ESS 411/511 Geophysical Continuum Mechanics Class \#20
Highlights from Class \#19 - Yiyu Ni
Today's highlights on Monday - John-Morgan Manos

Kinematics of Deformation and Motion
For Wednesday, please read MSM Chapter 4.1 through 4.6
Kinematics of Deformation and Motion
For Friday, please read MSM Chapter 4.7 and 4.8

- Infinitesimal strain
- Strain compatibility

Also check out 4.11 and 4.12

- Velocity gradient and strain rate
- Material derivatives of lines, areas, and volumes


## Indoor Icequakes

Researchers at Penn State have reproduced stick/slip failure between ice and bedrock with $\sigma_{N}=500 \mathrm{kPa}$, like under a 50 m thick glacier. The peer-reviewed paper was in GRL about a year ago, and is titled Application of Constitutive Friction Laws to Glacier Seismicity

Here's the link:
https://doi.org/10.1029/2020GLO 88964


## Problem Set \#4

- Brad is working on it


## Mid-term

- I'm working on it ...


## ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools - vectors, tensors, coordinate changes
- Stress - principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain - Finite strain; infinitesimal strains
- Moments - lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves


## Some terms

Kinematics

- Description without reference to forces

Concept of particle in a continuum

- Just an infinitesimal point in the material, labeled with a vector field $\boldsymbol{X}$, with an associated density

Displacement

- Vector mapping of an object from initial $\boldsymbol{X}$ to final configuration $\boldsymbol{x}$

Deformation

- Change of shape described by a displacement field

Rigid-body rotation and translation

- No deformation, but displacement can differ from point to point

Strain or distortion

- Elongation or shear

Homogeneous deformation

- Initially straight material lines stay straight

Finite strain

- Material lines can become curved


## Initial and Final Configurations



Drawing A shows a fossil trilobite, a creature related to crabs and lobsters, that lived on the sea floor many millions of years ago.


Trilobites as strain gauges


B


C


D

Drawings B,C and D are of fossils of the same species of trilobite that were found in rocks that have been squashed and folded.

## Rates of change in a continuum

When the material is being tracked through time, it is convenient to use two sets of coordinates:

## Material

The material coordinates $X_{\mathrm{A}}$ are the initial positions of a material particle $X$ in a coordinate system $I_{\mathrm{A}}$

- Although particle $X$ may move over time, the place $X_{\mathrm{A}}$ where it started from doesn't ever change.
- The coordinates $X_{\mathrm{A}}$ act as a label identifying particle $X$, wherever it goes.


## Spatial

- The spatial coordinates $x_{\mathrm{i}}(X, t)$ mark the current position of a material particle $X$ in a coordinate system $\hat{e}_{i}$
o Conversely, $X\left(x_{i}, t\right)$ indicates which particle $X$ is occupying location $x_{i}$ at time $t$.


## Temporal Derivatives

As we saw with the traffic on I-5, there are two types of temporal derivatives of some quantity $\phi$ in a continuum.

- Rate of change of any property $\phi\left(x_{i}, t\right)$ at a fixed point $x_{i}$ in space, can be written as

$$
\begin{equation*}
\frac{\partial \phi\left(x_{i}, t\right)}{\partial t} \tag{1}
\end{equation*}
$$

The partial derivative symbol $\partial$ indicates that position $x_{i}$ is held constant.

- Rate of change of $\phi\left(X_{\mathrm{A}}, \mathrm{t}\right)$ for a particle $X_{\mathrm{A}}$ in the moving material, can be written as $\frac{D \phi\left(X_{A}, t\right)}{D t}$ or $\frac{d \phi\left(X_{A}, t\right)}{d t}$
where " $D$ " or " $d$ " indicate a "total" or "material-following" derivative. The identity $X_{A}$ of a particle isn't changing through time (Calvin and Hobbs transmogrification isn't allowed),
So (2) is a function of a single variable $t$, and

$$
\frac{d \phi\left(X_{A}, t\right)}{d t}=\frac{\partial \phi\left(X_{A}, t\right)}{\partial t}
$$

## Material Derivatives

In the material coordinate system $I_{A}$, rate of change of $\phi$ for particle $X_{\mathrm{A}}$ as it moves along its trajectory is relatively simple:

$$
\frac{d \phi\left(X_{A}, t\right)}{d t}=\frac{\partial \phi\left(X_{A}, t\right)}{\partial t}
$$

However, it gets uglier if we want to express the material-following derivative in the fixed coordinate system $\hat{e}_{i}$
The rate of change of $\phi$ for the particle currently at $x_{i}$ as it moves along its trajectory depends on two things:

1. The rate of change of $\phi$ seen by an observer at position $x_{i}$

$$
\frac{\partial \phi\left(x_{i}, t\right)}{\partial t}
$$

2. The rate of change at which the flow $v$ carries gradients of $\phi$ past position $x_{i}$, even though $\phi$ may not be changing on the particles

$$
-\frac{\partial \phi\left(x_{i}, t\right)}{\partial x_{k}} \frac{\partial x_{k}}{\partial t}, \quad \frac{\partial x_{k}}{\partial t}=v_{k}
$$

## Ways to change $\phi$ at a point $x_{i}$

No motion
e.g. material warming in place

$\frac{\partial \phi\left(X_{A}, t\right)}{\partial t}=\frac{\partial \phi\left(x_{i}, t\right)}{\partial t}$

Motion uniform and constant e.g. seabed elevation as a seamount is carried past a point by ocean-plate motion


$$
\frac{\partial \phi\left(x_{i}, t\right)}{\partial t}=-\frac{\partial \phi\left(x_{i}, t\right)}{\partial x_{k}} \frac{\partial x_{k}}{\partial t}, \quad \frac{\partial x_{k}}{\partial t}=v_{k}
$$

## Putting it all together

However, it gets uglier if we want to express the material-following derivative in the fixed coordinate system $\hat{e}_{i}$

In the spatial coordinate system $\hat{e}_{i}$, rate of change of $\phi$ for a particle $X_{\mathrm{A}}$ as it passes through $x_{i}$ :


Rate of change of $\phi$ seen by the of $\phi$ seen at $x_{i}$ moving particle $X_{A}$

Correction for changes carried in by flow, without $\phi$ actually changing on the particles

For example, for the seamount, the two terms must cancel each other, because we know that $\phi$, the topography of the seamount, is not changing. Other situations can be more complicated

## Displacement and Finite Strain

Any two nearby points $\mathbf{P}$ and $\mathbf{Q}$ in the initial configuration are moved to $\mathbf{p}$ and $\mathbf{q}$ in the final configuration.
The displacement of point $\mathbf{P}$ is $\boldsymbol{u}_{\mathbf{P}}=\mathbf{p}-\mathbf{P}$ The displacement of point $\mathbf{Q}$ is $\boldsymbol{u}_{\mathbf{Q}}=\mathbf{q}-\mathbf{Q}$ Or in general, $\boldsymbol{u}=\boldsymbol{x}-\mathbf{X}$
Because $\mathbf{Q}$ is close to $\mathbf{P}, \boldsymbol{u}$ can be expanded as a Taylor series around the point $\mathbf{P}$. Here are the first-order terms:

$$
u_{Q}=u_{P}+\frac{\partial u_{i}}{\partial X_{A}} d X_{A}
$$



This can be arranged to find du

$$
d u=\left(u_{Q}-u_{P}\right)=\frac{\partial u_{i}}{\partial X_{A}} d X_{A}
$$

The small line element $d X_{A}$ in the initial configuration also gets deformed into a different line element $\mathrm{d} x_{i}$.

## Displacement and Finite Strain

The small line element $d X_{A}$ in the initial configuration gets deformed into a different line element $\mathrm{d} x_{i}$, which can also be expressed by the first-order terms of a Taylor series

$$
d x_{i}=\frac{\partial x_{i}}{\partial X_{A}} d X_{A}
$$

The derivatives form the deformation gradient tensor $F_{i A}$


$$
F_{i A}=\frac{\partial x_{i}}{\partial X_{A}}=x_{i, A}
$$

The deformation is reversible, so $F_{i A}$ has an inverse

$$
\left(F_{i A}\right)^{-1}=\frac{\partial X_{A}}{\partial x_{i}}=X_{A, i}
$$

## Class-prep: deformation tensor $\mathrm{F}_{i A}$ (Break-outs)

Figure 4.2 in text MSM shows that an initial small line element $\mathrm{d} X_{\mathrm{A}}$ between points $\mathbf{P}$ and $\mathbf{Q}$ in a body becomes a small line element $\mathrm{d} x_{i}$ between points $\mathbf{p}$ and $\mathbf{q}$ after deformation.
The deformation gradient tensor $F_{i A}$ characterizes the deformation in the vicinity of $P$ and $Q$ by relating $\mathrm{d} x_{i}$ to $\mathrm{d} X_{A}$, e.g. as expressed in Equation 4.39

$$
d x_{i}=\frac{\partial x_{i}}{\partial X_{A}} d X_{A}=x_{i, A} X_{A}=F_{i A} d X_{A}
$$



## Assignment

Write the general form of the tensor $F_{i A}$ in the equation above as a $3 \times 3$ matrix.
Find the $3 \times 3$ matrix $\mathrm{F}_{i A}$ for the particular deformation field defined by

$$
\begin{aligned}
& x_{1}=x_{1} \\
& x_{2}=2 x_{3} \\
& x_{3}=-1 / 2 x_{2}
\end{aligned}
$$

Find the vector $\mathrm{d} x_{i}$ resulting from deformation of the column vector

$$
\mathrm{d} X_{\mathrm{A}}=[1,1,1]^{\top}
$$

and comment on the results in terms of rotation and stretching.

Reference line element $\mathrm{dX}_{\mathrm{A}}$

$$
d X_{A}=\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right] \quad\left\{\begin{array}{c}
x_{1}=X_{1} \\
x_{2}=2 X_{3} \\
x_{3}=-\frac{1}{2} X_{1}
\end{array}\right\}
$$

$$
\left[F_{i A}\right]=\left[\begin{array}{lll}
\frac{\partial x_{1}}{\partial X_{1}} & \frac{\partial x_{1}}{\partial X_{2}} & \frac{\partial x_{1}}{\partial X_{3}} \\
\frac{\partial x_{2}}{\partial X_{1}} & \frac{\partial x_{2}}{\partial X_{2}} & \frac{\partial x_{2}}{\partial X_{3}} \\
\frac{\partial x_{3}}{\partial X_{1}} & \frac{\partial x_{3}}{\partial X_{2}} & \frac{\partial x_{3}}{\partial X_{3}}
\end{array}\right]=\left[\begin{array}{ccc} 
& & \\
1 & 0 & 0 \\
0 & 0 & 2 \\
0 & -\frac{1}{2} & 0
\end{array}\right]
$$

Finding current line element $\mathrm{d} x_{i}$

$$
F_{i A} d X_{A}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 2 \\
0 & -\frac{1}{2} & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=\left[\begin{array}{c}
1 \\
2 \\
-\frac{1}{2}
\end{array}\right]
$$

Luckily for us, there is no change in the $\mathrm{X}_{1}$ direction ©


A measure for strain $(\mathrm{d} x)^{2}-(\mathrm{dX})^{2}$

$$
\begin{aligned}
(\mathrm{d} x)^{2}-(\mathrm{dX})^{2} & =\left(x_{i, A} d X_{A}\right)\left(x_{i, B} d X_{B}\right)-\delta_{A B} d X_{A} d X_{B} \\
& =\left(x_{i, A} x_{i, B}-\delta_{A B}\right) \mathrm{d} X_{A} d X_{B} \\
& =\left(C_{A B}-\delta_{A B}\right) \mathrm{d} X_{A} d X_{B}
\end{aligned}
$$

Green's deformation tensor

$$
C_{A B}=x_{i, A} x_{i, B} \quad \text { or } \quad \mathbf{C}=\mathbf{F}^{\top} \cdot \mathbf{F}
$$

Lagrangian finite strain tensor

$$
2 E_{A B}=C_{A B}-\delta_{A B} \quad \text { or } \quad 2 E=C-I
$$

A measure for strain $(\mathrm{d} x)^{2}-(\mathrm{dX})^{2}$

$$
\begin{aligned}
(\mathrm{d} x)^{2}-(\mathrm{d} X)^{2} & =\delta_{i j} \mathrm{~d} x_{i} \mathrm{~d} x_{j}-\left(X_{A, i} \mathrm{~d} x_{i}\right)\left(X_{A, j} \mathrm{~d} x_{j}\right) \\
& =\left(\delta_{i j}-X_{A, i} x_{A, j}\right) \mathrm{d} x_{i} d x_{j} \\
& =\left(\delta_{i j}-c_{i j}\right) d x_{i} d x_{j}
\end{aligned}
$$

Cauchy deformation tensor

$$
c_{i j}=X_{A, i} X_{A, j} \quad \text { or } \quad \mathbf{c}=\left(\mathbf{F}^{-1}\right)^{\top} \cdot\left(\mathbf{F}^{-1}\right)
$$

Eulerian finite strain tensor

$$
2 e_{i j}=\left(\delta_{i j}-c_{i j}\right) \quad \text { or } \quad 2 \boldsymbol{e}=(\mathbf{I}-\mathbf{c})
$$

In terms of displacements $u_{i}=\left(x_{i}-X_{i}\right)$

$$
\begin{aligned}
& 2 E_{A B}=x_{i, A} x_{i, B}-\delta_{A B}=\left(u_{i, A}+\delta_{i A}\right)\left(u_{i, B}+\delta_{i B}\right)-\delta_{A B} \\
& 2 E_{A B}=u_{A, B}+u_{B, A}+u_{i, A} u_{i, B} \\
& 2 e_{i j}=\delta_{i j}-X_{A, i} X_{A, j}=\delta_{i j}-\left(\delta_{A i}-u_{A, i}\right)\left(\delta_{A j}-u_{A, j}\right) \\
& 2 e_{i j}=u_{i, j}+u_{j, i}-u_{A, i} u_{A, j}
\end{aligned}
$$

