ESS 411/511 Geophysical Continuum Mechanics Class #10

Highlights from Class #9 – Jensen DeGrande
Today's highlights on Friday – Jaylen Generoso

For Friday class

- Please read Mase, Smelser, and Mase, Ch 3 through Section 3.8

- Those of of you taking this class as ESS 511, please remember that on Friday, you will be giving a 60-second outline of your ideas so far about your term topic.

ESS 411/511 Geophysical Continuum Mechanics

## Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

# Problem Sets

Problem Set #2  due in Canvas today

Problem Set #3 in Problem session tomorrow

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**Warm-up (groups)**

Vectors and tensors get more interesting when they vary with time *t* and with position ***x*** inside a body.

Explain in words what is meant by:

- $\boldsymbol{v}(\boldsymbol{x},t)$ $\sigma_{ij}(\boldsymbol{x},t)$
- $v_i$ $v_{i,i}$ $v_{i,j}$ $\varepsilon_{ijk}v_{k,j}$
- $\sigma_{ij}$ $\sigma_{ii,k}$ $\sigma_{ij,k}$ $\sigma_{ij,kk}$ $\sigma_{ij,kl}$
- $x_{i,j} = \delta_{ij}$

# Class-prep for today

Choosing eigenvectors

Suppose you have found the 3 eigenvalues $l_{(q)}$ for 3x3 tensor $t_{ij}$, and then the 3 corresponding eigenvectors $\pm n_j^{(q)}$. The sign ambiguity can arise because the tensor projects an eigenvector parallel to itself, but the projection vector $t_{ij}\, n_j$ can point in either direction, i.e. either parallel to $n_j$ or antiparallel.

So the transformation matrix is

$$\begin{bmatrix} \pm n_1^{(1)} & \pm n_2^{(1)} & \pm n_3^{(1)} \\ \pm n_1^{(2)} & \pm n_2^{(2)} & \pm n_3^{(2)} \\ \pm n_1^{(3)} & \pm n_2^{(3)} & \pm n_3^{(3)} \end{bmatrix}$$

where each row is one of the eigenvectors.

- What criterion would you use to decide which sign combination of signs to use on each eigenvector?
- Suppose in a different problem, the 2nd and 3rd eigenvalues $\mathbf{l_{(2)}}$ **and** $\mathbf{l_{(3)}}$ were equal. You can still find the first eigenvector, but because of the nonuniqueness, you won't be able to find 2 other eigenvectors in the same way as before.
- What's going on? You know that the other 2 eigenvectors must be orthogonal to the first one. What can you do to complete the new basis set?

# Derivatives of tensors

A tensor can vary smoothly in space and time, so it has derivatives.

$\frac{\partial v_j}{\partial t}$ = rate at which velocity vector is changing at a point.

$\nabla \vec{v} = v_{i,j}$ = how a velocity vector is changing with position at a point.
There are 2 indices, so this is a 3x3 array.

$$\begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Elements show how each vector component varies in each spatial direction.

# Divergence

A tensor can vary smoothly in space and time, so it has derivatives.

Scalar - a scalar doesn't have divergence ☹

Vector - $\nabla \bullet \vec{v} = v_{i,i}$

Tensor - $\nabla \bullet T = t_{ij,i}$

$$\begin{bmatrix} t_{11,1} + t_{21,2} + t_{31,3} \\ t_{12,1} + t_{22,2} + t_{32,3} \\ t_{13,1} + t_{23,2} + t_{33,3} \end{bmatrix}$$

Each row is the divergence of the corresponding column of ***T***

A divergence is 1 rank lower than the original quantity.

# Gradient

A tensor can vary smoothly in space and time, so it has derivatives.

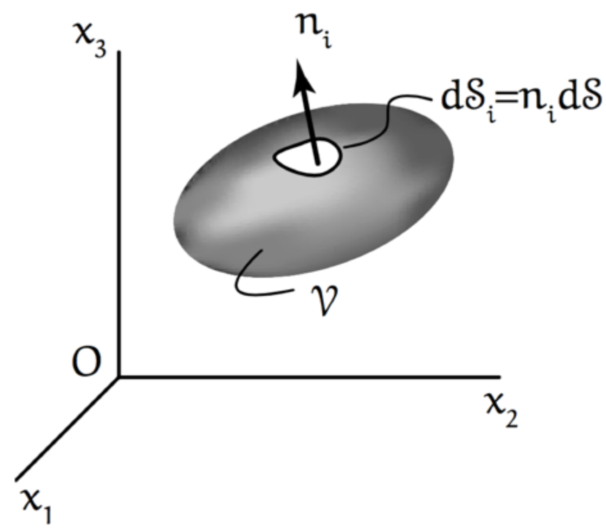
Scalar $$\frac{\partial \phi}{\partial x_j} = \phi_{,j}$$

Vector $$\nabla \vec{v} = \frac{\partial v_i}{\partial x_j} = v_{i,j}$$

Tensor etc …

A gradient is 1 rank higher than the original quantity.

# Integral theorems




We may want to know what is going on inside a body but have access only to its surface (or vice versa)

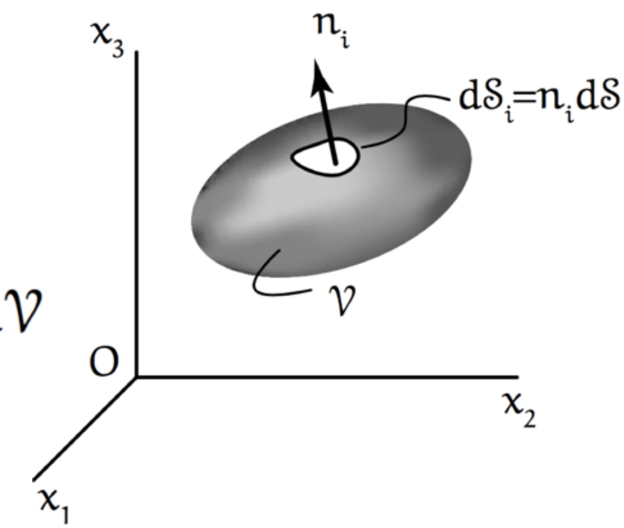
A volume $V$ has surface $S$.

- Each small patch d$S$ on the surface is defined by its normal vector $n_i$.

Divergence theorem $$\int_{\mathcal{S}} t_{ij\ldots k} n_q \, d\mathcal{S} = \int_{\mathcal{V}} t_{ij\ldots k,q} \, d\mathcal{V}$$

The total amount of $t_{ij\ldots k}$ directed out across $S$ is the same as the total amount of spreading (divergence) everywhere inside $V$.

# Special cases

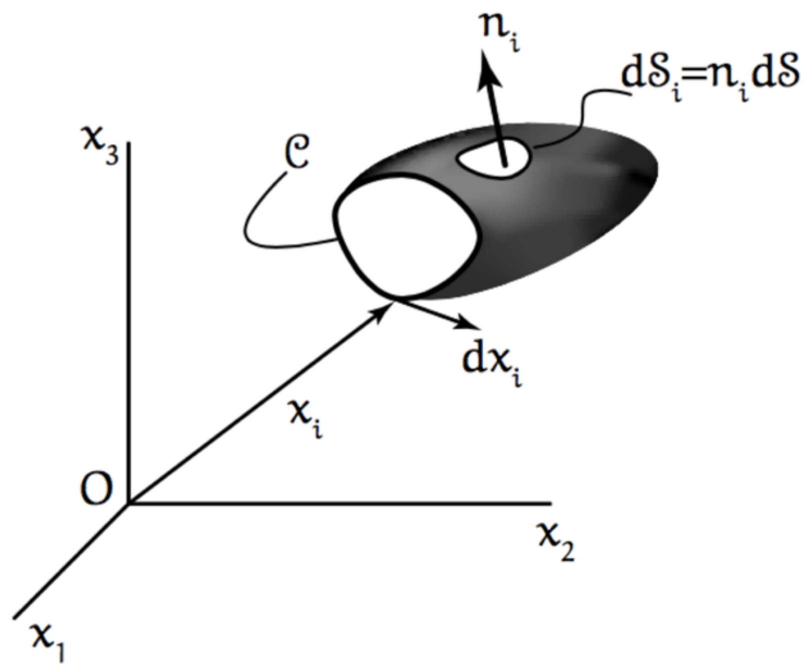



Divergence theorem $$\int_{\mathcal{S}} t_{ij\ldots k} n_q \, d\mathcal{S} = \int_{\mathcal{V}} t_{ij\ldots k,q} \, d\mathcal{V}$$

$$\int_{\mathcal{S}} v_q n_q \, d\mathcal{S} = \int_{\mathcal{V}} v_{q,q} \, d\mathcal{V} \quad \text{or} \quad \int_{\mathcal{S}} \boldsymbol{v} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = \int_{\mathcal{V}} \operatorname{div} \boldsymbol{v} \, d\mathcal{V}$$

If density $\rho$ is uniform, the total amount of "stuff" flowing out across $S$ with velocity *v* (the flux across $S$) is the same as the total amount of spreading (divergence) of that "stuff" everywhere inside $V$.

# Stokes theorem




$C$ is the perimeter of a cap on an open surface.

- d$\boldsymbol{x}$ is the tangent to the perimeter $C$.
- $\boldsymbol{v}$ is the material velocity.

$$\int_{\mathcal{S}} \varepsilon_{ijk} n_i v_{k,j}\, d\mathcal{S} = \int_{\mathcal{C}} v_k\, dx_k \quad \text{or} \quad \int_{\mathcal{S}} \hat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v})\, d\mathcal{S} = \int_{\mathcal{C}} \boldsymbol{v} \cdot d\boldsymbol{x}$$

If density $\rho$ is uniform, the total circulation of “stuff” (curl) within the cap (“churning”) is equal to the net flow along the perimeter $C$ (“the racetrack”).

($\varepsilon_{ijk} v_{j,k}$ is curl of $\boldsymbol{v}$)

# Definition of a tensor

In any rectangular coordinate system, a tensor is defined by components that transform according to the rule

$$R'_{ij\ldots k} = a_{iq} a_{jm} \cdots a_{kn} R_{qm\ldots n}$$

and where the basis vectors are related by

$$\hat{e}'_i = a_{ij} \hat{e}_j$$

$$\begin{bmatrix} \hat{e}'_1 \\ \hat{e}'_2 \\ \hat{e}'_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

# Forces in a continuum

Body forces $b_i$ force per volume
Surface forces $f_i$ force per area
(on exterior or interior surfaces)

Newton's second law ***F*** = m***a***

$$\int_V \rho(\vec{x}) b_i \, dV + \int_S t_i^{(\hat{n})} dS = \frac{d}{dt} \int_V \rho v_i \, dV$$