Continuum Mechanics ESS 411/511

Problem Set 3

MM = Mase and Mase 2nd Ed. Problems 2.17, 2.18, 2.23, 3.2, 3.4
or MSM = Mase, Smelser, and Mase 3rd Ed. Problems 2.19, 2.20, 2.25, 3.2, 3.4

1. MSM Problem 2.19

A tensor is called *isotropic* if its components have the same set of values in every Cartesian coordinate system at a point. Assume that **T** is an isotropic tensor of rank two with components t_{ij} relative to axes $Ox_1x_2x_3$. Let axes $Ox_1'x_2'x_3'$ be obtained with respect to $Ox_1x_2x_3$ by a right-hand rotation of 120° about the axis along $\hat{\boldsymbol{n}} = (\hat{e}_1 + \hat{e}_2 + \hat{e}_3)/\sqrt{3}$. Show by the transformation between these axes that $t_{11} = t_{22} = t_{33}$, as well as other relationships.

Further, let axes $Ox_1''x_2''x_3''$ be obtained with respect to $Ox_1x_2x_3$ by a right-hand rotation of 90° about x_3. Thus, show by the additional considerations of this transformation that if **T** is any isotropic tensor of second order, it can be written as $\lambda\mathbf{I}$, where λ is a scalar and **I** is the identity tensor.

Note for Problem 2.17 (MM) or 2.19 (MSM): To visualize a 120° rotation about the given vector, consider a physical model of 3 orthogonal basis vectors (e.g., perpendicular rods 1 unit long) face down on a table. Think about rotating the model by 120° about the vertical (relative to the table). Where do the axes end up? This may help you write down the matrix for the rotation. For Problem 2.19 in MSM (the 3rd edition), note the missing subscripts in $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) / \mathrm{sqrt}(3)$.

2. Problem 2.20 (MSM)

For a proper orthogonal transformation between axes $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$ show the invariance of δ_{ij} and ε_{ijk}. That is, show that

(a) $\delta'_{ij} = \delta_{ij}$, (b) $\varepsilon'_{ijk} = \varepsilon_{ijk}$.

Hint: This problem shows the power of indicial notation. In part (b), the third-rank tensor ε_{ijk} needs to be multiplied by 3 transformation matrices (see eq 2.5-14 in M&M, or 2.65 in MSM), but unlike in the rank-two case, there is no clear way to do this by writing out 3× 3 matrices, as was done in the solution to Example 2.5-1 (M&M) or 2.13 (MSM). Indicial notation allows one to do this.

For part (b) let $\varepsilon'_{ijk} = a_{iq}a_{jm}a_{kn}\varepsilon_{qmn}$ *and make use of Eq 2.4-12 (M&M), or Eq 2.43 (MSM).* $\varepsilon_{qmn} \det \mathbf{A} = \varepsilon_{ijk}A_{iq}A_{jm}A_{kn}$

*There are several ways to solve this. In general, they will all use the fact that **A** is proper and orthogonal and $a_{ij} = a^T_{ji}$ (by definition of transpose).*

3. Problem 2.25 (MSM)
Determine the principal values of the matrix

$$[K_{ij}] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 11 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{bmatrix}$$

and show that the principal axes $Ox_1^* x_2^* x_3^*$ are obtained from $Ox_1x_2x_3$ by a rotation of 60° about the x_1 axis.

Answer

$\lambda_1 = 4$, $\lambda_2 = 8$, $\lambda_3 = 12$.

4. Problem 3.2 (MSM)
Verify the result established in Problem 3.1 for the area elements having normals

$$\hat{n}_1 = \frac{1}{7}(2\hat{e}_1 + 3\hat{e}_2 + 6\hat{e}_3)$$

$$\hat{n}_2 = \frac{1}{7}(3\hat{e}_1 - 6\hat{e}_2 + 2\hat{e}_3)$$

if the stress matrix at **P** is given with respect to axes $Px_1x_2x_3$ by

$$[\sigma_{ij}] = \begin{bmatrix} 35 & 0 & 21 \\ 0 & 49 & 0 \\ 21 & 0 & 14 \end{bmatrix}$$

FYI Here is Problem 3.1
*At a point **P**, the stress tensor relative to axes $Px_1x_2x_3$ has components t_{ij}. On the area element $dS^{(1)}$ having the unit normal $\hat{n}_1$, the stress vector $t^{(\hat{n}_1)}$, and on area element $dS^{(2)}$ with normal $\hat{n}_2$ the stress vector is $t^{(\hat{n}_2)}$. Show that the component of $t^{(\hat{n}_1)}$ in the direction of $\hat{n}_2$ is equal to the component of $t^{(\hat{n}_2)}$ in the direction of $\hat{n}_1$.*

5. Problem 3.4 (MSM)
The stress tensor has components at point P in ksi units as specified by the matrix

$$[\sigma_{ij}] = \begin{bmatrix} -9 & 3 & -6 \\ 3 & 6 & 9 \\ -6 & 9 & -6 \end{bmatrix}$$

Determine:
(a) the stress vector on the plane at **P** whose normal vector is

$$\hat{n} = \frac{1}{9}(\hat{e}_1 + 4\hat{e}_2 + 8\hat{e}_3)$$

(b) the magnitude of this stress vector,
(c) the component of the stress vector in the direction of the normal,
(d) the angle in degrees between the stress vector and the normal.

Answer

(a) $t^{(\hat{n})} = (-5\hat{e}_1 + 11\,\hat{e}_2 - 2\hat{e}_3)$
(b) $t^{(\hat{n})} = \sqrt{150}$
(c) 23/9
(d) 77.96°