ESS 411/511 Geophysical Continuum Mechanics Class #11

Highlights from Class #10 – Jonathan Gates
Today's highlights on Monday – Jason Ott

For Monday class

- Please read Mase, Smelser, and Mase, Ch 3 through Section 3.11

For those of you taking this class as ESS 511, it is time for you to give a 60-second outline of your ideas so far about your term topic.

- Jensen
- Alysa
- Jonathan
- Anna
- Peter
- John-Morgan
- Yiyu
- Jason

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Class-prep from Wednesday

Choosing eigenvectors

Suppose you have found the 3 eigenvalues $l_{(q)}$ for 3x3 tensor t_{ij}, and then the 3 corresponding eigenvectors $\pm n_j^{(q)}$. The sign ambiguity can arise because the tensor projects an eigenvector parallel to itself, but the projection vector $t_{ij}\, n_j$ can point in either direction, i.e. either parallel to n_j or antiparallel.

So the transformation matrix is

$$\begin{bmatrix} \pm n_1^{(1)} & \pm n_2^{(1)} & \pm n_3^{(1)} \\ \pm n_1^{(2)} & \pm n_2^{(2)} & \pm n_3^{(2)} \\ \pm n_1^{(3)} & \pm n_2^{(3)} & \pm n_3^{(3)} \end{bmatrix}$$

where each row is one of the eigenvectors.

- What criterion would you use to decide which combination of signs to use on each eigenvector?
- Suppose in a different problem, the 2nd and 3rd eigenvalues $\mathbf{l_{(2)}}$ **and** $\mathbf{l_{(3)}}$ were equal. You can still find the first eigenvector, but because of the nonuniqueness, you won't be able to find 2 other eigenvectors in the same way as before.
- What's going on? You know that the other 2 eigenvectors must be orthogonal to the first one. What can you do to complete the new basis set?

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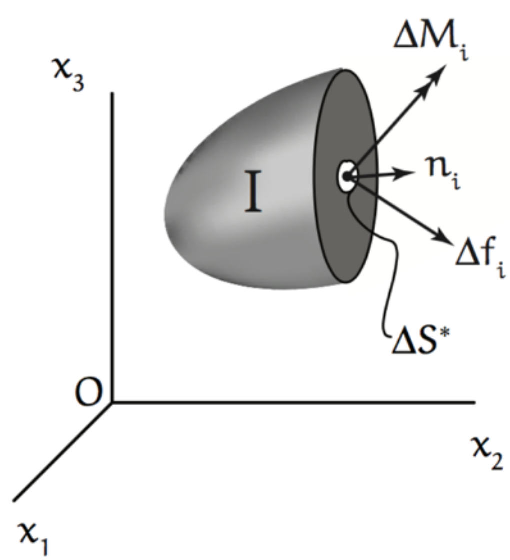
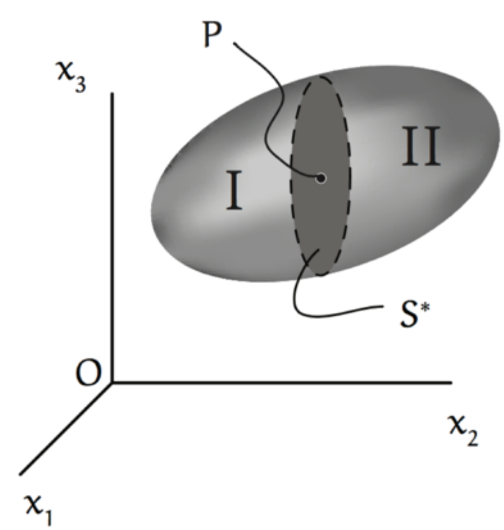
Class-prep questions for today

Tractions inside volumes

Imagine an arbitrary surface defined by its normal vector n_i inside a continuum under a stress σ_{ij}. The stress vector or traction vector $t_j^{(n)}$ expresses the force exerted on a unit area of that plane.

1) How would you find the traction vector on plane n_i?
2) How can you define the other side (the "back side") of that plane?
3) What might you expect the traction vector to be on that "back side" of the plane n_i?
4) What conservation law have you probably invoked in your answer to 3)?





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Prep for Monday class

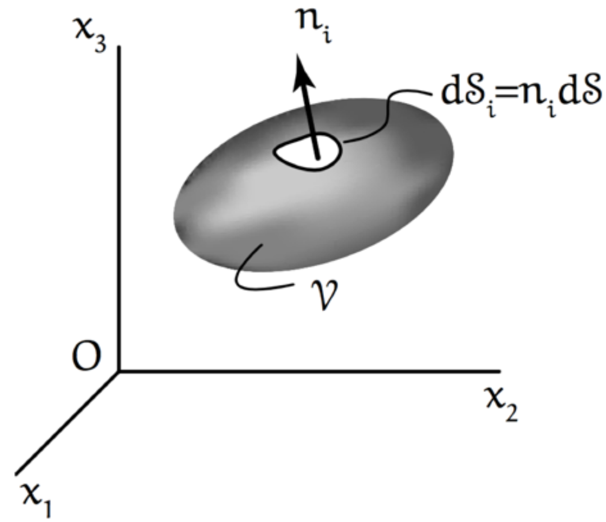
- Please read Mase, Smelser, and Mase, Ch 3 through Section 3.11

Traction vectors on principal planes, Mohr's circles

The traction vector $t_j^{(n)}$ expresses the force per unit area of the plane defined by n_i. Section 3.7 explains how to resolve the traction vector into its components s_N and s_S normal and parallel to the plane respectively, and shows how to find the maximum and minimum values of s_N and s_S.
With respect to faults at km scales, or layered rocks such as shales at the meter scale, describe how you think this might be useful.

Section 3.8 explains how to represent the state of stress at a point with Mohr's circles.
How do you think the Mohr's circle representation of stress t_{ij} might give potentially more insight than just the stress tensor t_{ij} itself?

Integral theorems



We may want to know what is going on inside a body but have access only to its surface (or vice versa)

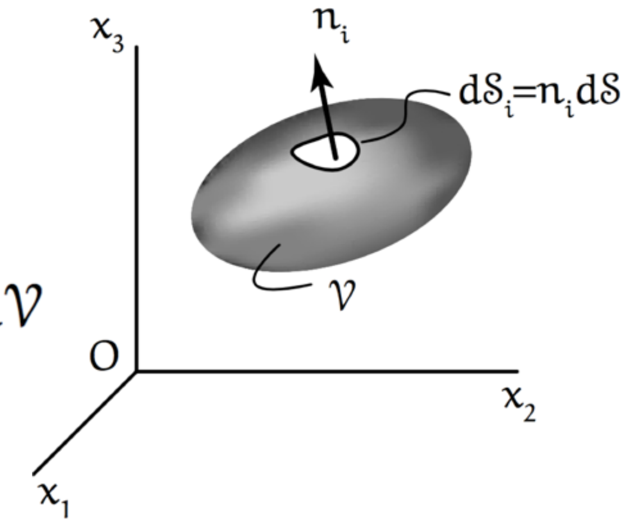
A volume V has surface S.

- Each small patch dS on the surface is defined by its normal vector n_i.

Divergence theorem $$\int_{\mathcal{S}} t_{ij\ldots k} n_q \, d\mathcal{S} = \int_{\mathcal{V}} t_{ij\ldots k,q} \, d\mathcal{V}$$

The total amount of $t_{ij\ldots k}$ directed out across S is the same as the total amount of spreading (divergence) everywhere inside V.

Special cases



Divergence theorem $\displaystyle\int_S t_{ij\ldots k} n_q \, dS = \int_{\mathcal{V}} t_{ij\ldots k,q} \, d\mathcal{V}$

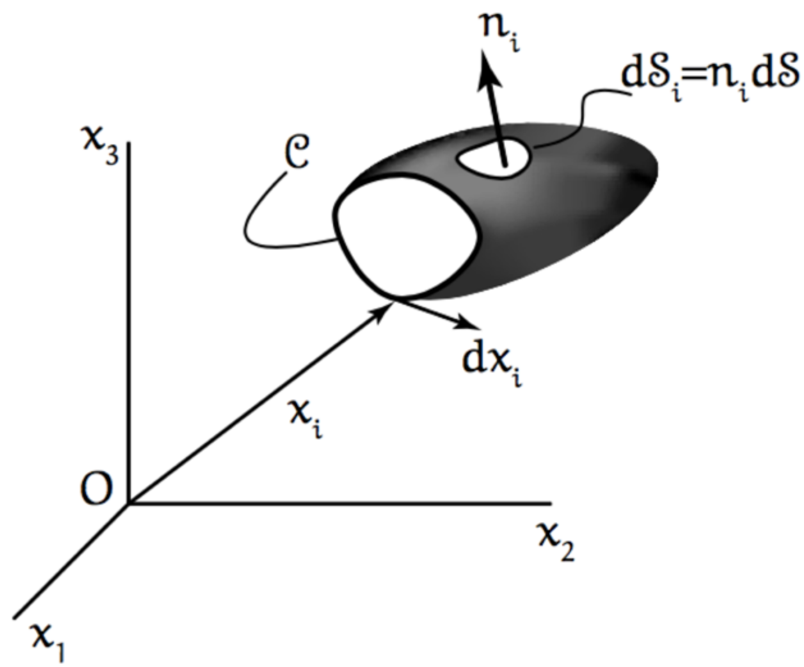
$$\int_S v_q n_q \, dS = \int_{\mathcal{V}} v_{q,q} \, d\mathcal{V} \quad \text{or} \quad \int_S \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \, dS = \int_{\mathcal{V}} \operatorname{div} \boldsymbol{v} \, d\mathcal{V}$$

If density ρ is uniform, the total amount of “stuff” flowing out across S with velocity *v* (the flux across S) is the same as the total amount of spreading (divergence) of that “stuff” everywhere inside V.

Stokes theorem

C is the perimeter of a cap on an open surface.

- d$\boldsymbol{x}$ is the tangent to the perimeter C.
- $\boldsymbol{v}$ is the material velocity.



$$\int_{\mathcal{S}} \varepsilon_{ijk} n_i v_{k,j} \, d\mathcal{S} = \int_{\mathcal{C}} v_k \, dx_k \quad \text{or} \quad \int_{\mathcal{S}} \hat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) \, d\mathcal{S} = \int_{\mathcal{C}} \boldsymbol{v} \cdot d\mathbf{x}$$

If density ρ is uniform, the total circulation of "stuff" (curl) within the cap ("churning") is equal to the net flow along the perimeter C ("the racetrack").

($\varepsilon_{ijk} v_{j,k}$ is curl of $\boldsymbol{v}$)

Definition of a tensor

In any rectangular coordinate system, a tensor is defined by 9 components that transform according to the rule

$$R'_{ij\ldots k} = a_{iq} a_{jm} \cdots a_{kn} R_{qm\ldots n}$$

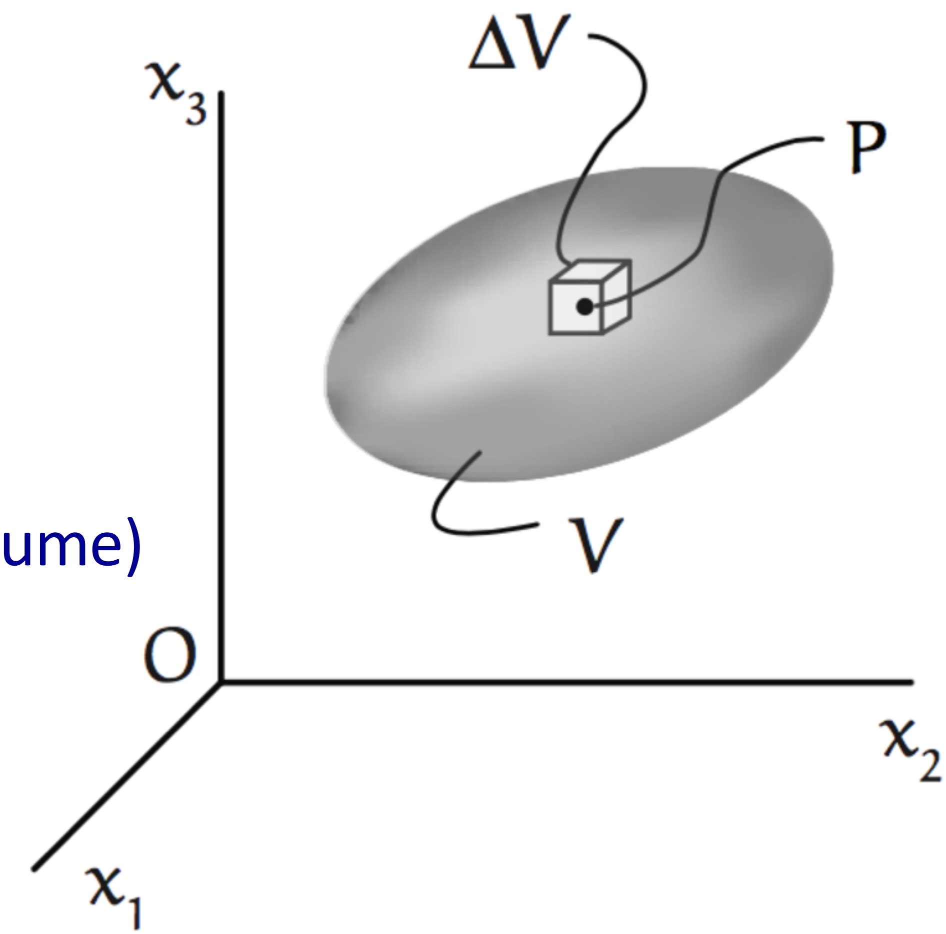
and where the basis vectors are related by

$$\hat{\mathbf{e}}'_i = a_{ij} \hat{\mathbf{e}}_j$$

or $$a_{ij} = \hat{e}'_i \cdot \hat{e}_j$$

Density ρ in a continuum

Δm is the mass in a small volume ΔV around the point P.



$\rho(\boldsymbol{x})$ = density (mass per unit volume)

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

Forces in a continuum

Body forces b_i force per mass (e.g. gravity)
Surface forces $t^{(n)}_i$ force per area or traction $\sigma_{ji}\, n_j$
(on exterior or interior surfaces)

Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$

In a continuum:

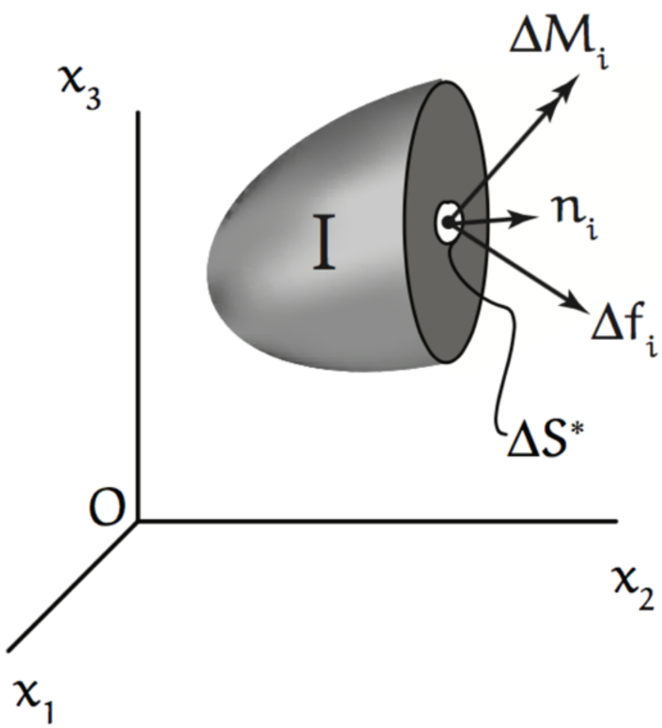
$$\int_V \rho(\vec{x}) b_i dV + \int_S t_i^{(\hat{n})} dS = \frac{d}{dt}\int_V \rho v_i dV$$

Traction and torque

f_i = force on ΔS^*

M_i = moment or torque on ΔS^*

(force x lever arm from center of ΔS^*)

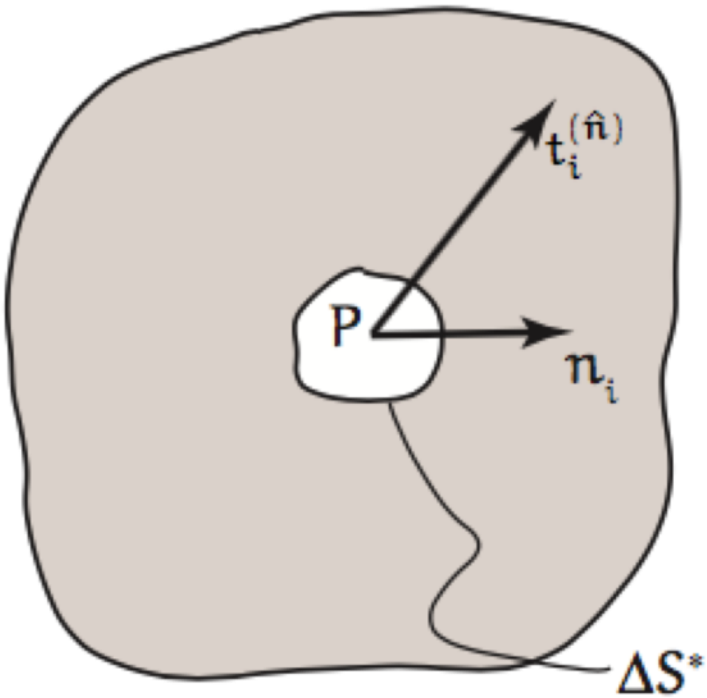


$$\lim_{\Delta S^* \to 0} \frac{\Delta f_i}{\Delta S^*} = \frac{df_i}{dS^*} = t_i^{(\hat{n})}$$

$$\lim_{\Delta S^* \to 0} \frac{\Delta M_i}{\Delta S^*} = 0 \,.$$

Traction

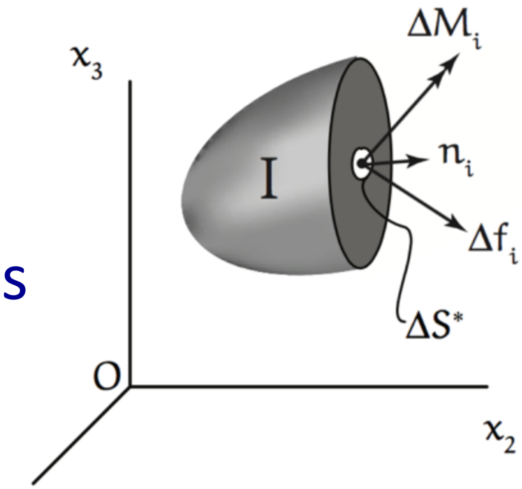
Traction vector acting at point P of plane element ΔS^* whose normal vector is n_i.



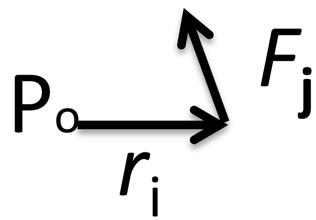
Torque

Torque causes material to spin

M_i = moment or torque on ΔS^* (force times lever arm from center of ΔS^*)



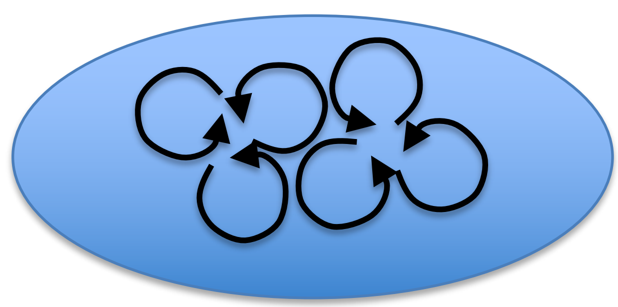
$$M_k = r_i \times F_j$$



$$\lim_{\Delta S^* \to 0} \frac{\Delta M_i}{\Delta S^*} = 0 \,.$$

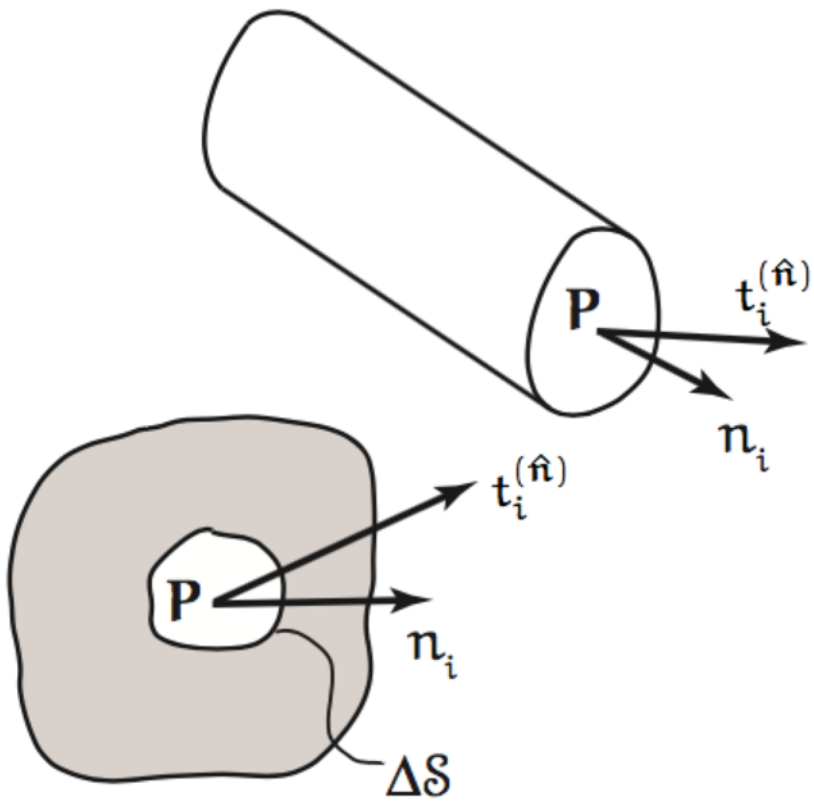
The torque or moment must go to zero because the lever arm must go to zero as ΔS^* gets smaller.

If it did not, the material would be churned and ripped apart internally …

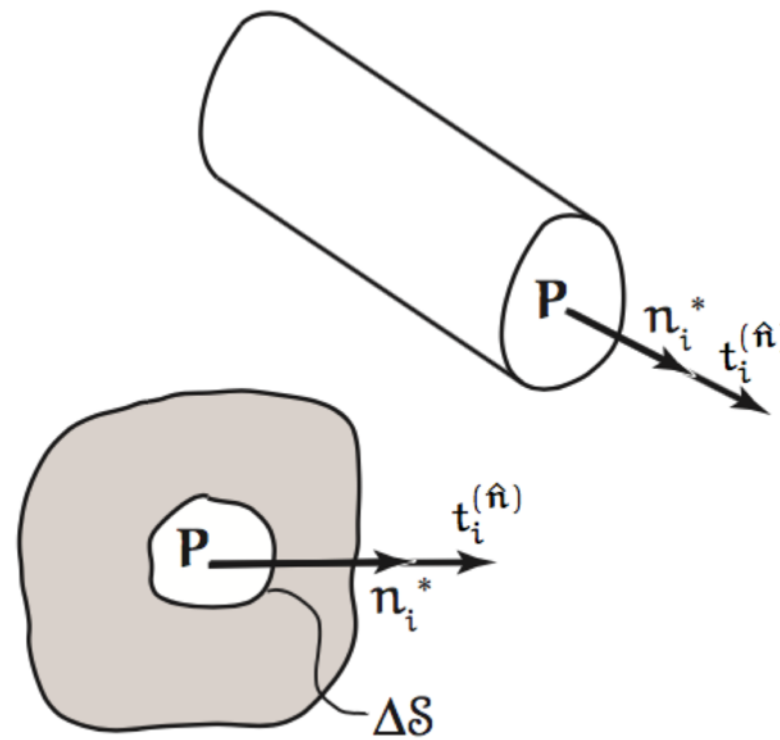


(This is also why the stress tensor must be symmetric.)

A stress tensor has principal coordinates in which the shear stress vanish, so that the traction vector on the principal planes is parallel to the normal vector



(a) Traction vector at point **P** for an arbitrary plane whose normal is n_i.

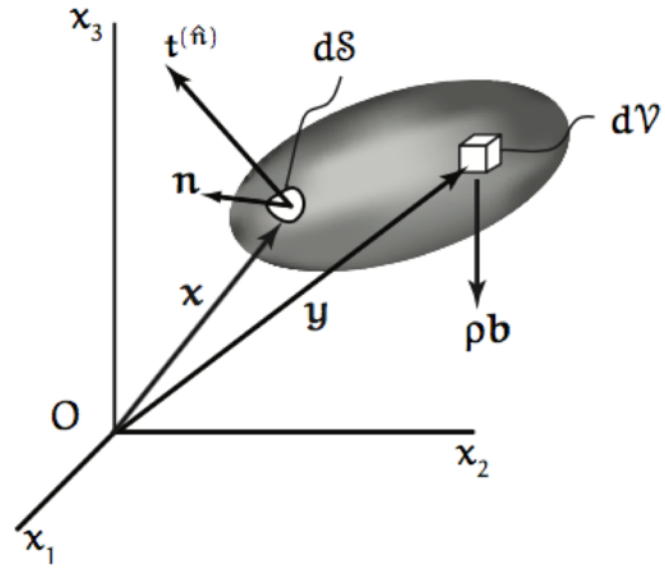


(b) Traction vector at point **P** for a principal plane whose normal is n_i^*.

Momentum Conservation Equation

Force equilibrium

$$\int_{\mathcal{S}} t_i^{(\hat{n})} \, d\mathcal{S} + \int_{\mathcal{V}} \rho b_i \, d\mathcal{V} = 0 \qquad (1)$$



Divergence theorem for first term

$$\int_{\mathcal{S}} t_{ji} n_j \, d\mathcal{S} = \int_{\mathcal{V}} t_{ji,j} \, d\mathcal{V}$$

Substitute in (1)

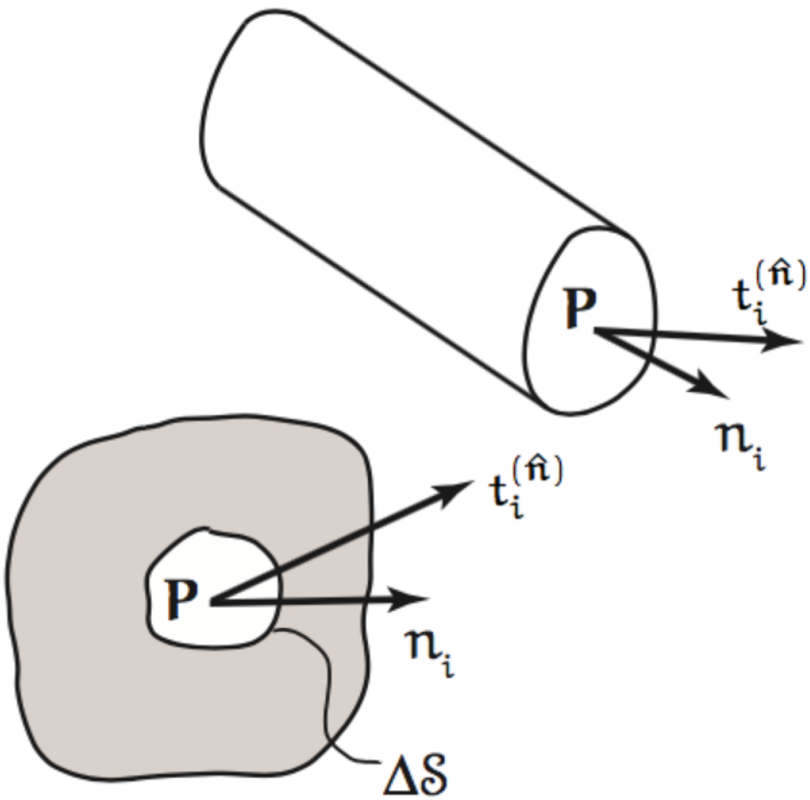
$$\int_{\mathcal{V}} \left(t_{ji,j} + \rho b_i \right) d\mathcal{V} = 0$$

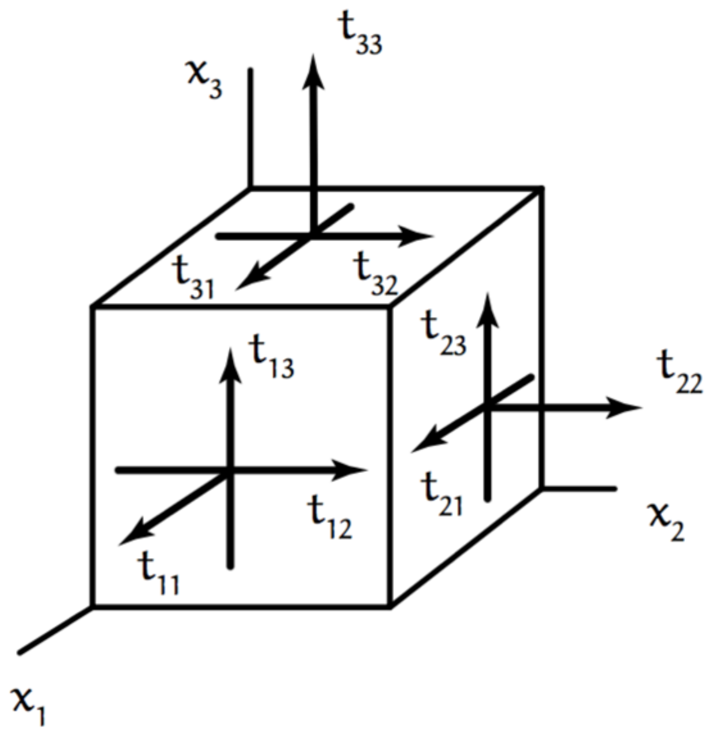
Volume *V* is arbitrary, so integrand must vanish for any *V*

$$t_{ji,j} + \rho b_i = 0$$

If a stress tensor ***is not*** expressed in its principal coordinates

- the traction vectors $\hat{t}_i^{(\hat{n})}$ on the coordinate planes may ***not*** be parallel to the coordinate vectors t_{11}^{*}, and
- off-diagonal elements of t_{ij} may be nonzero.

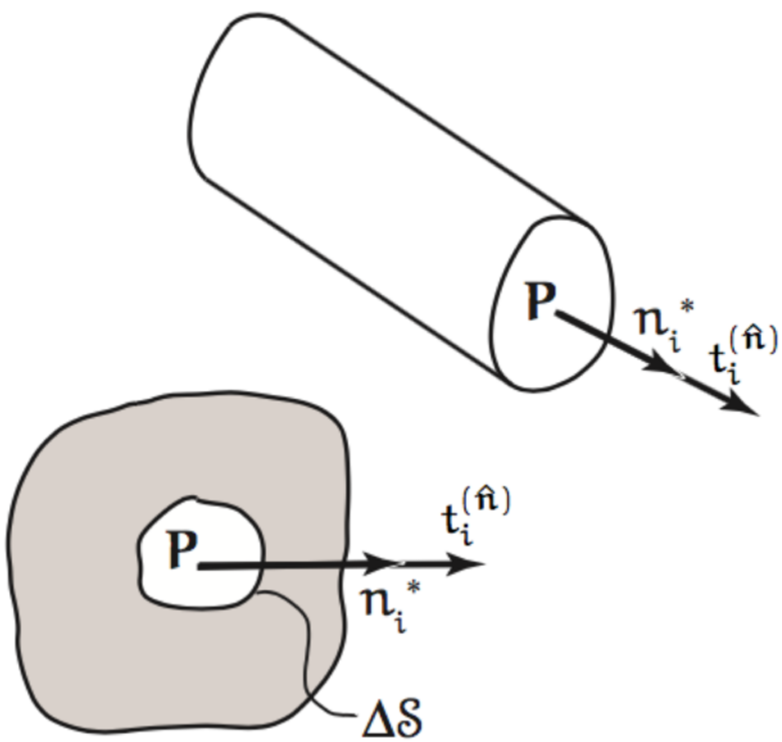


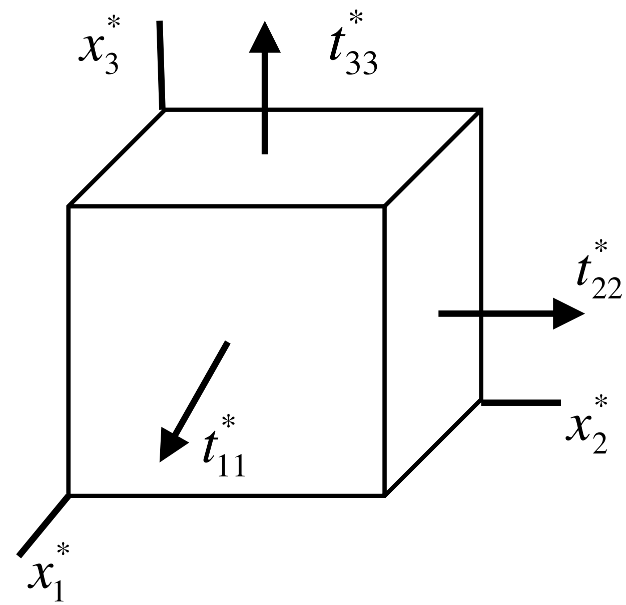


$$\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

If a stress tensor ***is*** expressed in its principal coordinates

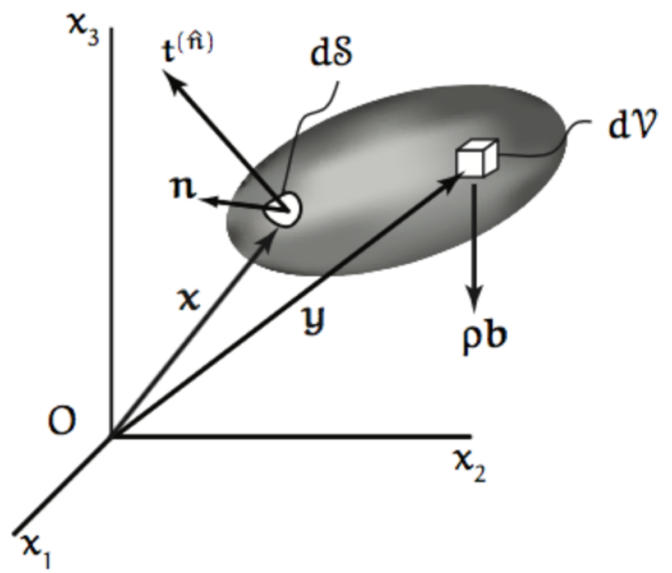
- the traction vectors $\hat{t}_i^{(\hat{n})}$ on the coordinate planes ***must*** be parallel to the coordinate vectors $\hat{n}_i^*$, and
- off-diagonal elements t^*_{ij} must be zero.





$$\begin{bmatrix} t^*_{11} & 0 & 0 \\ 0 & t^*_{22} & 0 \\ 0 & 0 & t^*_{33} \end{bmatrix}$$

Momentum Conservation Equation



Force equilibrium

$$\int_{\mathcal{S}} t_i^{(\hat{n})} d\mathcal{S} + \int_{\mathcal{V}} \rho b_i d\mathcal{V} = 0 \qquad (1)$$

Replace traction vector with stress tensor

$$t_i^{(\hat{n})} = t_{ji} n_j \qquad (2)$$

Apply Divergence Theorem (3) to change surface integral into volume integral

$$\int_{\mathcal{S}} t_{ji} n_j d\mathcal{S} = \int_{\mathcal{V}} t_{ji,j} d\mathcal{V} \qquad (3)$$

Collect volume terms

$$\int_{\mathcal{V}} (t_{ji,j} + \rho b_i)\, d\mathcal{V} = 0$$

Volume *V* is arbitrary, so integrand must vanish for any *V*

$$t_{ji,j} + \rho b_i = 0$$

Symmetry of stress tensor

As discussed in MSM, the stress tensor is symmetric because the moment on an infinitesimal surface element d*S* must go to zero. The text derives this more rigorously in Section 3.4.
The message to remember is that for stress -

$$t_{ij} = t_{ji}$$

The stress tensor can be reflected across its diagonal without changing.

Transformation laws for stress tensor

Section 3.5 goes over how to express the stress tensor (t_{ij} or σ_{ij}) in different coordinate systems.
This is mainly a repeat of earlier ideas in Chapter 2 about transforming any tensor.

Section 3.6 goes over how to find the principal values (eigenvalues) of the stress tensor, how to find the principal directions (eigenvectors), and how to find the 3 scalar invariants of the stress tensor.
This is also mainly a repeat of earlier ideas in Chapter 2 about transforming any tensor.

Notation

Principal stresses $\sigma_{I} > \sigma_{II} > \sigma_{III}$

$$[t^*_{ij}] = \begin{bmatrix} \sigma_{(1)} & 0 & 0 \\ 0 & \sigma_{(2)} & 0 \\ 0 & 0 & \sigma_{(3)} \end{bmatrix} \quad \text{or} \quad [t^*_{ij}] = \begin{bmatrix} \sigma_{I} & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

Conventions:

- Compressive stresses are negative
- Principal stresses are numbered from largest (most positive) to smallest
- Other conventions are also used in other texts and in research literature, but this convention is most versatile and correct in all situations

Notation

Scalar Invariants of the stress tensor

These are the coefficients in the cubic characteristic equation when solving for the eigenvalues

$$|t_{ij} - \lambda\delta_{ij}| = 0$$

$$\lambda^3 - I_{\mathbf{T}}\lambda^2 + II_{\mathbf{T}}\lambda - III_{\mathbf{T}} = 0$$

Trace $\quad I_{\mathbf{T}} = t_{ii} = \mathrm{tr}\,\mathbf{T}\ ,$

Second invariant $\quad II_{\mathbf{T}} = \frac{1}{2}\left[t_{ii}t_{jj} - t_{ij}t_{ij}\right] = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{T})^2 - \mathrm{tr}\,\mathbf{T}^2\right]$

Determinant $\quad III_{\mathbf{T}} = \varepsilon_{ijk}t_{1i}t_{2j}t_{3k} = \det\,\mathbf{T}\ .$