ESS 411/511 Geophysical Continuum Mechanics Class #4

Highlights from Class #3 – Alysa Fintel
Today's highlights on Friday – Anna Ledeczi

Warm-up question (break-out) –

- What is a creep function?
- What is a relaxation function?

Announcements

- When I set up the sign-up schedule for class highlights, I hadn't properly accounted for the November 11 Veteran's Day holiday. It should be correct now. If you had signed up to highlight a class around Nov 11 or later, please check that the revised schedule still works for you. Thanks.
- On the Canvas home page, I have added
  - a link to a calendar showing daily topics
  - links to FILES folders where you can find the slides from past lectures, and your Highlights reports from those lectures.
- Homework #1 is live on Canvas. Please check it out and bring questions to our HW Lab on Thursday.

ESS 411/511 Geophysical Continuum Mechanics Class #3

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

ESS 411/511 Geophysical Continuum Mechanics Class #4

Class-prep answers (break-out)

This was the assignment:

Energy and dissipation

If you manually compress a perfect spring and then release it back to its initial state (a strain cycle), no energy gets converted into heat, but if you manually drive a perfect dash-pot through a similar strain cycle, some energy is always converted to heat.

Without resorting to any equations, explain to a nonscientific family member why this has to be true, based your efforts expended in the two experiments.

Share with partners your explanations of energy dissipation (or not) in cyclical springs and dash-pots.

ESS 411/511 Geophysical Continuum Mechanics Class #4

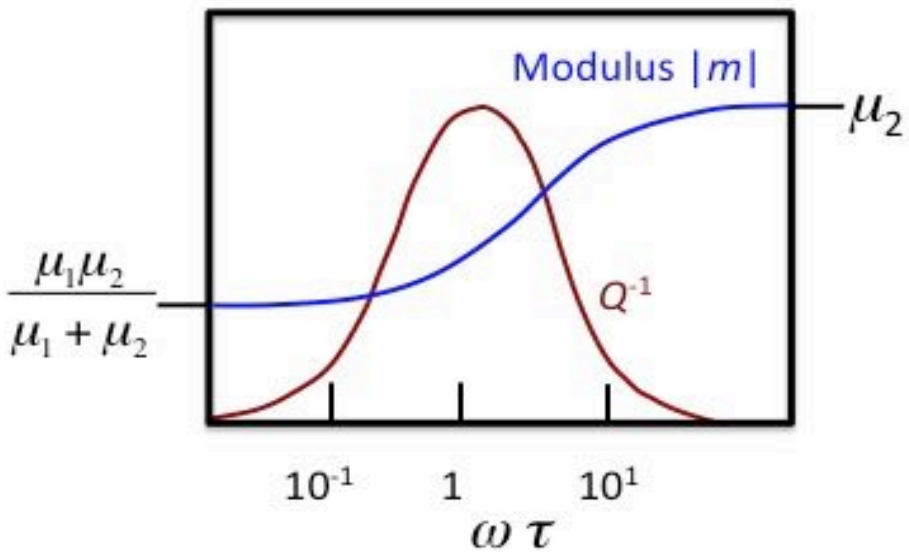
For Friday (class #5)

Please review the section in Raymond Ch 2, Section 2.9 on harmonic loading, and start reading Mase, Smelser, and Mase, Ch.2, through Section 2.3.

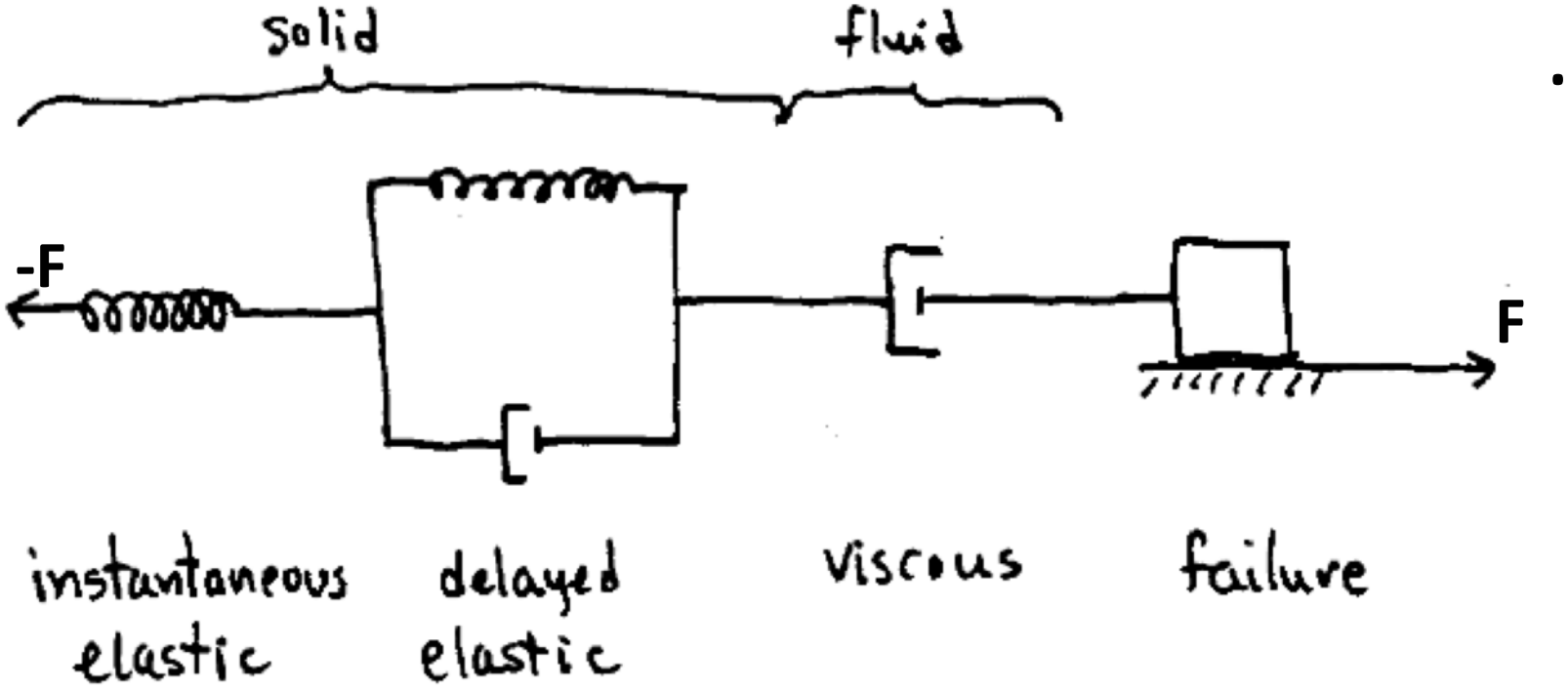
Your class-prep assignment on frequency-dependent attenuation

When a viscoelastic material is loaded harmonically at a frequency ω, the attenuation (Q^{-1}) of that sinusoidal signal depends on the relation between ω and a characteristic time τ of the material.

- With reference to the Standard Linear Solid model, please explain in words why energy loss (attenuation Q^{-1}) is minimal at high and low frequencies, but is high at mid-range.



A model for idealized real materials



Forces are balanced

- Each element feels the same force **F**

Rheological tests

Creep tests

- Apply a constant stress σ
  e.g. put a weight on top of a sample
- Measure strain *e(t)* or strain rate $\dot{e}(t)$

Relaxation tests

- Apply an abrupt strain *e,* then hold it constant
  e.g. abrupt shortening in a vice.
- Measure stress $\sigma(t)$ as sample adjusts.

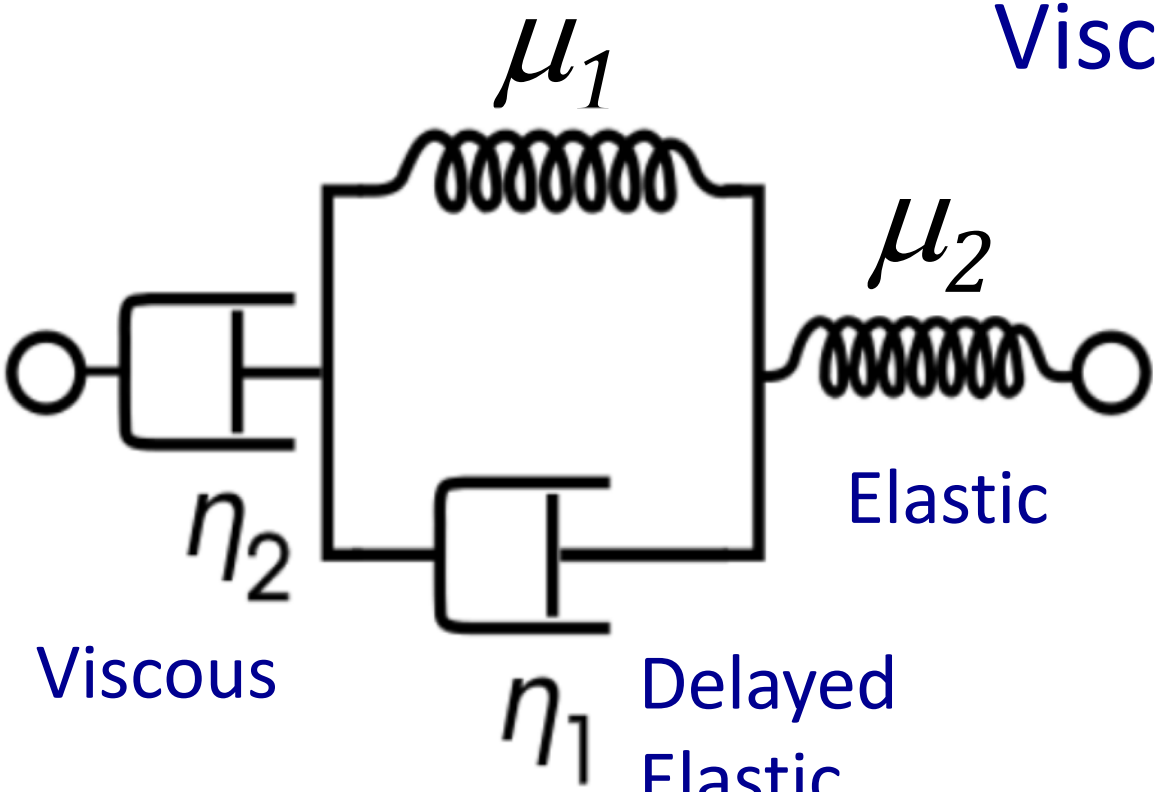
Constant strain-rate tests

- Apply a constant strain rate
  e.g. with a motor-driven vice
- Measure stress $\sigma(t)$

Models for linear solids

Those springs and dashpots …

Viscoelastic model

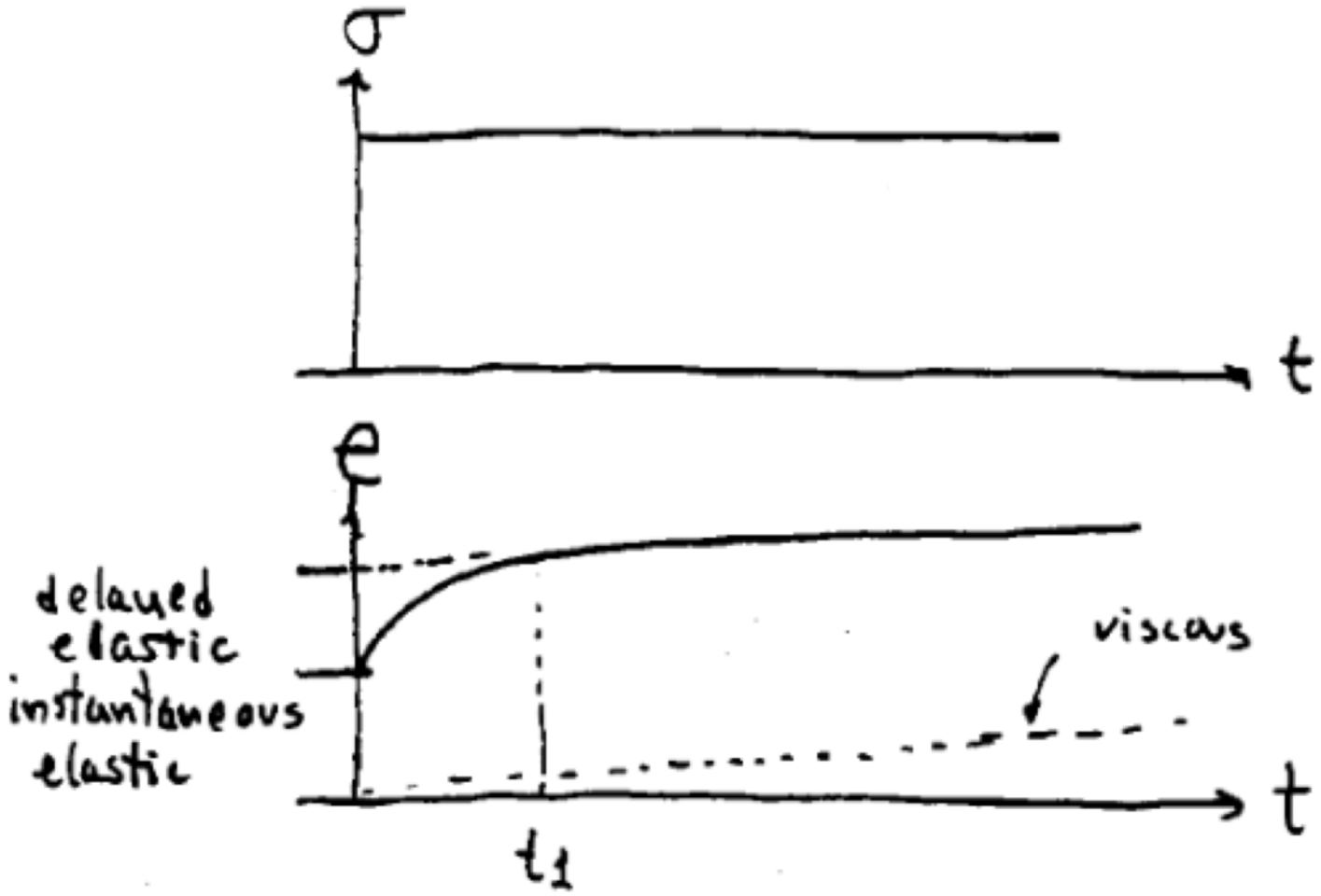


Called *Maxwell Solid*, if $\eta_1 = \infty$, $\mu_1 = \infty$

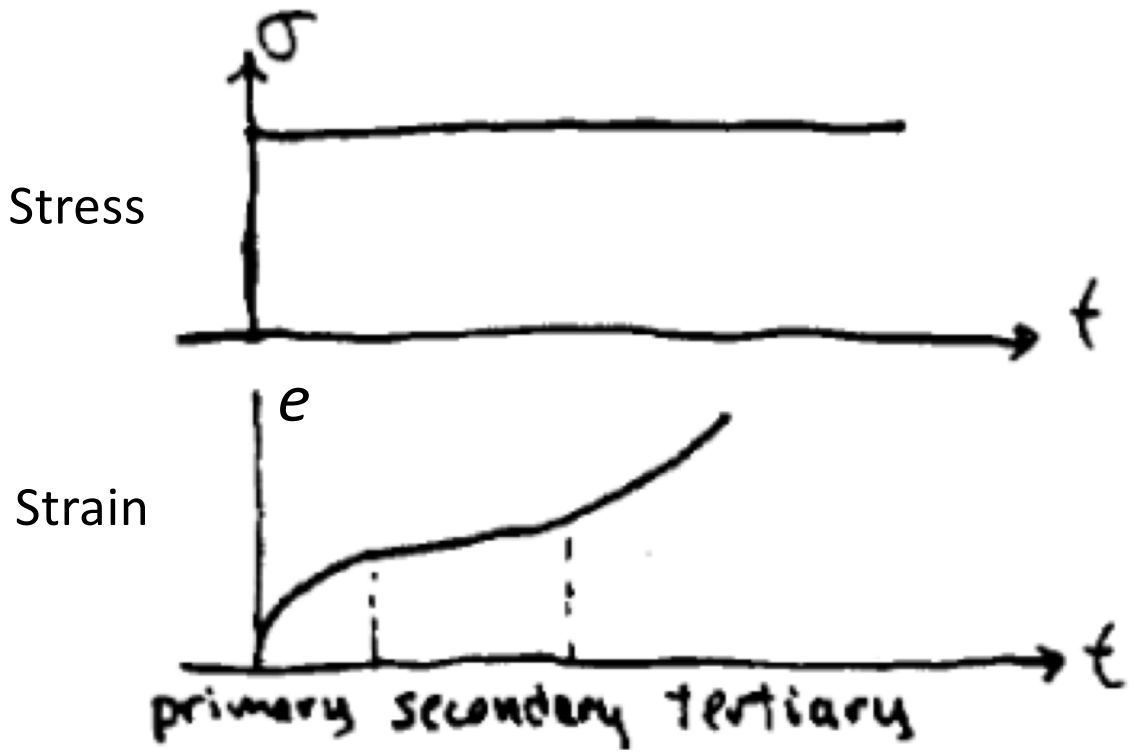
Called *Kelvin-Voigt Solid*, if $\eta_2 = \infty$, $\mu_2 = \infty$

Called *Standard Linear Solid*, if $\eta_2 = \infty$

Creep Test with Viscoelastic model



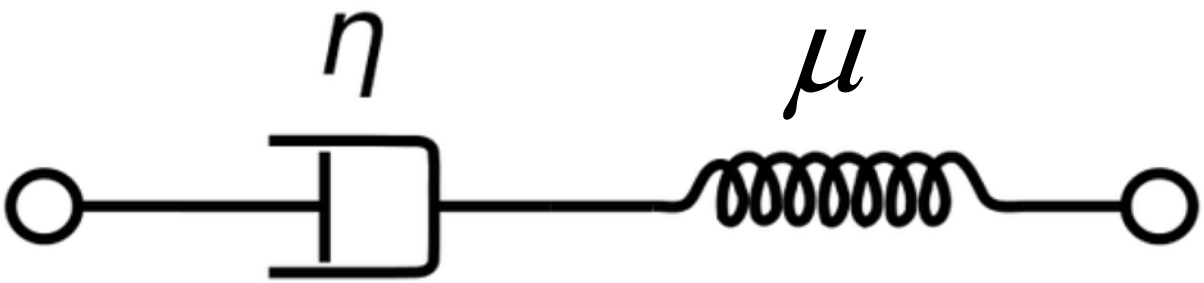
Viscoelastic behavior in real materials

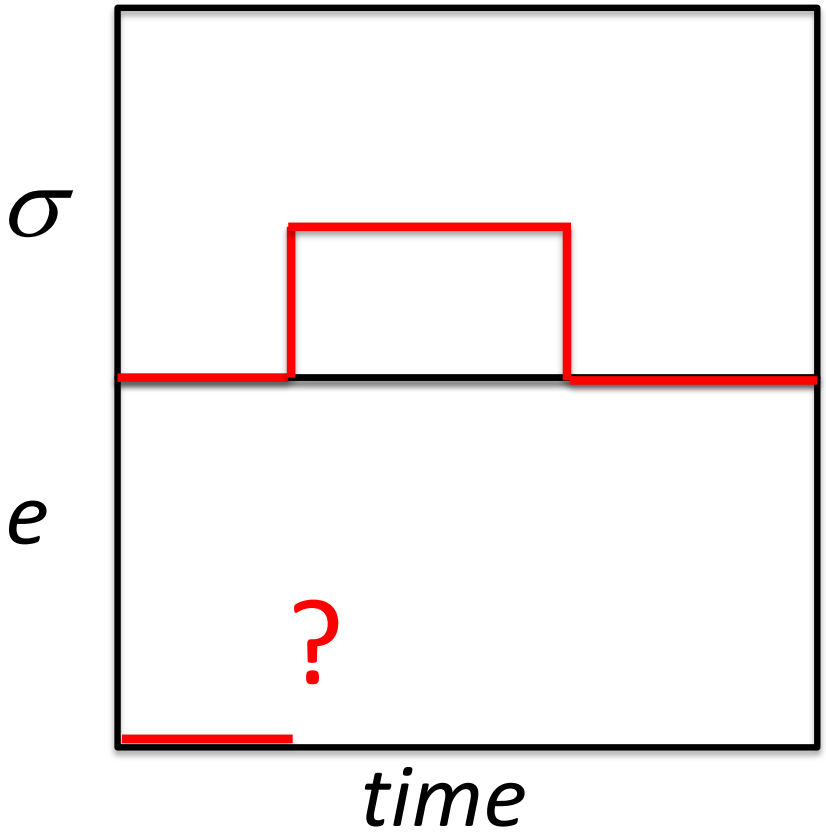


Changes in the microstructure at the crystal level inside the sample can alter the effective viscosity after significant strain as the test progresses.

- e.g. crystal basal planes align for easy glide
- microcracks may develop, allowing internal slip

Maxwell solid

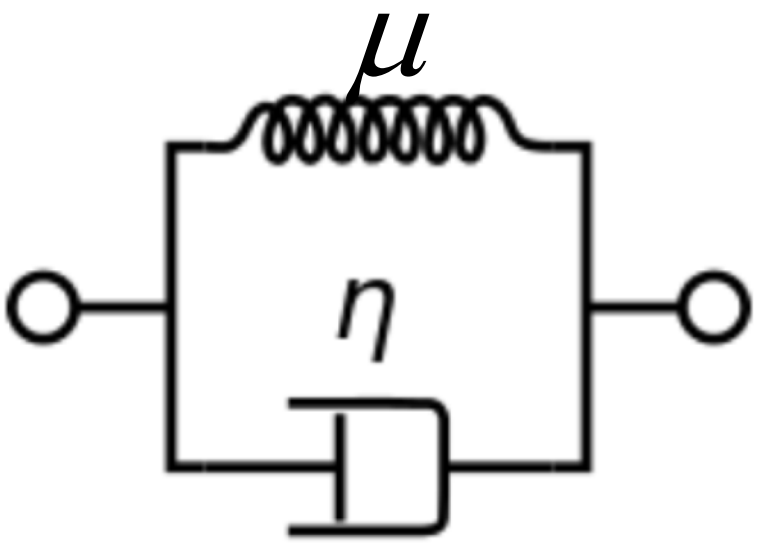


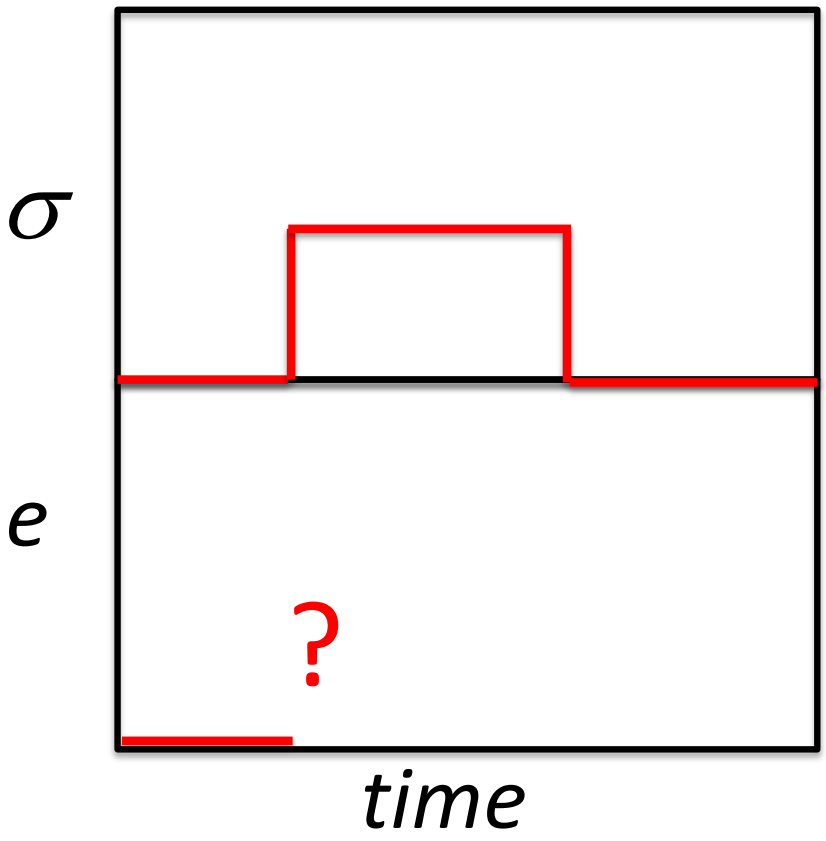


Also Homework set #1

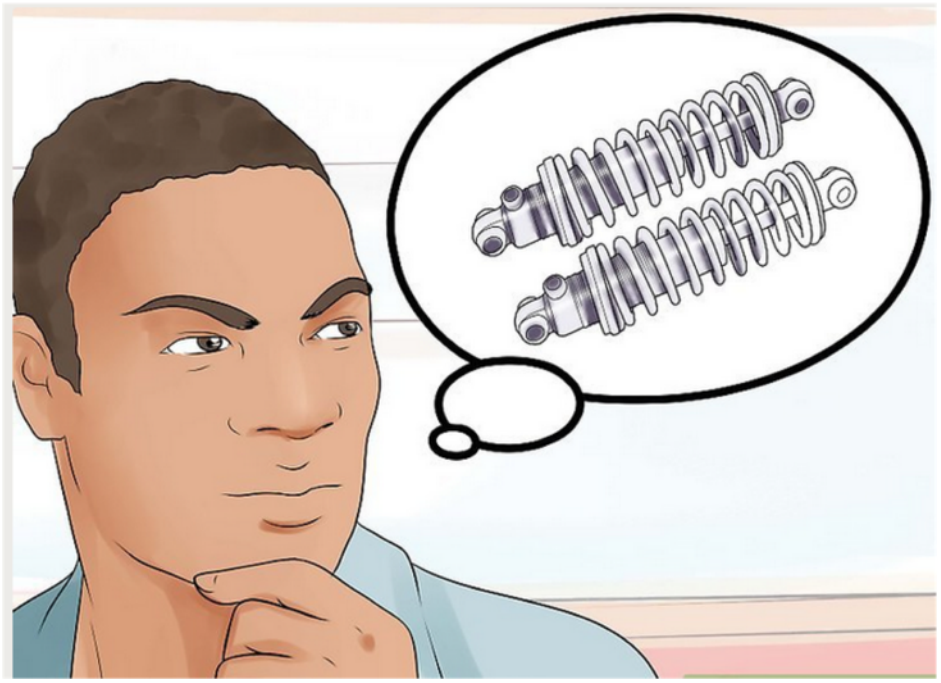
- Is there a characteristic time for the material?
- η/μ ?

Kelvin-Voigt solid

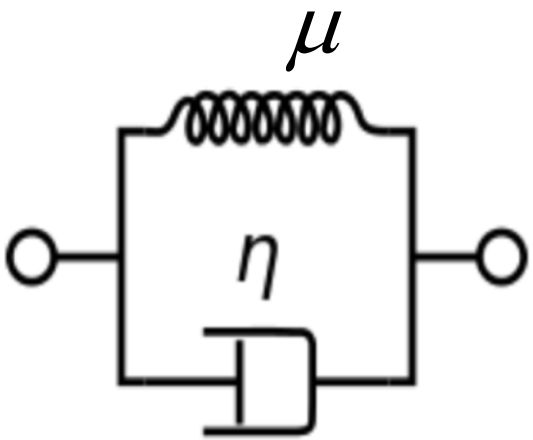




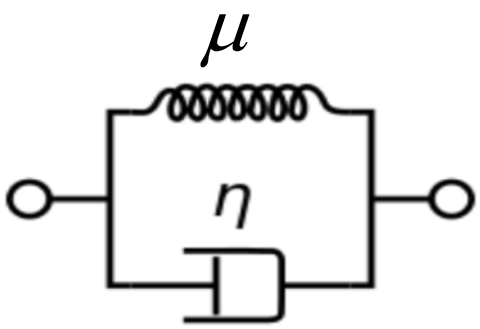
- Is there a characteristic time for the material?
- η/μ ?



A shock absorber can be modeled as a delayed elasticity Kelvin-Voigt solid.



Kelvin-Voigt Response



Spring and dashpot together support stress σ

$$\sigma(t) = \mu\ e(t) + \eta\ \dot{e}(t)$$

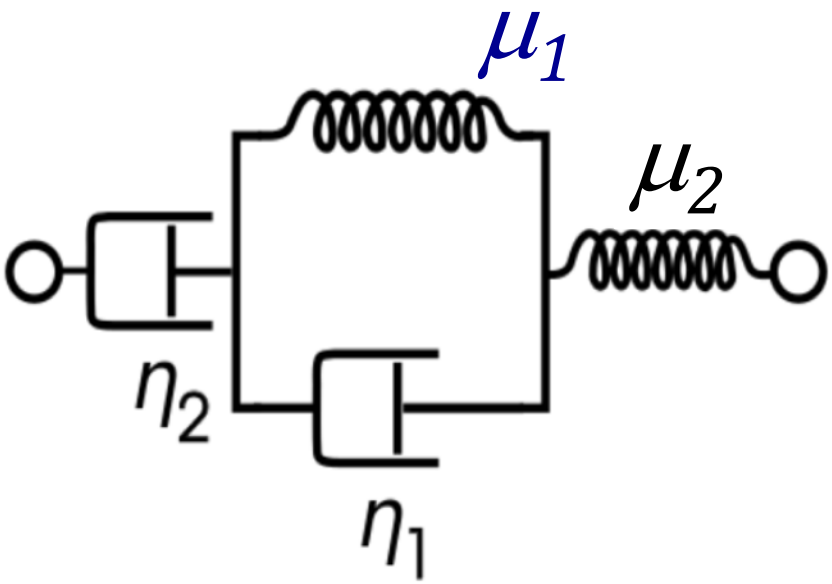
- At *t = 0*, spring hasn't shortened; dashpot supports all the stress σ, so
  $e(0) = 0 \quad (*)$
- At *t* = ∞, dashpot has stopped; spring supports all the stress σ, so
  $e(\infty) = \sigma/\mu\ (**)$
- The transition is probably a decaying exponential. (Let's try it …)
- $\tau = \eta/\mu$ must be the time constant defining the transition.

$$e(t) = \frac{\sigma}{\mu} + \mathrm{A}\exp\left(-\frac{\eta}{\mu}t\right)$$

With the boundary conditions (*) and (**), A can be found, and solution is …

$$e(t) = \frac{\sigma}{\mu}\left[1 - \exp\left(-\frac{\mu}{\eta}t\right)\right]$$

Generalized linear viscoelastic solid



Response to constant loading σ

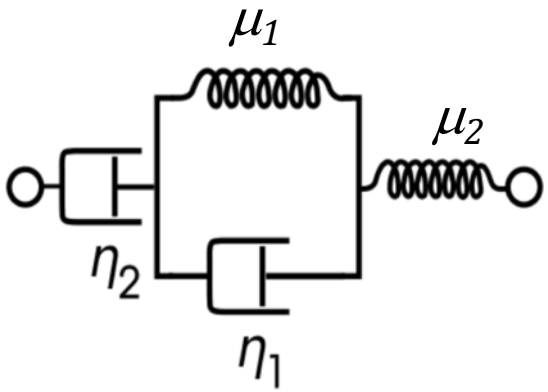
$$e(t) = \frac{\sigma}{\eta_2}t + \frac{\sigma}{\mu_2} + \frac{\sigma}{\mu_1}\left[1 - \exp\left(-\frac{\mu_1}{\eta_1}t\right)\right]$$

Viscous

Elastic

Delayed Elastic

How did we get that?!



$$e(t) = \frac{\sigma}{\eta_2}t + \frac{\sigma}{\mu_2} + \frac{\sigma}{\mu_1}\left[1 - \exp\left(-\frac{\mu_1}{\eta_1}t\right)\right]$$

Viscous Elastic Delayed Elastic

Each element feels the same stress σ,

- We just added up the strains in each element

The Raymond notes also give creep functions and relaxation functions for step changes in stress or strain

Creep test

$$\sigma(t) = \begin{cases} 0 & t=0 \\ 1 & t \geq 0 \end{cases} \qquad \sigma'(t) = \delta(t)$$

Applied stress σ

$$e(t) = \int_0^t C(t-t')\delta(t')dt' = C(t)$$

$C(t\text{-}t')$ is the creep function

Relaxation test

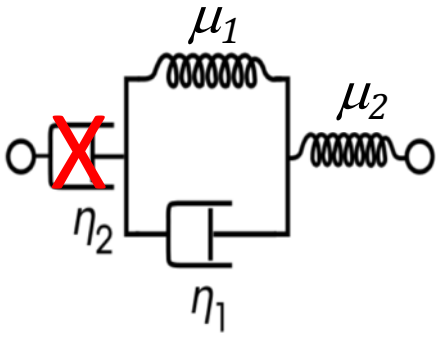
$$e(t) = \begin{cases} 0 & t=0 \\ 1 & t \geq 0 \end{cases} \qquad e'(t) = \delta(t)$$

Applied strain σ

$$\sigma(t) = \int_0^t k(t-t')\delta(t')dt' = k(t)$$

$k(t\text{-}t')$ is the relaxation function

Relaxation Function in Standard Linear Solid ($\eta_2 = \infty$)



e = constant (after t=0)

$$\sigma(t) = e\frac{\mu_1\mu_2}{\mu_1 + \mu_2}\left\{1 + \frac{\mu_2}{\mu_1}\exp\left(-\frac{\mu_1 + \mu_2}{\eta_1}t\right)\right\} \quad (*)$$

At t=0 :

- The spring μ_1 in the K-V element is prevented from deforming, due to η_1.
- All applied strain e is taken up initially in the spring μ_2. So $\sigma(0) = \mu_2 e$
  (Do you agree that (*) shows this?)
- Stress $\sigma(0)$ also acts on the K-V element, so it also begins to strain.

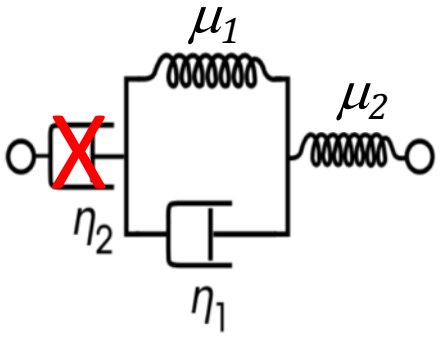
For a K-V element, $e(t) = \frac{\sigma}{\mu}\left[1 - \exp\left(-\frac{\mu}{\eta}t\right)\right]$ (strain $e(0) = 0$ ✓).

$\mu = \mu_1$
$\eta = \eta_1$

By differentiating with respect to time t, $\dot{e}(t) = \frac{\sigma}{\mu}\left[\left(\frac{\mu}{\eta}\right)\exp\left(-\frac{\mu}{\eta}t\right)\right] = \frac{\sigma}{\eta}\exp\left(-\frac{\mu}{\eta}t\right)$

At t=0, the strain *rate* in the K-V element is $\dot{e}(0) = \frac{\sigma}{\eta} = \frac{\mu_2 e}{\eta}$

Relaxation Function in Standard Linear Solid ($\eta_2 = \infty$)



e = constant

$$\sigma(t) = e\frac{\mu_1\mu_2}{\mu_1 + \mu_2}\left\{1 + \frac{\mu_2}{\mu_1}\exp\left(-\frac{\mu_1 + \mu_2}{\eta_1}t\right)\right\} \quad (*)$$

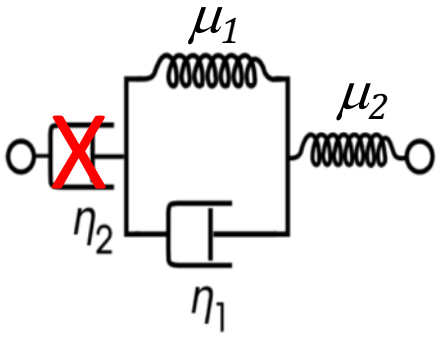
At t > 0

- The K-V element is starting to strain at the rate $\dot{e}(0) = \dfrac{\mu_2 e}{\eta}$,
- K-V begins to take over some of the strain from the spring μ_2.
- Strain e_1 increases in spring μ_1 and η_1, and strain e_2 decreases in spring μ_2

  $e_1 + e_2 = e$
- Because strain is decreasing in spring μ_2, stress $\sigma(t)$ must be decreasing.
- Strain e_1 in spring μ_1 cannot exceed σ_∞/μ_1
- Dash-pot η_1 must eventually stop moving.
- This means there is no stress in the dash-pot at $t=t_\infty$
- There will be a time constant τ that depends on μ_2, μ_1 and η_1

$$\tau = \left[\frac{\eta_1}{\mu_1 + \mu_2}\right]$$

Relaxation Function in Standard Linear Solid ($\eta_2 = \infty$)



e = constant

$$\sigma(t) = e\frac{\mu_1\mu_2}{\mu_1 + \mu_2}\left\{1 + \frac{\mu_2}{\mu_1}\exp\left(-\frac{\mu_1 + \mu_2}{\eta_1}t\right)\right\} \quad (*)$$

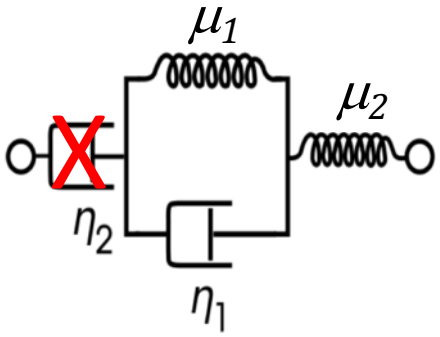
At t = ∞

- Strain e_1 in spring μ_1 cannot exceed σ_∞/μ_1
- Dash-pot η_1 must eventually stop moving.
- This means there is no stress in the dash-pot at $t=t_\infty$
- Both springs μ_1 and μ_2 then support the same stress σ_∞, so

$$e = \frac{\sigma_\infty}{\mu_1} + \frac{\sigma_\infty}{\mu_2} \quad \text{or} \quad \frac{\sigma_\infty}{e} = \left[\frac{1}{\mu_1} + \frac{1}{\mu_2}\right]^{-1}$$

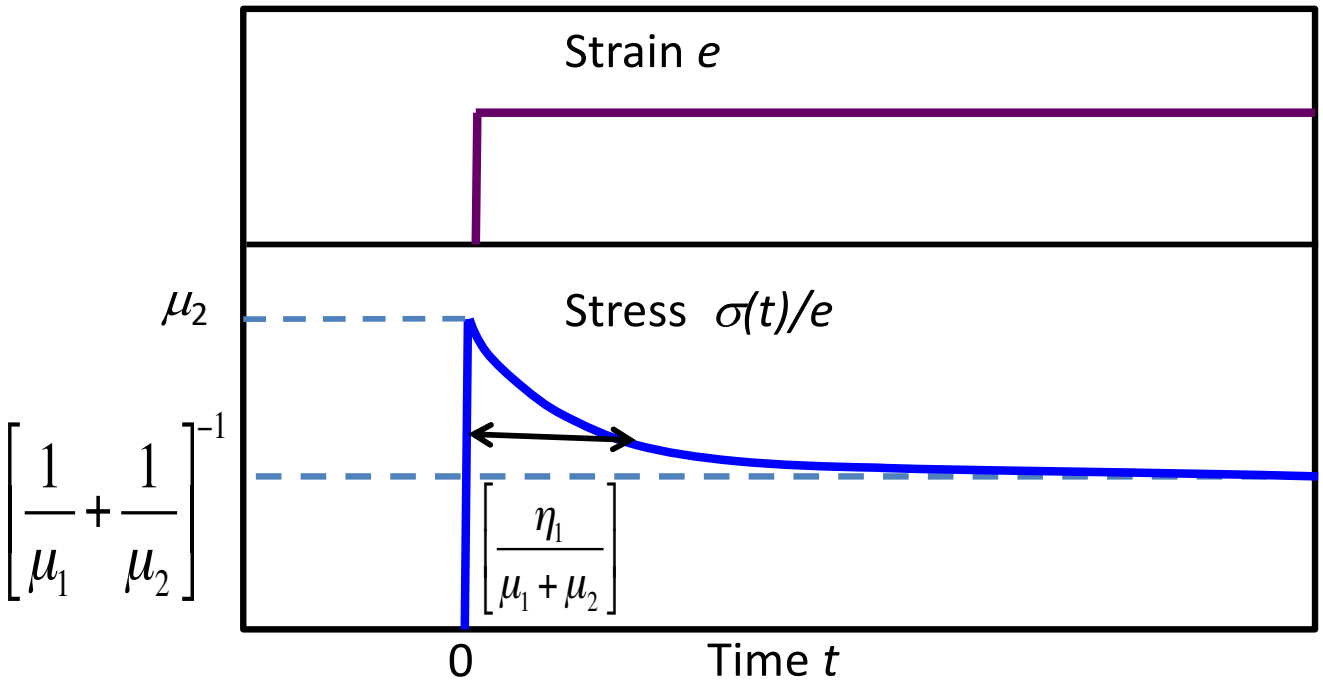
is the limiting stress at $t=t_\infty$

Relaxation Function in Standard Linear Solid ($\eta_2 = \infty$)



e = constant

$$\sigma(t) = e\frac{\mu_1\mu_2}{\mu_1 + \mu_2}\left\{1 + \frac{\mu_2}{\mu_1}\exp\left(-\frac{\mu_1 + \mu_2}{\eta_1}t\right)\right\} \quad (*)$$



Energy and Work

Work W is force ***F*** acting through a distance *d*

Work for point particles: W = ***F*** *d*

In Continuum – work done per unit volume:

$$\frac{W}{V} = \frac{Fd}{V} = \left(\frac{F}{A}\right) \bullet \left(\frac{d}{l}\right) = \sigma e$$ *= stress × strain*

Rate of doing work per unit volume

$$\frac{d}{dt}\left(\frac{W}{V}\right) = \frac{\dot{W}}{V} = \frac{F\dot{d}}{V} = \left(\frac{F}{A}\right) \bullet \left(\frac{\dot{d}}{l}\right) = \sigma \dot{e}$$ *= stress × strain rate*

(overdots indicate time derivatives)

Energy and Work

Total energy input between from time 0 to t

$$\Delta E(t) = \int_0^t \sigma(t')\dot{e}(t')dt'$$

For elastic material, substitute: $\sigma(t) = \mu e(t)$

$$\Delta E(t) = \int_0^t \mu e(t')\dot{e}(t')\ dt' = \frac{1}{2}\mu e^2(t) = \frac{\sigma^2(t)}{2\mu}$$

$\Delta E(t)$ returns to zero whenever σ returns to zero.

- All energy is recovered

Energy and Work

Total energy input between from time 0 to *t*

$$\Delta E(t) = \int_0^t \sigma(t')\dot{e}(t')dt'$$

For viscous material:

$$\sigma(t) = \eta\dot{e}(t)$$

$$\Delta E(t) = \int_0^t \eta\dot{e}(t)^2 dt'$$

The integrand is always positive.

- $\Delta E(t)$ can never return to zero if strain rate is ever nonzero
- energy is always lost if any strain has occurred.