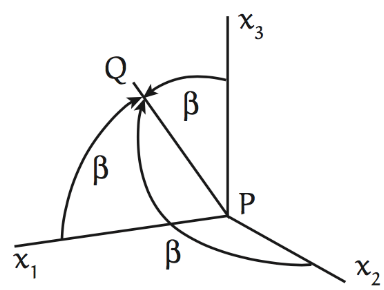
Continuum Mechanics ESS 411/511

Problem Set 4

1. MSM or M&M 3.10(a)

Rotated axes $Px_1'x_2'x_3'$ are obtained from axes $Px_1x_2x_3$ by a right-handed rotation about the line PQ that makes equal angles with respect to the $Px_1x_2x_3$ axes (see sketch). Determine the primed stress components for the stress tensor t_{ij} in (MPa) if the angle of rotation is 120º.

$$[t_{ij}] = \begin{bmatrix} 3 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & -3 \end{bmatrix}$$



Answer (a) $[t'_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 6 \\ 0 & 6 & 3 \end{bmatrix}$

Hint: 3.10(a) has an aspect similar to problem 2.17 (M&M), 2.19, MSM) that was assigned in Problem Set 2.

2. MSM or M&M 3.18

The stress tensor at P is given with respect to $Ox_1x_2x_3$ in matrix form with units of MPa by

$$[t_{ij}] = \begin{bmatrix} 4 & b & b \\ b & 7 & 2 \\ b & 2 & 4 \end{bmatrix}$$

where b is unspecified. If $\sigma_{III} = 3$ MPa and $\sigma_I = 2\,\sigma_{II}$, determine

(a) the principal stress values,

(b) the value of b,

(c) the principal stress direction of σ_{II} .

Answer

(a) σ_I =8MPa, σ_{II} =4MPa, σ_{III} =3MPa, (b) b=0, (c) $\hat{n}^{(2)} = \hat{e}_1$

Hint: It is helpful to use the fact that the trace and determinant of $[\sigma_{ij}]$ are invariant.

3. MSM or M&M 3.23

Sketch the Mohr's circles for the simple states of stress given by

(a) $[t_{ij}] = \begin{bmatrix} \sigma_0 & 0 & \sigma_0 \\ 0 & \sigma_0 & 0 \\ \sigma_0 & 0 & \sigma_0 \end{bmatrix}$ (b) $[t_{ij}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & 2\sigma_0 & 0 \\ 0 & 0 & -\sigma_0 \end{bmatrix}$

and determine the maximum shear stress in each case.

Answer (a) $(\sigma_S)_{max} = \sigma_0$ (b) $(\sigma_S)_{max} = 3/2\ \sigma_0$

Hint:The Mohr's circle construction readily shows the maximum shear stress σ_S. What is the orientation of the plane of maximum shear stress? What is the normal stress σ_N on that plane?

4. MSM or M&M 3.24

Relative to axes $Ox_1x_2x_3$, the state of stress at O is represented by the matrix

$$[t_{ij}] = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that, relative to principal axes $Ox_1^*x_2^*x_3^*$, the stress matrix is

$$[t_{ij}] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and that these axes result from a rotation of 45° about the x_3 axis. Verify these results by Eq 3.9-3 (M&M), or Eq 3.75 (MSM).

$$t'_{11} = \frac{t_{11} + t_{22}}{2} + \frac{t_{11} - t_{22}}{2} \cos 2\theta + t_{12} \sin 2\theta \ , \qquad (3.75a)$$

$$t'_{22} = \frac{t_{11} + t_{22}}{2} - \frac{t_{11} - t_{22}}{2} \cos 2\theta - t_{12} \sin 2\theta \ , \qquad (3.75b)$$

$$t'_{12} = -\frac{t_{11} - t_{22}}{2} \sin 2\theta + t_{12} \cos 2\theta \ . \qquad (3.75c)$$

Hint: Note the applicability of eq 3.76 (MSM) or 3.9-4 (MM). They are somewhat simpler because they involve the principal coordinate system.

5. Poor Lab Design

A very large cylinder of rock is being tested in an uniaxial experiment. It is near-vertical, but it tilts 20° to the north, because of restrictions on ceiling height. 900 MPa of compressive stress is applied along the cylinder axis. The cylinder sits in a cylindrical metal sleeve filled with oil pressurized to 600 MPa.

(a) Write the stress tensor in the principal coordinates (ignore the effect of gravity which is small compared to these applied stresses). What are the eigenvectors? Are they unique? Why or why not? Tip: draw a diagram with x_3^* along cylinder axis and x_1^* to the east.

(b) Find the transformation matrix to transform to the east-north-up coordinate system. This is a rotation of how much about the x_1^* axis? Note order, check to see that all coordinate systems are right-handed.

(c) Write the stress tensor in the east-north-up coordinate system.

(d) Now find the traction vector on a weak fine-grained plane P in the sample that strikes north-south through the sample and dips 30° below horizontal to the east. Thus, it has a complex relationship to the axis of the cylinder. Determine the normal to the plane in a convenient coordinate system (i.e., the coordinate system in which I described it), and compute the traction vector on plane P. Report it in the e-n-up coordinate system.

(e) Report the magnitude of the shear and normal stresses on that weak plane.

(f) Now the whole experiment is moved 300 m northwest to a bigger building and repeated with the sample cylinder upright. Report the magnitudes of the shear and normal stresses on that same fine-grained plane P in this new set-up.

6. Constructing Mohr Circles

You will need graph paper, protractor and compass. Read the problem through first and have it nearby to refer to.

Work through the example in Section 3.8 (M&M, or MSM) with a different unit normal vector: $\mathbf{n} = 1/7\ (3\mathbf{e}_1 + 2\mathbf{e}_2 + 6\mathbf{e}_3)$, which is differs from **n** in the example. Note that the original stress tensor $[\sigma_{ij}]$ is the same, the transformation matrix $[a_{ij}]$ is the same, the diagonalized stress tensor is the same. Only **n** is different.

(a) Obtain σ_N and σ_S as was done in the example. $\mathbf{t}^{(\mathbf{n})} = \mathbf{n}\ \boldsymbol{\sigma} = \boldsymbol{\sigma}\ \mathbf{n}$

(b) Get some graph paper, and a protractor and compass, and draw the Mohr circles (draw large ones) using the σ_I, σ_{II}, and σ_{III} (all the same as before). Plot the σ_I, σ_{II}, and σ_{III} in one color and the centers of the circles in another color. Now compute the angles ϕ, β, and θ between **n** and the principal coordinate directions $x_1^*\ \ x_2^*$, x_3^*. How? Get the components of **n** in principal coordinates using $[a_{ij}]$ as in the example. Those components are the cosines of the angles ϕ, β, and θ between **n** and the principal directions.

(c) Use the angles to plot points on the Mohr circle analogous to *k, d, e*, and *g* in the figures. On the Mohr circle construction, those angles are doubled and measured either clockwise or counter-clockwise from the relevant principal axis, so that they are on the circle corresponding to the plane in which the angle is measured; e.g., physical point K lies in the x_1^* , x_3^* plane, so *k* lies on the largest circle, the one running between σ_I and σ_{III}. And on that largest Mohr circle *k* lies at the point which is 2ϕ from σ_I . Thus you measure counter-clockwise from σ_I for that case. Physical point D lies in the x_1*-x_2* plane at angle ϕ from axis x_1* associated with σ_I. So *d* lies 2ϕ from σ_I on the Mohr circle running between σ_I and σ_{II}, again counter-clockwise.

The stress state on the surface corresponding to **n** must lie somewhere on a new circle that runs between *k* and *d*, and is larger than, but parallel to, the x_2^* - x_3^* circle. Carefully draw the *k-d* and *e-g* circles around the appropriate circle centers^. There is a third circle that is parallel and larger than the σ_I - σ_{II} circle, it should also intersect at the same point, but is redundant for our purpose. The intersection at *q* of these "concentric circles" (*k-d, e-g* and the other one) will give the stress state (σ_N, σ_S) on the plane normal to **n**. Estimate it from the plot and compare it to the analytic result that you got in part a).

The figures are better (though not perfect) in the newer third edition.

Suggestion: Don't try to do this in the middle of the night. Coffee or tea might have a helpful effect.

7. Mohr circle for stresses on a fault surface and its nodal plane

The shear and normal stress measured within a drill hole on a fault surface are found to be σ_S = 175 MPa and σ_N = -260 MPa (i.e., compressional normal stress). The trend of the fault plane is N45°E and the fault plane is dipping 25°SE. Based on regional observations of faulting, it is assumed that the principal direction $\hat{e}_1^*$ associated with the principal stress σ_1 is oriented vertically and the principal direction $\hat{e}_3^*$, which is associated with σ_3, is oriented horizontally along N45°W. Use the engineering convention in Mase and Mase ($\sigma_1 > \sigma_2 > \sigma_3$ and positive σ_i is tensional). Under that convention, by definition which is the most compressive stress?

(a) Draw a map view schematic of the fault in geographic coordinates, indicating the orientations of all 3 principal stress directions $\hat{e}_1^*$, $\hat{e}_2^*$, $\hat{e}_3^*$. Indicate the location of the cross section for part (b). Label north and east, angles, fault plane, principal stress directions, etc. What type of faulting is occurring - thrust, normal, or strike-slip?

(b) Draw a cross section through the fault surface, indicating the orientations of all three principal stresses, and the dipping fault plane.

(c) Add the normal to the fault plane to the cross section in b) (make sure to follow the right hand rule for drawing the normal – fingers point down the dip of the fault, thumb points along strike, normal comes out of back of hand). Indicate the angle between $\hat{e}_1^*$, and the normal to the fault plane. Label axes, angles, fault plane, normal, etc. Make note of the angle between $\hat{e}_2^*$, and the normal.

(d) To determine the magnitudes of the principal stresses σ_1 and σ_3, construct the Mohr circle between σ_1 and σ_3. Hint: you know a point on the circle (what point?) and you determined the angle θ between the fault-plane normal and $\hat{e}_1^*$, from the cross section in part c). Roughly sketch a right triangle in Mohr space containing the point on the circle, the center of the circle (even though you don't know it exactly) and the point (σ_N, 0). Compute the radius of the Mohr circle by writing trigonometric relationships between some of the following: 2θ, σ_N, σ_S, the radius, and the center of the Mohr circle. Then compute the center of the Mohr circle. Finally, compute σ_1 and σ_3, and draw the Mohr circle carefully. Label the relevant quantities.

(e) Use the Mohr circle to determine the normal and shear stress on the plane that 1) is perpendicular to the fault and 2) contains $\mathbf{e}_2$*. This is the nodal plane to the fault plane, which we will discuss further when we talk about earthquake focal mechanisms ("beachballs"). Plot and label this point on the Mohr circle. Is it more or less likely to slip than the original fault plane, given a typical frictional sliding line?

(f) Assume that the mean stress is -350 MPa. What is the differential stress for this stress state? What is the deviatoric stress tensor? What information do you have about σ_2?