ESS 411/511 Geophysical Continuum Mechanics Class #8

Highlights from Class #7 – Abigail Thienes
Today's highlights on Monday – Alexandria Vasquez-Hernandez

Warm-up (break-out rooms)

- Show that $a_{ik}\, b_{kj}$ is the same as multiplying two 3x3 matrices **A** and **B** together.

- Class- prep questions

  $\mathbf{Q}_{ij} = \mathbf{A}_{ij} + \mathbf{B}_{ij}$ (1)

  What property of $\mathbf{A}_{ij}$ makes it symmetric?

  How can you make $\mathbf{A}_{ij}$ from $\mathbf{Q}_{ij}$?

  What property of $\mathbf{B}_{ij}$ makes it anti-symmetric?

  How can you make $\mathbf{B}_{ij}$ from $\mathbf{Q}_{ij}$?

  Show that Equation (1) holds with your $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$.

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For Monday class

- Please read Mase, Smelser, and Mase, CH 2 through Section 2.8
  (tensor fields; Gauss' theorem)
  Please also read Mase, Smelser, and Mase, CH 3 through Section 3.3

Your short CR/NC Pre-class prep writing assignment (1 point) in Canvas

- It will be due in Canvas at the start of class.
- I will send another message when it is posted in Canvas.

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Problem Sets

You are working on Problem Set #2

- Due in Canvas on Wednesday

Vector algebra

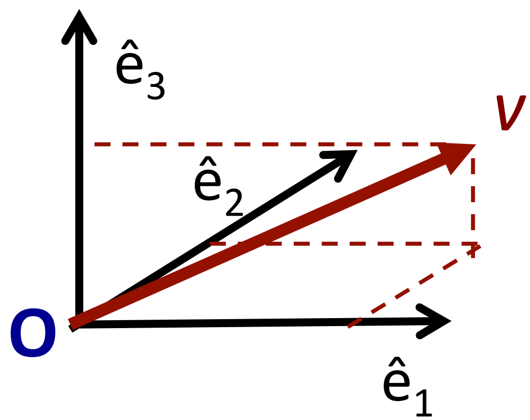
Lots of details in CH 2

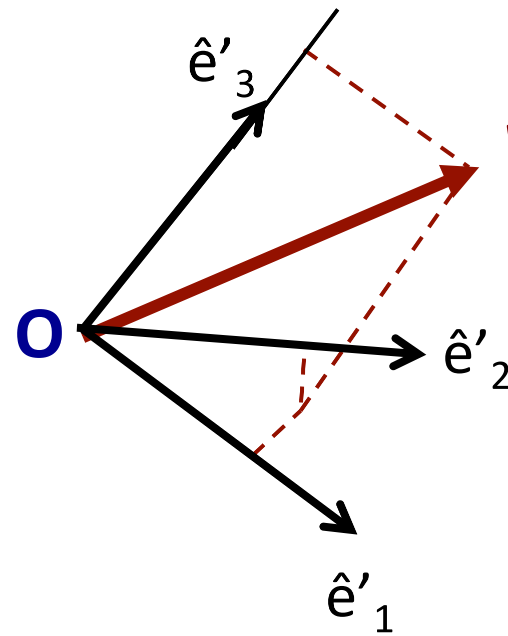
- Are there points that are unclear?
- Please let me know if there are things you would like us to look at specifically.

Transformation of Cartesian Coordinates

An object such as vector ***v*** is represented as v_j in coordinate system $Ox_1x_2x_3$ with unit coordinate vectors $\hat{e}_j$

The same object (e.g. ***v***) is represented as v_j‘ in coordinate system $Ox_1'x_2'x_3'$ with unit coordinate vectors $\hat{e}'_j$





v is ***not*** rotated –

- its coordinates are just expressed in a different coordinate system

Transformation matrix

The new coordinate vectors $\hat{e}'_j$ can be expressed in terms of the old coordinate vectors $\hat{e}_j$

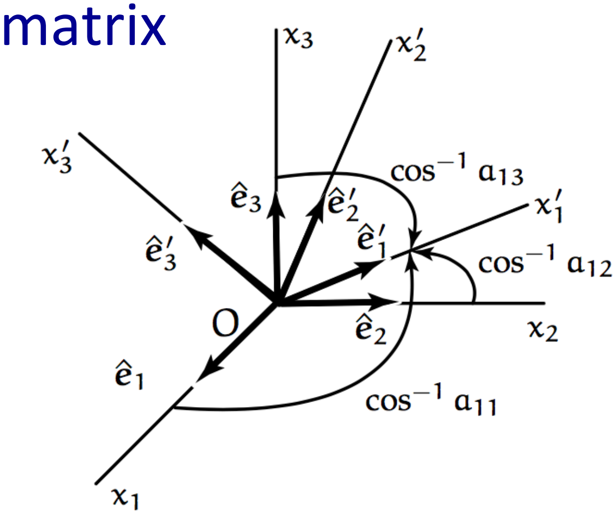
$$\hat{e}'_1 = a_{11}\hat{e}_1 + a_{12}\hat{e}_2 + a_{13}\hat{e}_3 = a_{1j}\hat{e}_j$$
$$\hat{e}'_2 = a_{21}\hat{e}_1 + a_{22}\hat{e}_2 + a_{23}\hat{e}_3 = a_{2j}\hat{e}_j$$
$$\hat{e}'_3 = a_{31}\hat{e}_1 + a_{32}\hat{e}_2 + a_{33}\hat{e}_3 = a_{3j}\hat{e}_j$$

or,

$$\hat{e}'_i = a_{ij}\hat{e}_j$$

$$\begin{bmatrix} \hat{e}'_1 \\ \hat{e}'_2 \\ \hat{e}'_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

A is the transformation matrix



a_{ij} is just the projection of the i^{th} new axis unit vector $\hat{e}'_i$ onto the j^{th} old axis unit vector $\hat{e}_j$ through their dot product $\hat{e}'_i \bullet \hat{e}_j$

Change of coordinate system for any order tensor $R_{qm\ldots n}$

$$R'_{ij\ldots k} = a_{iq} a_{jm} \cdots a_{kn} R_{qm\ldots n}$$

Multiply by transformation matrix A once
for each order in the tensor $R_{qm\ldots n}$

Examples

- 0^{th} order tensor (scalar) – no a_{pq} factors $\theta' = \theta$
- 1^{st} order tensor (vector) – 1 a_{pq} factor $u'_i = a_{ij} u_j$
- 2^{nd} order tensor – 2 a_{pq} factors $t'_{ij} = a_{im} a_{jn} t'_{mn}$

Proper and Improper changes of coordinates

det(**A**) = 1   is a rotation (proper)

det(**A**) = -1   is a reflection (improper)
(right-handed coordinate system becomes a left-handed coordinate system (generally not good …)

We will use only right-handed coordinates

Principal values and directions (Eigenvalues and eigenvectors)

A 2nd order tensor s_{ij} maps a vector u_j onto another vector v_i

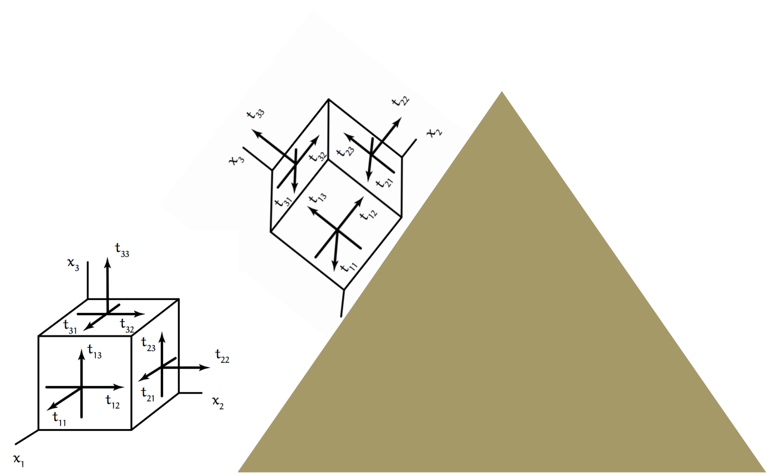
$$s_{ij}u_j = v_i$$

In general u_j and v_i point in different directions.

It would be nice if we could find some special vectors u_j that mapped onto vectors v_i that were parallel to u_j.
That could help us to find a coordinate system in which s_{ij} could be expressed more simply.

For example, stress in the rocks on a mountain side.
We know that there is no shear stress on the sloping surface.

- Maybe the stress tensor would be simpler using a coordinate system aligned with the mountain surface.



Finding eigenvectors

When t_{ij} is symmetric with real components, there will be some vectors n_j that *do* map onto a parallel vector.

$$t_{ij} n_j = \lambda n_i \qquad \text{or} \qquad \mathbf{T} \cdot \hat{\mathbf{n}} = \lambda \hat{\mathbf{n}}$$

When n_j is a unit vector, it defines a principal direction or ***eigenvector*** of the tensor t_{ij}, and λ is called a principal value or ***eigenvalue*** of t_{ij}.

$$t_{ij}\, n_j - \lambda\, n_i = 0$$

Since $n_i = \delta_{ij}\, n_j$

$$t_{ij}\, n_j - \lambda\, \delta_{ij}\, n_j = 0$$

or

$$(t_{ij} - \lambda\, \delta_{ij})\, n_j = 0 \quad \text{or in symbolic form,} \quad (\mathbf{T} - \lambda \mathrm{I}) \bullet \boldsymbol{n} = 0$$

Finding eigenvectors

$(t_{ij} - \lambda\,\delta_{ij})\,n_j = 0$ or in symbolic form, $(\mathbf{T} - \lambda \mathrm{I})\bullet \boldsymbol{n} = 0$

$$(t_{11} - \lambda)\,n_1 + t_{12}n_2 + t_{13}n_3 = 0$$
$$t_{21}n_1 + (t_{22} - \lambda)\,n_2 + t_{23}n_3 = 0$$
$$t_{31}n_1 + t_{32}n_2 + (t_{33} - \lambda)\,n_3 = 0$$

Obviously these equations are satisfied if $n_1 = n_2 = n_3 = 0$. But that is no help because we said n_j is a unit vector

Nontrivial solutions can exist (the equations are not independent) if the determinant = 0

$$|t_{ij} - \lambda\delta_{ij}| = 0$$

Evaluating the determinant produces a cubic equation in λ

$$\lambda^3 - I_T\lambda^2 + II_T\lambda - III_T = 0$$

This is called the ***Characteristic Equation***, and the 3 coefficients are the ***first, second, and third invariants*** of the tensor t_{ij}.

Finding eigenvectors

$$\lambda^3 - I_T\lambda^2 + II_T\lambda - III_T = 0$$

This is called the ***Characteristic Equation***, and the 3 coefficients are the ***first, second, and third invariants*** of the tensor t_{ij}.

$$I_T = t_{ii} = \mathrm{tr}\ \mathbf{T}$$

$$II_T = \frac{1}{2}\left(t_{ii}t_{jj} - t_{ij}t_{ji}\right) = \frac{1}{2}\left[(\mathrm{tr}\ \mathbf{T})^2 - \mathrm{tr}\left(\mathbf{T}^2\right)\right]$$

$$III_T = \varepsilon_{ijk}t_{1i}t_{2j}t_{3k} = \det \mathbf{T}$$

No matter what coordinate system we use to express the tensor **T**, these 3 special quantities always have the same 3 values.

Finding eigenvectors

$$\lambda^3 - I_{\mathbf{T}}\lambda^2 + II_{\mathbf{T}}\lambda - III_{\mathbf{T}} = 0$$

This is called the ***Characteristic Equation***, and the 3 coefficients are the ***first, second, and third invariants*** of the tensor t_{ij}.

The cubic equation has 3 solutions $\lambda_{(1)}$, $\lambda_{(2)}$, and $\lambda_{(3)}$, which are all real for a symmetric tensor **T** whose elements are real.

There is a direction $n_i^{(q)}$ associated with each eigenvalue $\lambda_{(q)}$.

We can find $n_i^{(q)}$ by solving

$$\left[t_{ij} - \lambda_{(q)}\delta_{ij}\right] n_i^{(q)} = 0 \qquad (q = 1, 2, 3)$$

$$n_i^{(q)} n_i^{(q)} = 1 \qquad (q = 1, 2, 3).$$

To see how to do this, check out MSM Example 2.14 on page 32.