## ESS 411/511 Problem Set 2

Mase, Smelser, and Mase, 3rd Edition: Problems 2.4, 2.5, 2.7, 2.10, 2.13, 2.16 or Mase and Mase, 2nd Edition: Problems 2.4, 2.5, 2.7, 2.9, 2.11, 2.14

## 1. (MSM Problem 2.4)

Using the notations $\left.\mathrm{A}_{(\mathrm{ij}}\right)=1 / 2\left(\mathrm{~A}_{\mathrm{ij}}+\mathrm{A}_{\mathrm{ji}}\right)$ and $\mathrm{A}_{[\mathrm{ij}]}=1 / 2\left(\mathrm{~A}_{\mathrm{ij}}-\mathrm{A}_{\mathrm{ji}}\right)$ show that
(a) the tensor A having components $\mathrm{A}_{\mathrm{ij}}$ can always be decomposed into a sum of its symmetric $\mathrm{A}_{(\mathrm{ij})}$ and skew-symmetric $\mathrm{A}_{[\mathrm{ij}]}$ parts, respectively, by the decomposition,

$$
\mathrm{A}_{\mathrm{ij}}=\mathrm{A}(\mathrm{ij})+\mathrm{A}[\mathrm{ij}],
$$

(b) the trace of A is expressed in terms of $\mathrm{A}_{(\mathrm{ij})}$ by

$$
\mathrm{A}_{\mathrm{ii}}=\mathrm{A}(\mathrm{ii}),
$$

(c) for arbitrary tensors A and B,

$$
\mathrm{A}_{\mathrm{ij}} \mathrm{~B}_{\mathrm{ij}}=\mathrm{A}_{(\mathrm{ij})} \mathrm{B}_{(\mathrm{ij})}+\mathrm{A}_{[\mathrm{ij}]} \mathrm{B}_{[\mathrm{ij}]}
$$

Problem 2.4c goes faster if you use the fact that the product of a symmetric and an antisymmetric tensor is zero, which was proved in MSM Chapter 2 . Be aware of the perfectly legitimate "trick" used at the bottom of pg. 17 of MSM, $3^{\text {rd }}$ ed. (top of page 15 of MM, 2 nd ed.).

## 2. (MSM Problem 2.5)

Expand the following expressions involving Kronecker deltas, and simplify where possible. (a) $\delta_{\mathrm{ij}} \delta_{\mathrm{ij}}$, (b) $\delta_{\mathrm{ij}} \delta_{\mathrm{jk}} \delta_{\mathrm{ki}}$, (c) $\delta_{\mathrm{ij}} \delta_{\mathrm{jk}}$, (d) $\delta_{\mathrm{ij}} \mathrm{A}_{\mathrm{ik}}$

## Answer

(a) 3, (b) 3 , (c) $\delta_{j k}$, (d) $A_{j k}$

## 3. (MSM Problem 2.7)

By summing on the repeated subscripts determine the simplest form of
(a) $\varepsilon_{3 j \mathrm{k}} \mathrm{a}_{j} \mathrm{a}_{\mathrm{k}}$,
, (b) $\varepsilon_{\mathrm{ijk}} \delta_{\mathrm{kj}}$,
(c) $\varepsilon_{1 j \mathrm{k}} \mathrm{a}_{2} \mathrm{~T}_{\mathrm{kj}}$, (d) $\varepsilon_{1 \mathrm{jk}} \delta_{3 \mathrm{j}} \mathrm{v}_{\mathrm{k}}$.

## Answer

(a) 0 , (b) 0 , (c) $\mathrm{a}_{2}\left(\mathrm{~T}_{32}-\mathrm{T}_{23}\right)$, (d) $-\mathrm{v}_{2}$

## 4. (MSM Problem 2.10)

If $A_{i j}=\delta_{i j} B_{k k}+3 B_{i j}$, determine $B_{k k}$ and using that solve for $B_{i j}$ in terms of $A_{i j}$ and its first invariant, $\mathrm{A}_{\mathrm{ii}}$.

## Answer

$$
\mathrm{B}_{\mathrm{kk}}=1 / 6 \mathrm{~A}_{\mathrm{kk}} ; \mathrm{B}_{\mathrm{ij}}=1 / 3 \mathrm{~A}_{\mathrm{ij}}-1 / 18 \delta_{\mathrm{ij}} \mathrm{~A}_{\mathrm{kk}}
$$

## 5. (MSM Problem 2.13)

Show by direct expansion (or otherwise) that the box product $=\varepsilon_{i \mathrm{ijk}} \mathrm{a}_{\mathrm{i}} \mathrm{b}_{\mathrm{j}} \mathrm{c}_{\mathrm{k}}$ is equal to the determinant

$$
\left[\begin{array}{lll}
a_{1} & a_{2} & a_{3} \\
b_{1} & b_{2} & b_{3} \\
c_{1} & c_{2} & c_{3}
\end{array}\right]
$$

Thus, by substituting $A_{1 i}$ for $a_{i}, A_{2 j}$ for $b_{j}$ and $A_{3 k}$ for $c_{k}$, derive

$$
\begin{equation*}
\operatorname{det} \mathrm{A}=\varepsilon_{\mathrm{ijk}} \mathrm{~A}_{\mathrm{i} 1} \mathrm{~A}_{\mathrm{j} 2} \mathrm{~A}_{\mathrm{k} 3}=\varepsilon_{\mathrm{ijk}} \mathrm{~A}_{1 \mathrm{i}} \mathrm{~A}_{2 \mathrm{j}} \mathrm{~A}_{3 \mathrm{k}} \tag{Eq2.42}
\end{equation*}
$$

in the form det $\mathrm{A}=\varepsilon_{\mathrm{ijk}} \mathrm{A}_{1 \mathrm{i}} \mathrm{A}_{2 \mathrm{j}} \mathrm{A}_{3 \mathrm{k}}$ where $\mathrm{A}_{\mathrm{ij}}$ are the elements of $\mathbf{A}$.

## 6. (MSM Problem 2.16)

Show that the square matrices
$\left[B_{i j}\right]=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{array}\right]$ and $\left[C_{i j}\right]=\left[\begin{array}{cc}5 & 2 \\ -12 & -5\end{array}\right]$
are both square roots of the identity matrix.

## 7. Back to Real Science Stuff

A geeky gnome made a spell catcher in the shape of a parallelogram from a couple of vectors $\mathbf{a}$ and $\mathbf{b}$ (in units of wandles, which are wand lengths). Then he held it out to catch some magic streaming by with velocity $\mathbf{c}$ wandles/eon. Note that $\mathbf{c}$ is a vector and not necessarily perpendicular to the spell catcher. If the density of magic is $\rho$ mags/wandle ${ }^{3}$, how much magic did he catch in one eon (a period of about 50 minutes, i.e. a UW class period ())? So, what is the flux of magic through his spell catcher? Give answer in correct magical units. This could also apply to solar collectors or sediment flux in a river... Imagine if any or all of the three vectors were changing in time.

## 8. Are you thirsty yet?

A cement-lined aqueduct brings water to a major Arizona city from a distant valley across mountains and desert. This was an engineering project, so naturally we must start with archaic units. :) Water depth $h$ is 10 feet in a section where the channel width $W$ is 15 feet. Water flows eastward at an average speed of $\mathbf{v}=1.5$ feet per second.

(a) Find the water flux to the city in cfs (cubic feet per second), in gallons per year, and in $\mathrm{m}^{3} \mathrm{~s}^{-1}$.
(b) You probably didn't use the box product in (a), so now find the water flux in $\mathrm{m}^{3} \mathrm{~s}^{-1}$ using the box product. Show all vector quantities and all steps in your calculation. Be sure to get the water flux moving eastward (in the same direction as $\mathbf{v}$ ) and not westward.
(c) Verify that you get the same magnitude for the flux in both (a) and (b).

## 9. How big is it?

Oh no, despite all those parallel sides, the parallelepiped below has no right angles. Quick what's its volume? What if any two of the vectors were collinear?


