ESS 411/511 Geophysical Continuum Mechanics Class #6

Highlights from Class #5 – Peter Lindquist
Today's highlights on Wednesday – John-Morgan Manos

Remember we are looking for just 2 or 3 *highlights*, not a summary of the entire class.   (What did you think was most important?)

EARTH SCIENCE WEEK

https://www.earthsciweek.org/

Since October 1998, the American Geosciences Institute has organized this national and international event to help the public gain a better understanding and appreciation for the Earth sciences and to encourage stewardship of the Earth. This year's Earth Science Week will be held from October 10 - 16, 2021 and will celebrate the theme "Water Today and for the Future." The coming year's event will focus on the importance of learning how to understand, conserve, and protect water, perhaps Earth's most vital resource.

ESS 411/511 Geophysical Continuum Mechanics

Prep for Wednesday class #7

- Please read Mase, Smelser, and Mase, CH 2 through Section 2.6

Class_07 assignment on Vector products

Equation 2.12, vector cross product

$\boldsymbol{u} \times \boldsymbol{v} = - \boldsymbol{v} \times \boldsymbol{u}$

$= |\boldsymbol{u}|\ |\boldsymbol{v}| \sin(\theta)\, \hat{e}$

$= e_{ijk}\, u_i\, v_j\, \hat{e}_k$

What is θ?

Where does the vector $\hat{e}$ point?

Can you give a simple geometric interpretation for $\boldsymbol{u} \times \boldsymbol{v}$?

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Problem Sets

- Problem Set #1 – due in Canvas on Wednesday
- Problem Set #2 – in Lab on Thursday

ESS 411/511 Geophysical Continuum Mechanics Class #6

Warm-up question (break-out) –

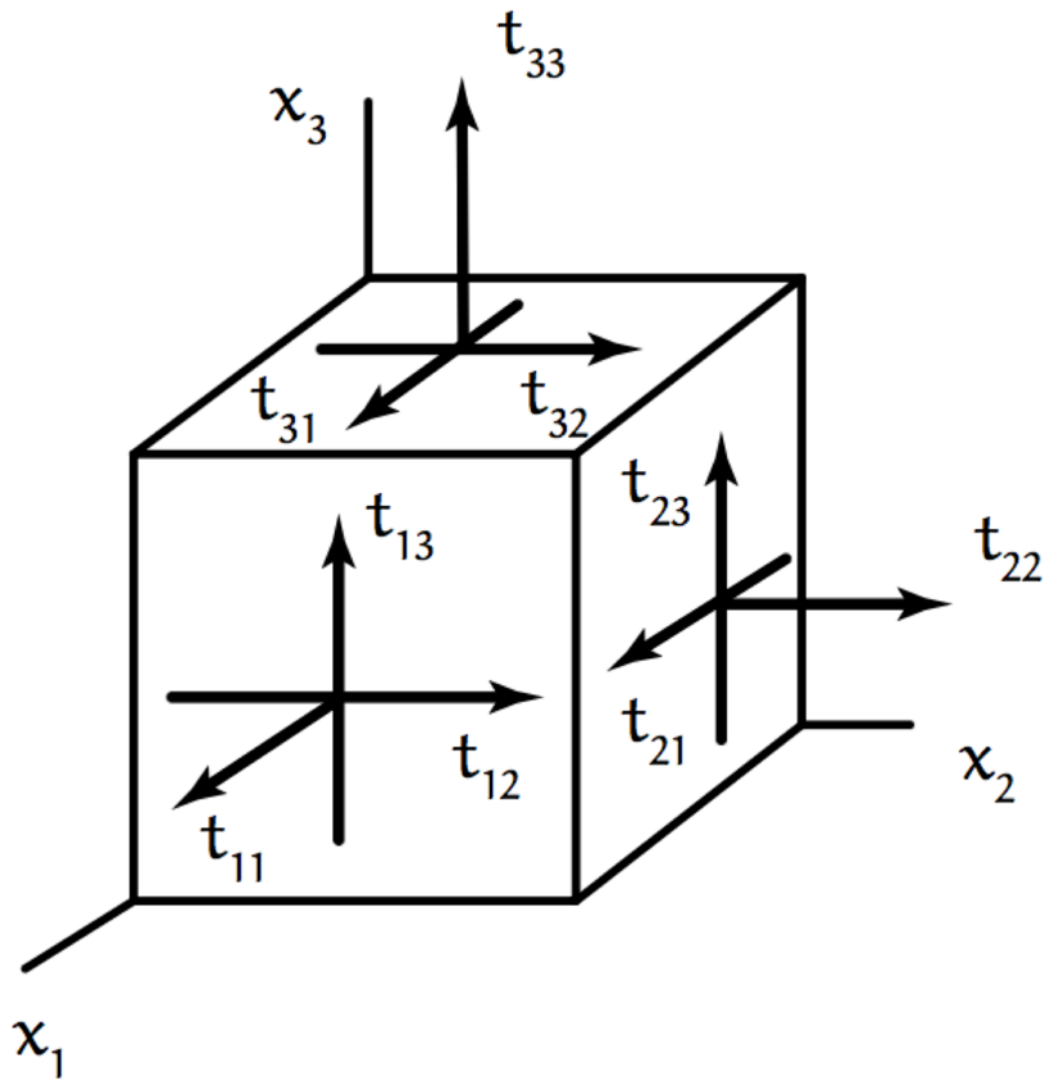
The Summation Convention is your Best Friend in Continuum

- Expand $t_{ii} v_j \hat{e}_j$ according to the summation convention.
- What is the tensor rank of the result?
- How would you interpret the result?

Class-prep answers (break-out)

- Expressing stress as vector? Why or why not?

Why stress has 9 components



Scalars, vectors, and tensors

Scalar (zero-order tensor)

- α, β, γ, θ, *a, b, r,* etc. lower-case italic greek or roman letters.
- Temperature, pressure, density, …

Vector (first-order tensor)

- ***u, v, w, x, y, z*** etc. Bold italic lower-case roman letters.
- Velocity, force, geothermal flux, …

Higher-order tensor

- **A, T, S**, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, E, etc. Bold letters, often upper-case roman, but many exceptions.
- Stress, strain rate, anisotropic material properties, …

Scalars, vectors, and tensors

Symbolic notation:

A tensor, or linear transformation **T**, assigns any vector **v** to another vector **T*v*** such that

$\mathbf{T}(\boldsymbol{v} + \boldsymbol{w}) = \mathbf{T}\boldsymbol{v} + \mathbf{T}\boldsymbol{w}$ (distributive property)

$\mathbf{T}(\alpha\boldsymbol{v}) = \alpha\mathbf{T}\boldsymbol{v}$ (associative property)

$(\mathbf{T}+\mathbf{S})\boldsymbol{v} = \mathbf{T}\boldsymbol{v} + \mathbf{S}\boldsymbol{v}$

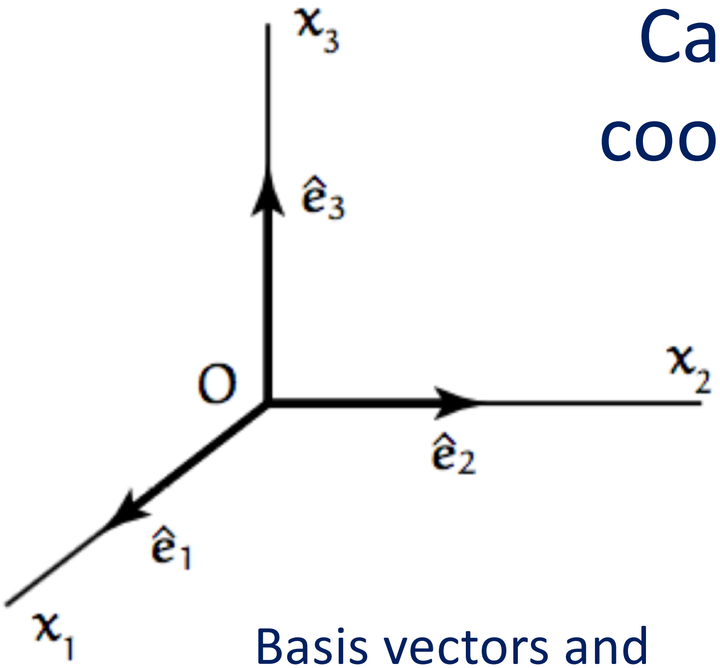
$(\alpha\mathbf{T})\boldsymbol{v} = \alpha(\mathrm{T}\boldsymbol{v})$

for all ***v*** and ***w***.

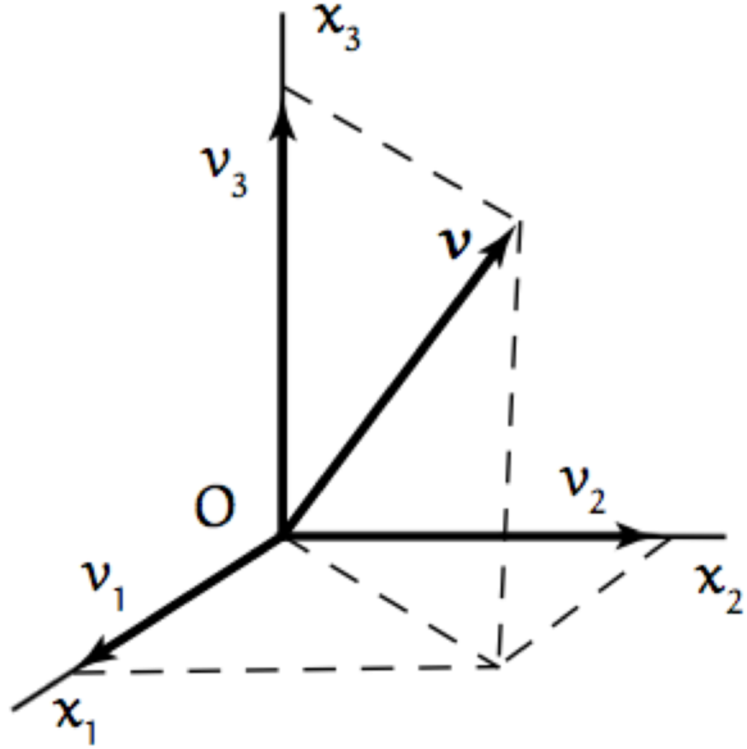
Not necessarily commutative though …

- Things like **T*v*** and ***v*T** get messier.
- May not be equal
- May not even exist …

Cartesian coordinates



Basis vectors and coordinate axes



A vector $\boldsymbol{v}$ exists independent of a particular coordinate system.

- However, after we choose a coordinate system, $\boldsymbol{v}$ can be expressed through its components in that coordinate system.

  $\boldsymbol{v} = (v_1, v_2, v_3)$

$$v = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 = \Sigma_{i=1}^{3} v_i\hat{e}_i$$

Index notation

- $u_i \Longleftrightarrow (u_1, u_2, u_3)$ vector (3 elements)
- $u_{ij} \Longleftrightarrow (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33})$
  Second-order tensor (9 elements)
- $u_{ijk} \Longleftrightarrow (u_{111}, u_{112}, u_{113}, u_{121}, u_{122}, u_{123}, \ldots u_{331}, u_{332}, u_{333},).$
  Third-order tensor (27 elements)
- $u_{ijkl} \Longleftrightarrow (u_{1111}, u_{1112}, u_{1113}, \ldots u_{3331}, u_{3332}, u_{3333},).$
  Fourth-order tensor (81 elements)

Summation Convention

- Whenever an index i is repeated in a term, a summation is implied, in which i takes the values 1, 2 ,3.

$$v_i\hat{e}_i \equiv \Sigma_{i=1}^{3} v_i\hat{e}_i$$

- In an implied summation, index i is a *dummy* index, i.e.

$$v_i\hat{e}_i = v_k\hat{e}_k$$

or

$$u_i v_i = u_k v_k$$

Kronecker delta

Since the base vectors $\hat{\mathbf{e}}_i$ ($i = 1,2,3$) are unit vectors and orthogonal

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{cases} 1 & \text{if numerical value of } i = \text{numerical value of } j \\ 0 & \text{if numerical value of } i \neq \text{numerical value of } j \end{cases}$$

Therefore, if we introduce the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if numerical value of } i = \text{numerical value of } j \\ 0 & \text{if numerical value of } i \neq \text{numerical value of } j \end{cases}$$

we see that

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \qquad (i, j = 1, 2, 3) \ .$$

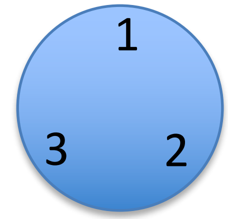
Note that δ_{ij} is like a 3x3 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is the value of δ_{jj} ?

Permutation symbol

Imagine the numbers 1, 2, 3 are written clock-wise around the circumference of a wheel



$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if the numerical values of } i,j,k \text{ are in clockwise order} \\ -1 & \text{if the numerical values of } i,j,k \text{ are anti-click-wise order} \\ 0 & \text{if the numerical values of } i,j,k \text{ are in any other order} \end{cases}$$

Cross products of basis vectors

$$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk}\hat{e}_k \qquad (i, j, k = 1, 2, 3)$$

Switching indices

$$\varepsilon_{ijk} = -\varepsilon_{kji} = \varepsilon_{kij} = -\varepsilon_{ikj}$$

Determinants

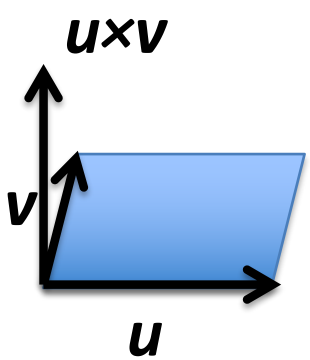
$$\det \boldsymbol{\mathcal{A}} = |\mathcal{A}_{ij}| = \begin{vmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{vmatrix}$$

$$\begin{aligned} \det \boldsymbol{\mathcal{A}} &= \mathcal{A}_{11} \begin{vmatrix} \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{32} & \mathcal{A}_{33} \end{vmatrix} - \mathcal{A}_{12} \begin{vmatrix} \mathcal{A}_{21} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{33} \end{vmatrix} + \mathcal{A}_{13} \begin{vmatrix} \mathcal{A}_{21} & \mathcal{A}_{22} \\ \mathcal{A}_{31} & \mathcal{A}_{32} \end{vmatrix} \\ &= \mathcal{A}_{11} (\mathcal{A}_{22}\mathcal{A}_{33} - \mathcal{A}_{23}\mathcal{A}_{32}) - \mathcal{A}_{12} (\mathcal{A}_{21}\mathcal{A}_{33} - \mathcal{A}_{23}\mathcal{A}_{31}) \\ &\quad + \mathcal{A}_{13} (\mathcal{A}_{21}\mathcal{A}_{32} - \mathcal{A}_{22}\mathcal{A}_{31}) \ . \end{aligned}$$

$$\varepsilon_{qmn} \det \boldsymbol{\mathcal{A}} = \varepsilon_{ijk} \mathcal{A}_{iq} \mathcal{A}_{jm} \mathcal{A}_{kn}$$

Box product

cross (vector) product of two vectors, Eq 2.12:

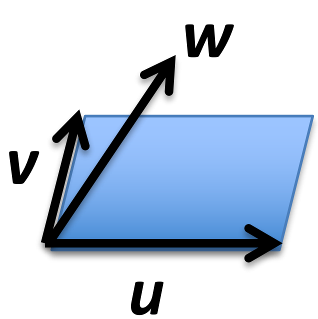


$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} = (uv \sin\theta)\, \hat{\mathbf{e}} = \varepsilon_{ijk} u_i v_j \hat{\mathbf{e}}_k$$

What does ***u×v*** measure?

triple scalar product (box product), Eq 2.13:

$$\begin{aligned} [\mathbf{u}, \mathbf{v}, \mathbf{w}] &= u_i \hat{\mathbf{e}}_i \cdot (v_j \hat{\mathbf{e}}_j \times w_k \hat{\mathbf{e}}_k) = u_i \hat{\mathbf{e}}_i \cdot \varepsilon_{jkq} v_j w_k \hat{\mathbf{e}}_q \\ &= \varepsilon_{jkq} u_i v_j w_k \delta_{iq} = \varepsilon_{ijk} u_i v_j w_k \end{aligned}$$



What does [***u,v,w***] measure?

Vector algebra