estimate of cost of borrow and of haul of excavated material. The cost of excavating the wasted material is included with other costs of excavation.

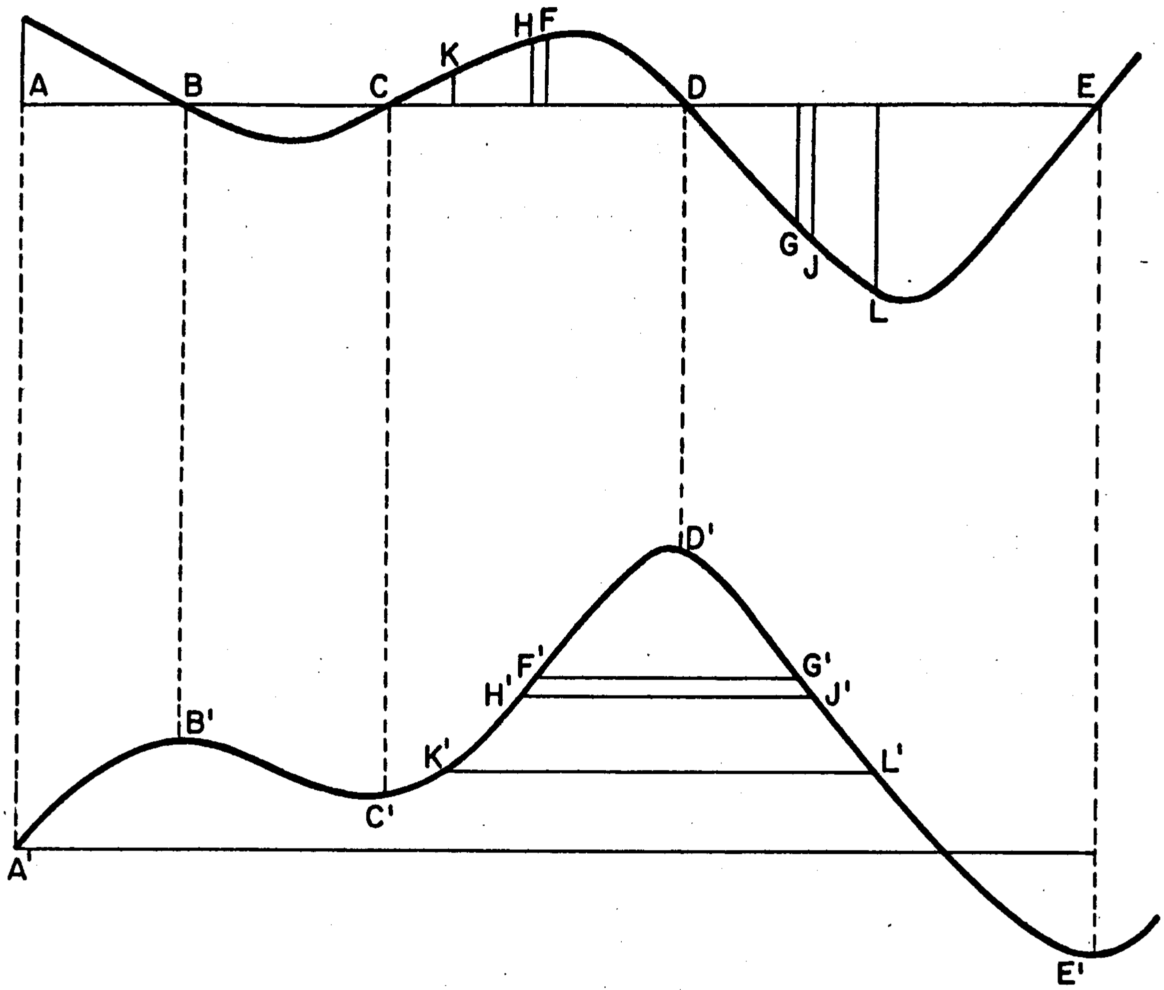

Fig. 1— Centerline profile and mass diagram of earthwork volumes.

2. Properties of Curve

Some of the properties of the mass curve and the relationship between that curve and the profile are illustrated in Fig. 1.

a. An upward slope on the mass curve indicates excavation (cut), the slope of the curve varying with the volume of excavation per linear foot along the line.

b. A downward slope on the mass curve indicates embankment (fill), the slope, as before, varying with volume per linear foot.

c. A maximum point on the mass curve, like B' or D', indicates a grade point, with change from cut to fill.

d. Similarly a minimum point on the mass curve shows a grade point and change from fill to cut.

e. A horizontal line intersecting the mass curve at two points, like the line F'G', shows a balance of cut and fill volumes between the two points of intersection, since the difference in ordinate, F' to D', is just equal to the difference in ordinate D' to G'. Transferring these points to the corresponding positions (stations) on the profile, F and G, the cut FD makes the fill DG. The same is true for the line H'J', closely adjacent to F'G', and volumes HD and DJ.

f. By difference, the small cut volume HF makes the fill GJ. Similarly, quantities intercepted between <u>any</u> two horizontal lines, like F'G' and K'L', balance.

g. The area between a horizontal line F'G' and the curve F'D'G' is a measure of total haul (volume x distance) necessary to move the cut volume FD to make the fill DG. The area between the two horizontal lines F'G' and K'L', is a measure of the total haul (volume x distance) necessary to move excavated material from KF to fill GL.

3. Areas

Since abscissas represent distance and ordinates represent volumes, areas on the curve in all cases represent <u>haul</u>, which is the product of volume by distance.

4. Free Haul and Overhaul

In 1933 the following specifications for overhaul were adopted by the American Railway Engineering Association:

"Unless otherwise specified, the contract price per cubic yard covers all haul which may be necessary. When an allowance for overhaul is provided for in the contract, it shall be handled as follows:

"The overhaul shall be calculated in the unit of one cubic yard, measured in excavation, moved one hundred feet.

"No overhaul shall accrue until the material must be transported a distance exceeding the free haul distance as provided in the contract, and on such material overhaul accures as follows:

"All material hauled ABC feet or less shall be paid for at the flat rate of X cents per yard. Overhaul of Y cents per yard per each one hundred (100) feet hauled over the ABC feet of free haul— will be paid in addition to the flat rate as provided above ----." (<u>Proceedings</u>, American Railway Engineering Association, Vol. 33, p. 315.)

These specifications have since been shortened, but the longer form is retained here for clearness. See also <u>Standard Specifications of the California Division of Highways</u>, Sec. 12 (cc), April 1945, for a different form of statement.

To establish the two points fixing the limits of free haul the procedure is to fix on the mass diagram a line like F'G', Fig. 1, which will intersect the curve at two points which are (to scale) as far apart as the distance between free haul limits. In Fig. 2, if AB represents to scale the limiting distance for free haul, the material excavated between A and C may be deposited in fill between C and B without special payment to the contractor for transportation of material. The same is true if AB is any distance less than the limiting distance for free haul. Such a balancing line may be placed within each loop of the curve, if so desired.

5. Limiting Distance for Economic Haul

This section is based on material in <u>Proceedings</u>, American Railway Engineering Association, Vol. 7, pp. 374-375.

Material excavated from roadway cuts but not used for forming embankments is termed waste. Material needed for formation of embankments, secured not from roadway excavation but elsewhere, is termed borrow, and is said to be obtained from borrow pit. The construction contract names a price per cubic yard to be paid for excavation of each class of material (earth, loose rock, solid rock, etc.) in roadway cuts and provides also prices per cubic yard for borrow. Also, a price per cubic yard per 100-ft station for transportation of excavated material beyond limit of free haul is provided.

The cost of waste and borrow may be increased by the necessity of hauling the excavated material, or may perhaps be increased by other considerations, such as cost of necessary additional land. If all such items are included, then suppose:

A = cost of excavating and wasting 1 cu yd
B = cost of borrowing 1 cu yd
C = cost of excavating 1 cu yd for hauling to embankment, included any free haul
D = cost of overhaul per sta yd
x = length of pay overhaul distance
x_1 = maximum economical overhaul distance
y = limiting distance for free haul

Then

$$C + x_1 D = A + B$$

$$x_1 = \frac{A + B - C}{D}$$

and

$$(x_1 + y) = \text{maximum limit of economical haul}$$

In case $A = C$ then $x_1 = \frac{B}{D}$ as assumed below.

For example, using the contract form quoted above,and assuming that waste or borrow may be accomplished within maximum free haul distance, if the price for borrow is 50¢ per cu yd, the price for overhaul 2¢ per station yard, and the free haul limiting distance 400 ft; then 50/2 + 4 = 29 stations or 2900 ft is the maximum economical haul distance and 2500 ft is the maximum economical pay overhaul distance. Beyond that limit it is cheaper to waste the excavated material and borrow to make the embankment, assuming that waste and borrow are possible within limits of free haul. The maximum economical pay overhaul distance plus the limiting free haul distance equals the limiting distance for economical haul, in the example above equal to 2900 ft. For any mass of material that is hauled beyond the limiting free haul distance, the pay overhaul distance equals the total average haul distance minus the maximum free haul distance.

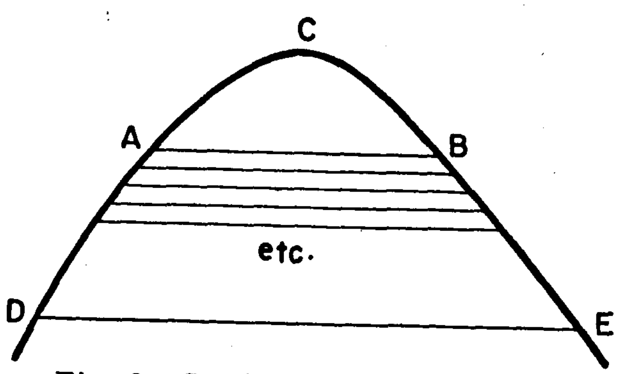

Fig. 2— Single loop of mass curve.

If Fig. 2 represents any single isolated loop of the mass curve (that is, represents the situation at a grade point having a long cut on one side and a long fill on the other), AB equals, to scale, the limiting distance for free haul, and DE represents a distance not greater than the limiting distance for economical haul, the material excavated between D and A will make the fill between B and E, but all of it will be transported over a longer distance than the limiting distance for free haul. The contractor is to be paid for all hauling of material in excess of the limiting distance for free haul; such hauling is termed overhaul, and the pay quantity is expressed in station yards. The yardage is the difference in ordinate between line DE and line AB. The average total haul distance is the distance between center of gravity of the mass excavated between D and A and the center of gravity of the fill constructed between B and E.

6. Actual Average Haul Distance

If a series of closely spaced horizontal lines be drawn between AB and DE, the curved lines DA and DE may be assumed to be straight between any two adjacent horizontal lines. This is equivalent to assuming that the area ABEDA is the sum of a large number of small trapezoids. If these are of equal altitude, the mean difference of abscissas between curve DA and curve BE is the average horizontal length of all the trapezoids. This is conveniently found by measuring the area ABEDA by planimeter and dividing by the altitude or difference of ordinates DE and AB. The resulting horizontal distance is the measure of the distance between centers of gravity (actually centers of volume) of the masses of excavation and embankment involved. This distance reduced by the limiting free haul distance is the required average pay overhaul distance.

7. Swell and Shrinkage

To this point it has tacitly been assumed that one yard of excavation will in all cases make one yard of embankment. That is rarely true. Ordinary earth makes a smaller volume of settled embankment than volume in place before excavation, and is said to shrink about 9% or 10%. Rock, after excavation, occupies a larger volume than it did in original place, and is said to swell, 20% or 30%, or some other amount; as a result, the fill volumes or the cut volumes or both must be modified by application of a shrinkage factor or swell factor before the true volumes can properly be balanced one against the other.

In case the excavated material is all of one character, it is possible to make the balance by increasing or decreasing the fill quantities before computing mass diagram ordinates. The resulting ordinates represent cut volumes and volumes of excavation necessary to build the fills. Increasing fills 10% allows for about 9% shrinkage, assuming all material to be ordinary earth.

If the excavated material is of several kinds the procedure is to apply the proper factor to each volume excavated and use the actual fills. In this case the ordinates represent volumes in fills and volumes from cuts available for filling.

Sometimes both cuts and fills are adjusted, swell factors being applied to cut volumes, and fill volumes adjusted for shrinkage of ordinary earth. The ordinates then represent the amount of earth excavation necessary to make the fills. The swell factors used in this procedure do not equal those used when cut volumes only are adjusted and the meaning of the ordinate is modified.

8. Ordinates

From the cut and fill volumes, adjusted by one of the plans described above, ordinates for the mass diagram are computed from the beginning of the diagram, cut volumes being considered positive and fill volumes negative. The ordinate at any station is the algebraic sum of volumes to that station.

At a grade point the cut and fill volumes may terminate in wedges, the line of zero cut and zero fill crossing the center line at right angles. In such a case there is only cut on one side of the grade point and only fill on the other side of the grade point. All material hauled from excavation to embankment passes the grade point, and appears on the mass diagram.

In general the line of no cut and no fill does not cross the center line at right angles, and "side hill" work results, with both cut and fill volumes between adjacent cross sections. In such a case the cut and fill are algebraically added, and the algebraic sum is the increment applied to the mass curve ordinate. The cut and fill volumes, balanced out and not appearing at all in the mass diagram, are entirely within free haul limits and no payment for overhaul becomes due to the contractor. But such excavation is included in the excavation paid for and must so appear in the estimate prepared by the engineer.

9. Plotting

By the use of the ordinates prepared as just described, the mass curve is plotted to the same distance scale, generally, as the profile and to as large a volume scale as may be convenient considering the maximum positive and maximum negative ordinates. The curve is sometimes shown as a series of straight lines but is preferably drawn as a smooth curve. At each maximum and each minimum point on the curve the computed ordinate is written for later use.

II. BALANCING PROCEDURES

10. General

Given the mass diagram, plotted in accordance with the plan outlined above. How is it to be used? What is the specific problem to be solved by its use?

In general the answer is that by a series of trials, using horizontal balancing lines in a variety of positions, it is possible to determine:

a. The earthwork distribution plan that will result in the minimum cost for overhaul plus borrow.
b. The economical expenditure for overhaul.
c. The economical expenditure for borrow.

The details of procedure in a few specific cases will now be discussed.

11. Single Large Loop

Consider again Fig. 2. This figure represents the situation at a change from a cut to a fill, both being long, so that at the left there will certainly be waste of excavated material and at the right there will certainly be borrow to make the fill. In this case the procedure is relatively simple.

a. Using the proper scale, place on the curve the horizontal balancing line AB using in length to the limit of free haul. Between the station at which A falls and the station at which B falls, the excavation quantity equals the embankment quantity, and furthermore there is no separate payment for hauling the excavated material. The ordinate of the line AB so fixed on the diagram is then determined graphically and written upon the line for future use. It is to be remembered that the diagram already shows definitely the computed ordinate at the maximum point. (See Section 9, last sentence.)

b. Compute the limiting distance for economic haul. (See Section 5, above.) Call this distance $x_1 + y$. Place the line DE on the curve, equal in length, to scale, to the quantity $x_1 + y$. Determine graphically the ordinate of the line DE and write it on that line. The points D and E then fix the stations limiting the overhaul, and to the left of D the material excavated will be wasted while to the right of E the fill will be made from borrow. The excavated earth and rock obtained between D and A is to be hauled to the right to make the fill between B and E. The actual average haul distance is determined as explained in Section 6; this distance reduced by the maximum free haul distance is the required pay overhaul distance for the material excavated between D and A. The number of cubic yards of free haul is the difference between the ordinate at C and the ordinate at the line AB. The number of cubic yards of material overhauled is the difference between the ordinate at the line AB and ordinate at the line DE. This last yardage is multiplied by the pay overhaul distance expressed in stations to find the number of station yards of overhaul.

The plan of work is the same in case of a minimum point on the curve, except that the haul is to the left instead of to the right. Should the located line be composed of long cuts and long fills, with waste from the central part of each cut and borrow for the central part of each fill, the balanced mass curve will show only balancing of this character.

Fig. 3— Mass curves with pair of loops.

12. One Pair of Loops

Given a mass curve with a pair of loops, as shown in Fig. 3a and 3b, in the one case with borrow at each end, and in the other case with waste at each end. Under these particular conditions raising or lowering the continuous balancing line ABC neither increases nor decreases the amount of waste or borrow, but merely changes the amount of haul. The problem therefore reduces to finding the position of that balancing line which will give a minimum total haul and that condition is obtained when the sum of the two areas between curve and balancing line is a minimum. This is accomplished when the balancing line is so placed that the two intercepts AB and BC are of equal length provided that each intercept, AB and BC, is in length not less than the limiting distance for free haul and not greater than the limiting distance for economical haul.

The truth of this rule is demonstrated by drawing another horizontal line closely adjacent to the one with equal intercepts and comparing the areas between such a line and the loops for the curve with those resulting when the intercepts are equal. In every case it appears that the least area occurs when the two intercepts on the balancing line equal one another.

The free haul limiting lines are not shown on Fig. 3 but will in general lie between the lines AB and BC and the loops of the curve.

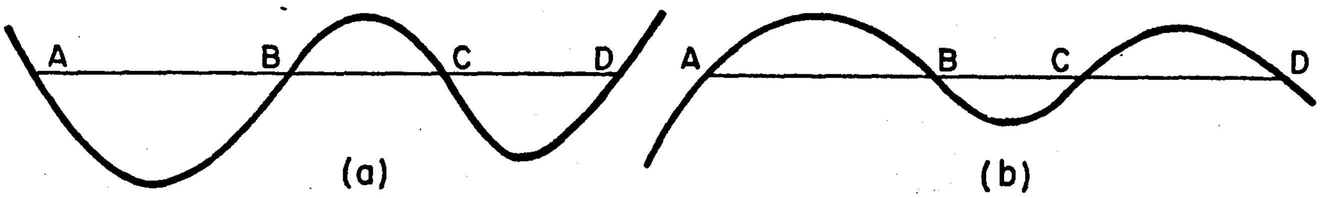

Fig. 4— Mass curves with three loops.

13. Triple Loop

For a series of three loops of the character shown in Fig. 4a and 4b with waste at one end and borrow at the other end, the condition of least cost is obtained if the sum of the two intercepts AB and CD is equal to the intercept BC plus the limiting distance for economical haul, provided no one of the three distances AB, BC, or CD is less than the limiting distance for free haul nor greater than the limiting distance for economical haul. As in the previous case, the free haul limiting lines will be shown in addition to the line AD of Fig. 4.

The truth of the rule here stated is demonstrated by comparing areas and waste and borrow with those obtained by the use of lines slightly above and slightly below those resulting from the use of the line satisfying the rule. Under certain conditions of lengths of the intercepts, the balancing line should be placed tangent to the curve, between B and C.

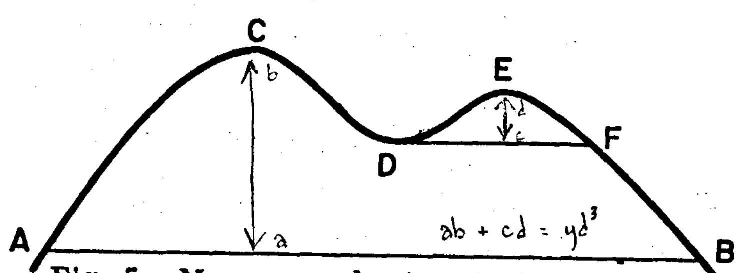

Fig. 5— Mass curve having single loop within free haul limit.

14. Small Loops Within Free Haul Limit

Fig. 5 illustrates a case that is not uncommon; the line AB represents the limiting distance for free haul. There is a maximum point at C, a minimum point at D, and a maximum point at E. The total yardage of free haul is equal to the difference of ordinates AB and C plus the difference of ordinates DF and E. This is demonstrated by use of the horizontal balancing line DF where the difference of ordinate between that and the point E indicates yardage additional to the yardage for a simple loop such as is shown in Fig. 2.

15. Other Cases

A number of other special cases might be listed, but the ones here shown are perhaps the most common ones. For more complicated cases estimates should be prepared for a variety of balancing lines, the most economical position of the balancing lines in each part of the problem being found by trial.

Such trial is conveniently made by using a narrow horizontal strip representing a unit volume, and for that unit volume comparing the costs of the different available possibilities: hauling to the right, hauling to the left, waste and borrow. A test so made will show whether the balancing line should be placed above or below the test strip. In general the problem is to place the balancing lines in such positions as will reduce the sum of overhaul cost and borrow cost to a minimum. This is the general solution of the problem.

16. Horizontal Strip Method

In this method test strips one cubic yard wide across the mass curve are used to find the best position of the balancing line. Along the strip distances are scaled between intersections with the curve. Then, for the area within the strip only, costs are estimated for the different possible alternative solutions. Such solutions involve hauls, wastes, or borrows, perhaps both waste and borrow. Comparison of those alternative costs shows whether the balancing line should be placed above or below that test strip.

Then the same kind of test is made with a test strip at a different ordinate. When, by trial, a strip is found for which the costs of the alternative solutions are equal, the best position for the balancing line has been found.

To illustrate this method by a comparatively simple example, assume cost of all roadway excavation, including waste, and also cost of borrow, to be 50 cents per cu yd, maximum free-haul distance 500 ft, and cost of overhaul 2 cents per station yd. Then the maximum economical haul distance is (50 ÷ 2) + 5 stations or 30 stations or 3,000 ft. As long as these prices apply, no single cubic yard of excavated earth will be hauled more than 3,000 ft because waste and borrow will be cheaper than a longer haul.

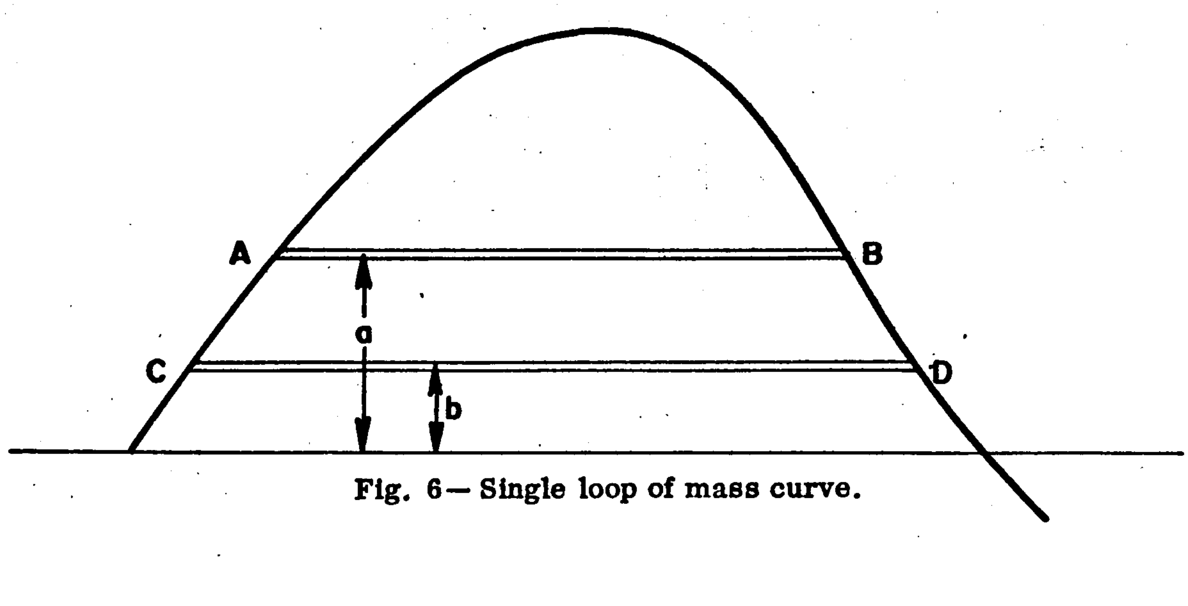

Fig. 6— Single loop of mass curve.

Now consider a single loop like the one shown in Fig. 6. The test strip AB, one cu yd wide, is at ordinate (a) selected for trial. If distance AB is 2,600 ft by scale, the haul of one cu yd from A to B costs 2(26-5) = 42 cents; the excavation at A costs 50 cents, and the total cost within the strip is 92 cents. The waste at A and borrow at B costs 100 cents. So overhaul is cheaper and the balancing line should be placed at some ordinate less than (a).

The test strip CD at ordinate (b) is found by scaling to be 3,300 ft, C to D. Cost of overhaul, C to D, is 2(33-5) = 56 cents and the excavations at C costs 50 cents or a total of 106 cents. But waste at C plus borrow at D costs only 100 cents. So the balancing line should be somewhere above CD.

By trial a strip is found for which cost of excavation and overhaul is equal to cost of waste and borrow, and the balancing line is placed there. The length of that strip across the loop is of course 3,000 feet.

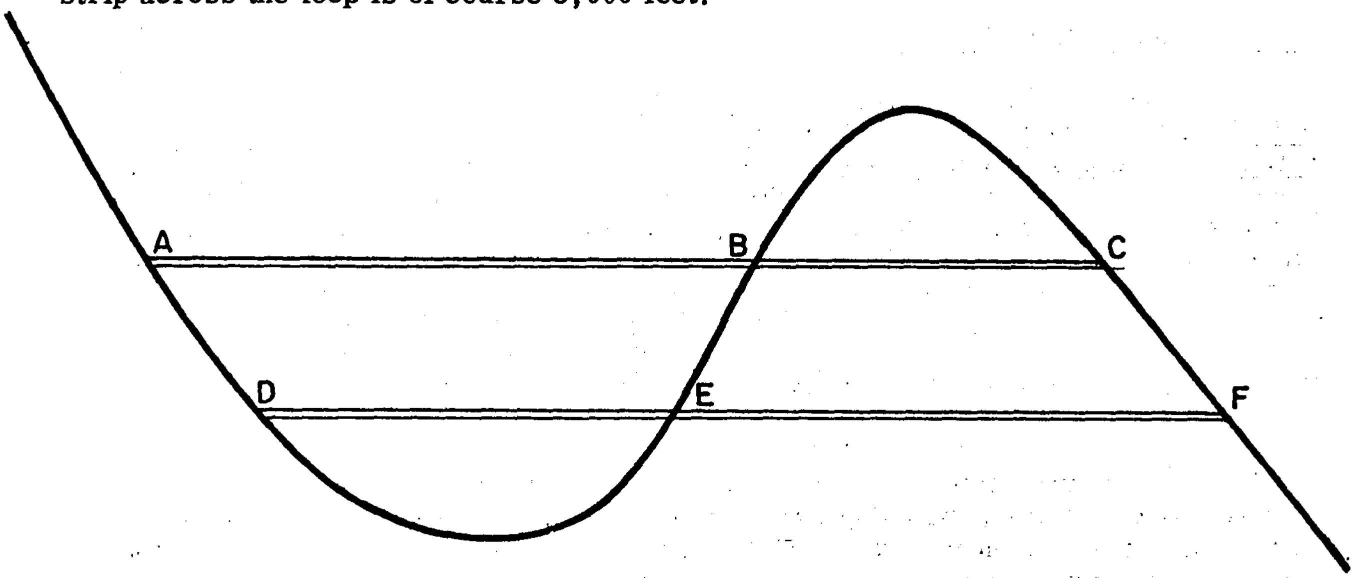

Fig. 7— Mass curve with pair of loops.

Consider now the situation with the two loops shown on this figure, and use the same unit costs as before.

In the test strip ABC, say that AB scales 2,500 ft and BC is 1,800 ft. The two alternatives are:

1.	Excavate at B	50 cents
	Haul to A	40 cents
	Borrow at C	50 cents
		140 cents
2.	Borrow at A	50 cents
	Excavate at	50 cents
	Haul to C	26 cents
		126 cents

The second alternative is the cheaper, so the balancing line should be somewhere below ABC.

For the test strip DEF, say DE is 1,900 ft to scale and EF is 2,400 ft. The two alternatives are:

1.	Excavate at E	50 cents
	Haul to D	28 cents
	Borrow at F	50 cents
		128 cents

2.	Excavate at E	50 cents
	Borrow at D	50 cents
	Haul E to F	38 cents
		138 cents

Since haul to the left is cheaper than haul to the right, the balancing line should be above the strip DEF.

At some certain ordinate between ABC and DEF the two alternatives will be equal in cost, and the best position for the balancing line has been found. Obviously this is where the two distances are equal. Remember that there should be no haul for a distance longer than the calculated maximum economical haul distance.

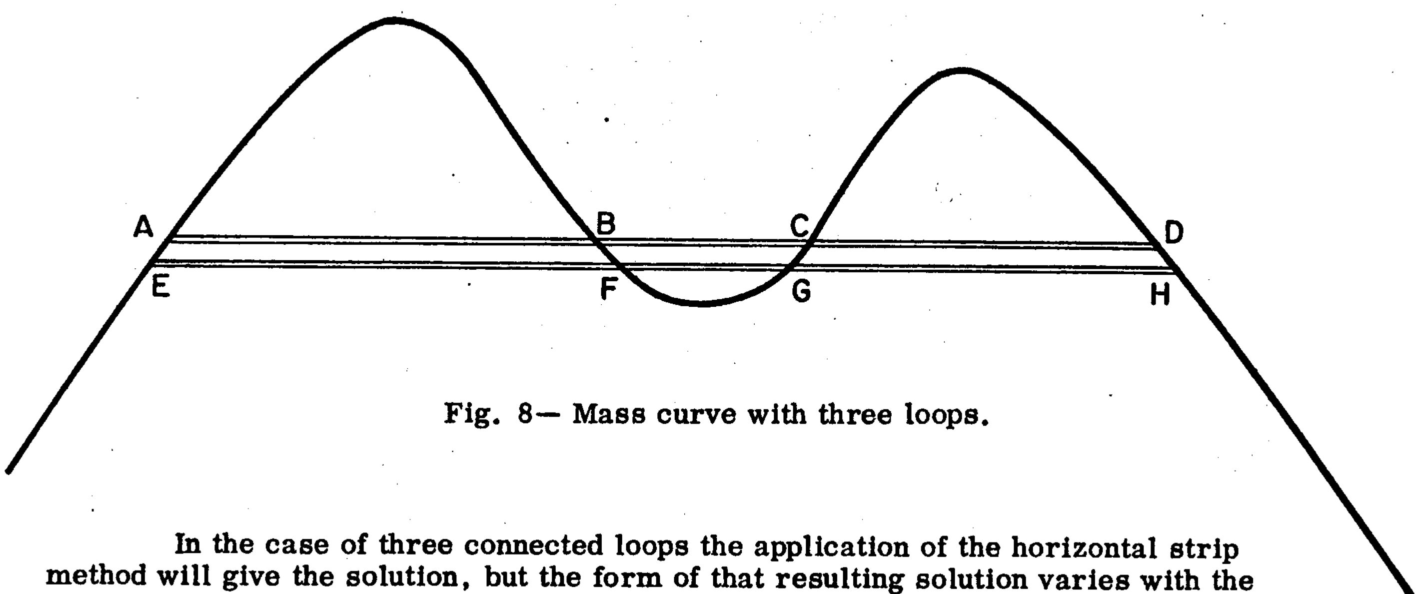

Fig. 8— Mass curve with three loops.

In the case of three connected loops the application of the horizontal strip method will give the solution, but the form of that resulting solution varies with the shape and dimensions of the curve. This will now be illustrated.

Consider the test strip ABCD and use the same unit prices as before. Suppose that by use of a scale the distances are found to be: AB 1,800 ft, BC 700 ft, CD 1,400 ft. Then, the alternatives are:

1.	Excavate at A	50 cents
	Haul A to B	26 cents
	Excavate at C	50 cents
	Haul C to D	18 cents
		144 cents

2.	Excavate and waste at A	50 cents
	Excavate at C	50 cents
	Haul C to B	4 cents
	Borrow at D	50 cents
		154 cents

The first alternative is the cheaper, so the balancing line should be lower than the strip ABCD.

By trying several test strips, say that finally the strip EFGH show EF = 2,000 ft, FG = 600 ft, and GH = 1,600 ft. Here the two alternative costs are equal at 152 cents each, and here the balancing line should be placed.

The general rule here illustrated may be stated as follows: Place the balancing line so that the sum of the overhaul distances in the first and third loops is equal to the sum of the overhaul distance in the middle loop and the maximum distance of economical overhaul (in this case 2,500 ft), all provided no single intercept exceeds the maximum limit of economical haul distance and there is waste at one end and borrow at the other.

The dimensions may not permit the solution just described. Suppose that each trial strip shows that the balancing line should be further lowered, even to tangency to middle loop, and that neither of the side intercepts exceeds maximum distance for economical haul. In that case, if the specifications permit, the balancing lines may be lowered to tangency.

In the case of a balancing line tangent to the middle loop, the middle intercept is zero, and if the sum of the other two intercepts is less than the maximum distance for economical haul, another overhaul item may be used providing for haul all the way across the three loops.

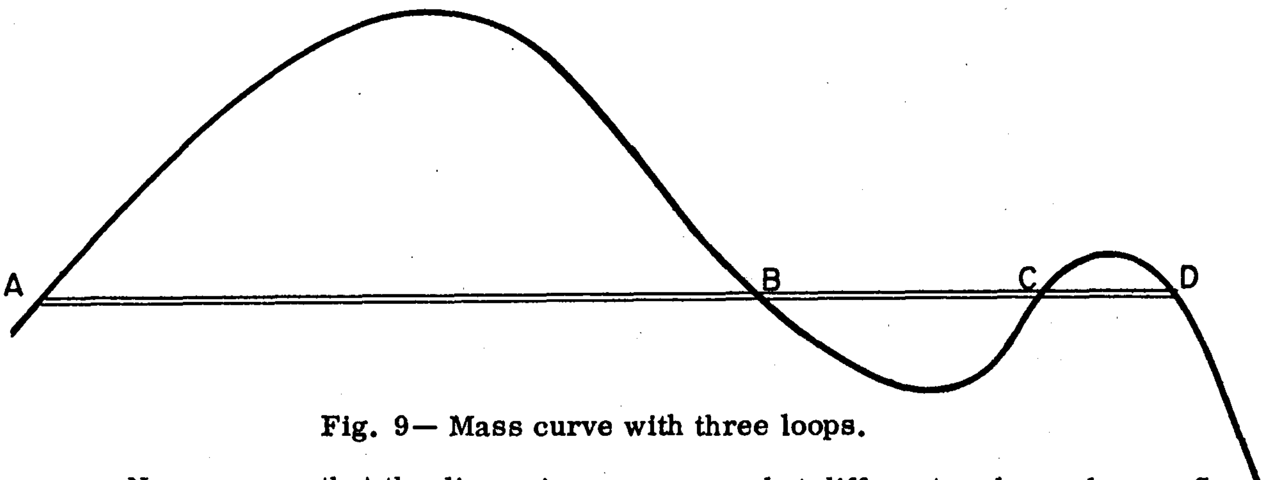

Fig. 9— Mass curve with three loops.

Now suppose that the dimensions are somewhat different as here shown. Say that AB = 3,000 ft, BC = 1,200 ft, and CD = 600 feet. Then the alternative costs are:

1.	Excavate at A	50 cents
	Haul A to B	50 cents
	Excavate at C	50 cents
	Haul C to D	2 cents
		152 cents

2.	Excavate and waste at A	50 cents
	Excavate at C	50 cents
	Haul C to B	10 cents
	Borrow at D	50 cents
		160 cents

This would seem to call for a balancing line lower than strip ABCD, but that would involve an uneconomically long haul across the first loop. So the balancing line across that first loop would be placed at AB, 3,000 ft long, the line BCD would be lowered to where its two intercepts are equal, and there would be some borrow immediately beyond B.

In the cases that have been discussed here it has been assumed that the section of the curve being considered is separated by waste or borrow from adjacent parts of the curve.

In a particular case unit cost of waste may not be equal to cost of roadway excavation for hauling, and cost of borrow may be still different. These various unit costs are taken account of in estimating the costs of the different possible solutions. This matter has been discussed in connection with the calculation of limiting distance for economical haul, and what has been stated there about costs of waste and borrow applies here also.

The situations used here to illustrate the use of the horizontal strip method lend themselves to solution by simple rules. Their inclusion here is to illustrate this method, which can be used for solution of more difficult cases.

17. Arithmetical Checks

There are several arithmetical checks that should be applied to the mass curve ordinates and to the quantities calculated from the curve. The statement of these checks here given assumes that the checks will be made upon volumes of excavation and embankment, adjusted for swell and shrinkage, as may be required by the conditions of the case, using any one of the several possible methods of making that adjustment.

The most important checks are the following:

a. The final ordinate is equal to the difference between total cut yardage and total fill yardage.
b. The final ordinate is equal to the difference between total yardage of waste and total yardage of borrow.
c. Making allowance for side hill volumes not appearing on the mass curve, the difference in ordinate between two adjacent grade points is the volume of excavation or embankment between any two grade points. (A similar check may be made between any two points on the curve.)
d. Again with allowance for side hill work, the sum of the volumes of material to be taken from any one cut or to be placed in any one fill is equal to the total volume of such cut or fill.

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Proceedings, American Railway Engineering Association, Vol. 7, pp. 374-375; Vol. 33, p. 315.

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