Rates of change in a continuum

When the material is being tracked through time, it is convenient to use two sets of coordinates

Material

The material coordinates X_A are the initial positions of a material particle X in a coordinate system I_A

- Although particle X may move over time, the place X_A where it started from doesn't ever change.
- The coordinates X_A act as a ***label*** identifying particle X, wherever it goes.

Spatial

- The spatial coordinates $x_i(X,t)$ mark the current position of a material particle X in a coordinate system $\hat{e}_i$
  - Conversely, $X(x_i,t)$ indicates which particle X is occupying location x_i at time t.

Temporal Derivatives

As we saw with the traffic on I-5, there are two types of temporal derivatives of some quantity ϕ in a continuum.

- Rate of change of any property $\phi(x_i, t)$ at a fixed point x_i in space, can be written as

$$\frac{\partial \phi(x_i, t)}{\partial t} \qquad (1)$$

The partial derivative symbol ∂ indicates that position x_i is held constant.

- Rate of change of $\phi(X_A, t)$ for a particle X_A in the moving material, can be written as

$$\frac{D\phi(X_A, t)}{Dt} \text{ or } \frac{d\phi(X_A, t)}{dt} \qquad (2)$$

where "*D*" or "*d*" indicate a "total" or "material-following" derivative.

The identity *A* of a particle isn't changing through time

(Calvin and Hobbs transmogrification isn't allowed),

So (2) is a function of a single variable *t,* and

$$\frac{d\phi(X_A, t)}{dt} = \frac{\partial \phi(X_A, t)}{\partial t}$$

Material Derivatives

In the material coordinate system I_A, rate of change of ϕ for particle X_A as it moves along its trajectory is relatively simple:

$$\frac{d\phi(X_A,t)}{dt} = \frac{\partial\phi(X_A,t)}{\partial t}$$

However, it gets uglier if we want to express the material-following derivative in the fixed coordinate system $\hat{e}_i$

The rate of change of ϕ for the particle currently at x_i as it moves along its trajectory depends on two things:

1. The rate of change of ϕ seen by an observer at position x_i

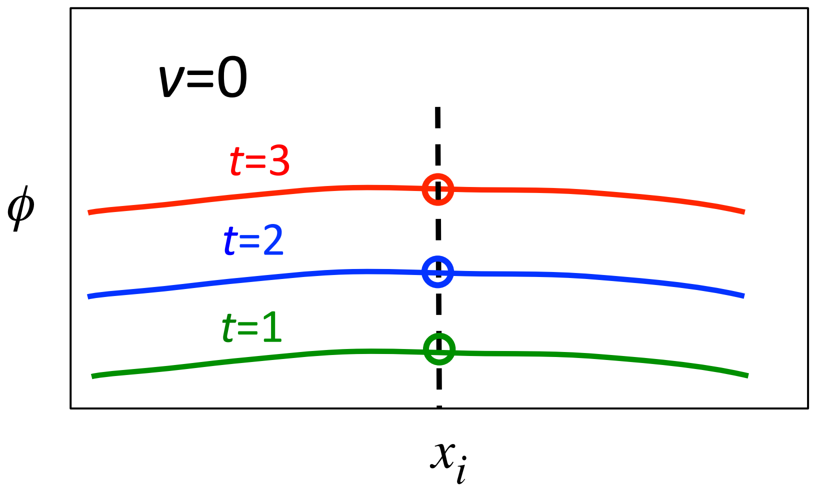
$$\frac{\partial\phi(x_i,t)}{\partial t}$$

2. The rate of change at which the flow *v* carries gradients of ϕ past position x_i, even though ϕ may not be changing on the particles

$$-\frac{\partial\phi(x_i,t)}{\partial x_k}\frac{\partial x_k}{\partial t}, \quad \frac{\partial x_k}{\partial t} = v_k$$

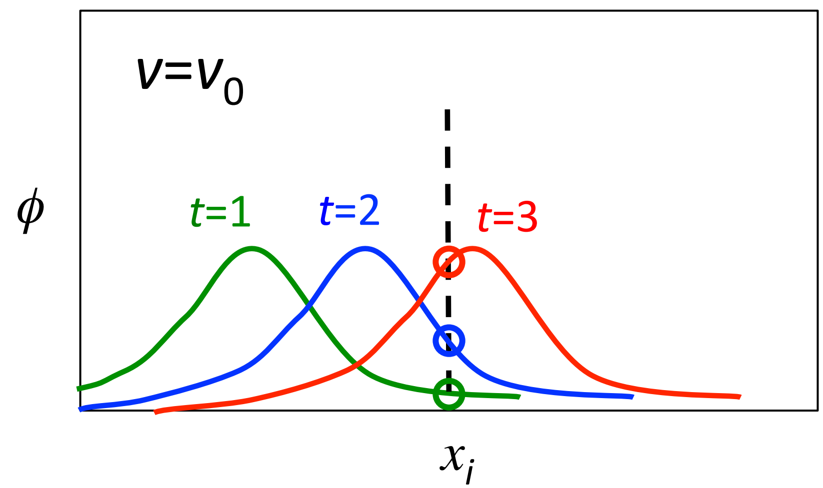
Ways to change ϕ at a point x_i

No motion
e.g. material warming in place



$$\frac{\partial \phi(X_A,t)}{\partial t} = \frac{\partial \phi(x_i,t)}{\partial t}$$

Motion uniform and constant
e.g. a seamount carried by ocean plate



$$\frac{\partial \phi(x_i,t)}{\partial t} = -\frac{\partial \phi(x_i,t)}{\partial x_k}\frac{\partial x_k}{\partial t}, \quad \frac{\partial x_k}{\partial t} = v_k = v_0$$

Putting it all together

However, it gets uglier if we want to express the material-following derivative in the fixed coordinate system $\hat{e}_i$

In the spatial coordinate system $\hat{e}_i$, rate of change of ϕ for a particle X_A as it passes through x_i :

$$\frac{d\phi(x_i,t)}{dt} = \frac{\partial\phi(x_i,t)}{\partial t} + \frac{\partial\phi(x_i,t)}{\partial x_k}\frac{\partial x_k}{\partial t}$$

Change seen at x_i

Correction for changes carried in by flow without ϕ changing on the particles