ESS 411/511 Geophysical Continuum Mechanics Class #9

Highlights from Class #8 – Alexandria Vasquez-Hernandez
Today's highlights on Wednesday – Chloe Mcburney

For Wednesday class

- Please read Mase, Smelser, and Mase, Ch 3 through Section 3.6

Your short CR/NC Pre-class prep writing assignment (1 point) in Canvas

- It will be due in Canvas at the start of class.
- I will send another message when it is posted in Canvas.

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

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How I see your class-prep assignments

- According to pedagogy experts, you learn better by doing your own reading, writing and discussing compared to listening to me repeat material in the text.
- The class-prep assignments are meant to help you get an initial idea about the topics coming up.
- Sometimes your answers before the class may miss the main point, or may be incomplete.
- That's OK. I am giving you the point anyway for thinking about it and writing down your thoughts. You are creating value in your education.
- I am not annotating your documents in Canvas, because I think you are figuring it out on your own in our subsequent break-out rooms and plenary discussions in class, without me adding comments again in Canvas ☺.

Problem Sets

Problem Set #2 due in Canvas on Wednesday

Problem Set #3 in Thursday session

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Warm-up (break-out rooms)

Finding the eigenvalues and eigenvectors of a 3x3 tensor is complicated and "mathy".

- Explain in words why anyone would want to find the eigenvalues and eigenvectors of a 3x3 tensor. Why bother?

Class- prep questions (break-out rooms)

- Stress *vector*. What the blazes is it, anyway?

Principal values and directions (Eigenvalues and eigenvectors)

A 2nd order tensor s_{ij} maps a vector u_j onto another vector v_i

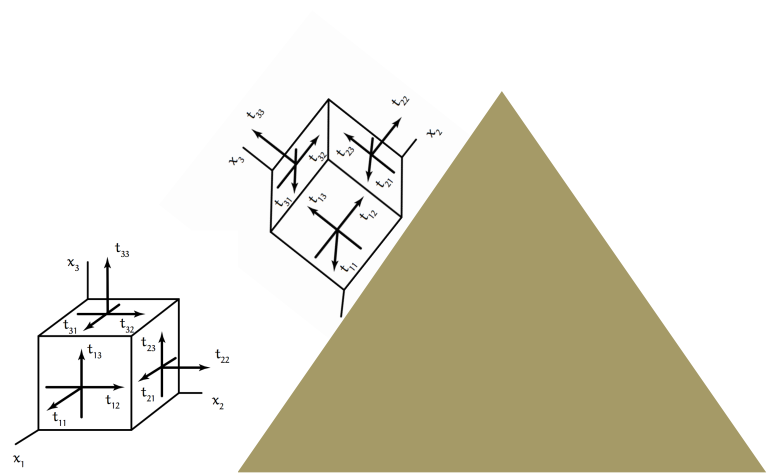
$$s_{ij}u_j = v_i$$

In general u_j and v_i point in different directions.

It would be nice if we could find some special vectors u_j that mapped onto vectors v_i that were parallel to u_j.
That could help us to find a coordinate system in which s_{ij} could be expressed more simply.

For example, stress in the rocks on a mountain side.
We know that there is no shear stress on the sloping surface.

- Maybe the stress tensor would be simpler using a coordinate system aligned with the mountain surface.



Finding eigenvectors

When t_{ij} is symmetric with real components, there will be some vectors n_j that *do* map onto a parallel vector.

$$t_{ij} n_j = \lambda n_i \qquad \text{or} \qquad \mathbf{T} \cdot \hat{\mathbf{n}} = \lambda \hat{\mathbf{n}}$$

When n_j is a unit vector, it defines a principal direction or ***eigenvector*** of the tensor t_{ij}, and λ is called a principal value or ***eigenvalue*** of t_{ij}.

$$t_{ij}\, n_j - \lambda\, n_i = 0$$

Since $n_i = \delta_{ij}\, n_j$

$$t_{ij}\, n_j - \lambda\, \delta_{ij}\, n_j = 0$$

or

$$(t_{ij} - \lambda\, \delta_{ij})\, n_j = 0 \quad \text{or in symbolic form,} \quad (\mathbf{T} - \lambda \mathrm{I}) \bullet \boldsymbol{n} = 0$$

Let's work through an example

Example 2.14

Determine the principal values and principal directions of the second-order tensor **T** whose matrix representation is

$$[t_{ij}] = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

In order for n_i to be an eigenvector, n_i and $t_{ij}\, n_j$ must be parallel

$$t_{ij}\, n_j - \lambda\, n_i = 0$$

Since $n_i = \delta_{ij}\, n_j$

$$t_{ij}\, n_j - \lambda\, \delta_{ij}\, n_j = 0$$

Factor out n_j: $(t_{ij} - \lambda\, \delta_{ij})\, n_j = 0$

Let's work through an example

$$(t_{11} - \lambda) n_1 + t_{12} n_2 + t_{13} n_3 = 0$$
$$t_{21} n_1 + (t_{22} - \lambda) n_2 + t_{23} n_3 = 0$$
$$t_{31} n_1 + t_{32} n_2 + (t_{33} - \lambda) n_3 = 0$$

Obviously these equations are satisfied if $n_1 = n_2 = n_3 = 0$.
But that is no help because we said n_j is a unit vector

Nontrivial solutions can exist
(the equations are not independent)
if the determinant = 0

$$|t_{ij} - \lambda\delta_{ij}| = 0$$

$$\begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

So, expanding on the third row,

$$(3-\lambda)\left(10 - 7\lambda + \lambda^2 - 4\right) = 0$$

which factors into

$$(3-\lambda)(6-\lambda)(1-\lambda) = 0$$

Let's work through an example

$$(3-\lambda)(6-\lambda)(1-\lambda)=0$$

So, the eigenvalues are: $\lambda_{(1)}=3 \quad \lambda_{(2)}=6, \quad \lambda_{(3)}=1$

Now, find the 3 corresponding eigenvectors n_j by solving the equations $(t_{ij} - \lambda\,\delta_{ij})\, n_j = 0$ with each value of λ in turn:

$$\begin{aligned}(t_{11}-\lambda)\,n_1 + t_{12}n_2 + t_{13}n_3 &= 0\\ t_{21}n_1 + (t_{22}-\lambda)\,n_2 + t_{23}n_3 &= 0\\ t_{31}n_1 + t_{32}n_2 + (t_{33}-\lambda)\,n_3 &= 0\end{aligned}$$

and

$$[t_{ij}] = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

With $\lambda_{(1)}$=3:
$$\begin{aligned}2n_1 + 2n_2 &= 0\\ 2n_1 - n_2 &= 0\end{aligned}$$

The solution is: $n_1 = n_2 = 0$ and since n_j must be a unit vector, $n_3 = 1$

and the first eigenvector is: (0, 0, ±1), i.e. $\hat{e}'_3 = \hat{e}_3$
So the new axes are just the old axes rotated about $\hat{e}_3$

Let's work through an example

Similarly, using $\lambda_{(2)}$=6,

$$\begin{aligned} -n_1 + 2n_2 &= 0 \\ 2n_1 - 4n_2 &= 0 \\ -3n_3 &= 0 \end{aligned}$$

$n_1 = 2n_2$ and since $n_3 = 0$,

And the unit-vector criterion gives

$$(2n_2)^2 + n_2^2 = 1, \text{ or } n_2 = \pm 1/\sqrt{5} \quad \text{and} \quad n_1 = \pm 2/\sqrt{5}$$

So, the second eigenvector is: $(\pm 2/\sqrt{5}, \pm 1/\sqrt{5}, 0)$

Let's work through an example

Similarly, using $\lambda_{(3)} = 1$,

$$\begin{aligned} 4n_1 + 2n_2 &= 0 \\ 2n_1 + n_2 &= 0 \end{aligned}$$

$n_1 = 2n_2$ and since $n_3 = 0$,

And the unit-vector criterion gives

$(2n_2)^2 + n_2^2 = 1$, or $n_2 = \pm 1/\sqrt{5}$ and $n_1 = \pm 2/\sqrt{5}$

So, the third eigenvector is: $(\pm 1/\sqrt{5}, \mp 2/\sqrt{5}, 0)$

And the Transformation matrix is:

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm \frac{2}{\sqrt{5}} & \pm \frac{1}{\sqrt{5}} & 0 \\ \pm \frac{1}{\sqrt{5}} & \mp \frac{2}{\sqrt{5}} & 0 \end{bmatrix}$$

Each row is an eigenvector

Let's work through an example

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm\frac{2}{\sqrt{5}} & \pm\frac{1}{\sqrt{5}} & 0 \\ \pm\frac{1}{\sqrt{5}} & \mp\frac{2}{\sqrt{5}} & 0 \end{bmatrix}$$

The rows provide 2 sets of eigenvectors, depending on the upper/lower +/- choices.

Delayed warm-up (break-out rooms)

- How would you decide which set to use?
- Suppose in a different problem, the 2nd and 3rd eigenvalues were equal. There will be some remaining ambiguity you won't be able to find 3 eigenvectors in the same way.
- What's going on? What have you learned about the directions in which $a_{ij}n_j$ points in the same direction as n_j?