ESS 411/511 Geophysical Continuum Mechanics Class #21

Highlights from Class #20 – John-Morgan Manos
Today's highlights on Friday – Alysa Fintel

Kinematics of Deformation and Motion

For Friday, please read MSM Chapter 4.11 and 4.12

- Velocity gradient and strain rate
- Material derivatives of lines areas, and volumes

Problem Set #4

- Brad is working on it ☺

Mid-term

- I’m working on it …

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Class-prep for Class 22 on Friday: Strain Compatibility

For small strains, the strain is defined (Equation 4.62) as

$$2\epsilon_{ij} = \frac{\partial u_i}{\partial X_A}\delta_{Aj} + \frac{\partial u_j}{\partial X_B}\delta_{Bi} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = u_{i,j} + u_{j,i}$$

where $u_i = x_i - X_A\,\delta_{iA}$ is displacement, the difference between current and initial positions. (The Kroenecker delta δ_{iA} just makes it clear how to relate the "A" subscripts in the initial coordinate system to the "i" subscripts in the current coordinate system.)

Section 4.8 describes strain compatibility, and the 81 strain-compatibility equations that relate the various second derivatives.

$$\epsilon_{ij,km} + \epsilon_{km,ij} - \epsilon_{ik,jm} - \epsilon_{jm,ik} = 0$$

Assignment

81 equations– that's a lot of equations!

- Why are only 6 of them needed in practice?
- Compatibility with what? What would *strain incompatibility* look like?

On page 129 – "*It may be shown that the compatibility equations, either Eq 4.90 or Eq 4.91, are both necessary and sufficient for a single-valued displacement f*ield of a body occupying a simply connected domain."

- How would you paraphrase this sentence into simple concepts?

A Lagrangian measure for strain $(dx)^2 - (dX)^2$

$$
\begin{aligned}
(dx)^2 - (dX)^2 &= (x_{i,A}\, dX_A)(x_{i,B}\, dX_B) - \delta_{AB}\, dX_A\, dX_B \\
&= (x_{i,A} x_{i,B} - \delta_{AB})\, dX_A\, dX_B \\
&= (C_{AB} - \delta_{AB})\, dX_A\, dX_B
\end{aligned}
$$

Green's deformation tensor

$$C_{AB} = x_{i,A} x_{i,B} \qquad \text{or} \qquad \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$$

Lagrangian finite strain tensor

$$2E_{AB} = C_{AB} - \delta_{AB} \qquad \text{or} \qquad 2\mathbf{E} = \mathbf{C} - \mathbf{I}$$

- Is C_{AB} symmetric?
- Is E_{AB} symmetric?

An Eulerian measure for strain $(\mathrm{d}x)^2 - (\mathrm{d}X)^2$

$$
\begin{aligned}
(\mathrm{d}x)^2 - (\mathrm{d}X)^2 &= \delta_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j - (X_{A,i}\,\mathrm{d}x_i)(X_{A,j}\,\mathrm{d}x_j) \\
&= (\delta_{ij} - X_{A,i}X_{A,j})\,\mathrm{d}x_i\,\mathrm{d}x_j \\
&= (\delta_{ij} - c_{ij})\,\mathrm{d}x_i\,\mathrm{d}x_j
\end{aligned}
$$

Cauchy deformation tensor

$$
c_{ij} = X_{A,i}X_{A,j} \qquad \text{or} \qquad \mathbf{c} = (\mathbf{F}^{-1})^{\mathsf{T}} \cdot (\mathbf{F}^{-1})
$$

Eulerian finite strain tensor

$$
2e_{ij} = (\delta_{ij} - c_{ij}) \qquad \text{or} \qquad 2\mathbf{e} = (\mathbf{I} - \mathbf{c})
$$

- Is c_{ij} symmetric?
- Is e_{ij} symmetric?

In terms of displacements $u_i = (x_i - X_i)$

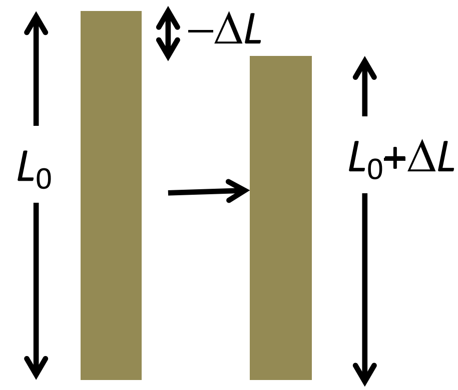
$$2E_{AB} = x_{i,A}x_{i,B} - \delta_{AB} = (u_{i,A} + \delta_{iA})(u_{i,B} + \delta_{iB}) - \delta_{AB}$$

$$2E_{AB} = u_{A,B} + u_{B,A} + u_{i,A}u_{i,B}$$

$$2e_{ij} = \delta_{ij} - X_{A,i}X_{A,j} = \delta_{ij} - (\delta_{Ai} - u_{A,i})(\delta_{Aj} - u_{A,j})$$

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{A,i}u_{A,j}$$

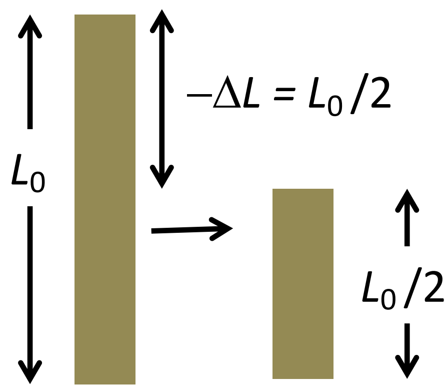
Finite Strain in 1-D



We typically first see strain expressed as $\varepsilon = \dfrac{\Delta L}{L_0}$ and when there is shortening, $\Delta L < 0$

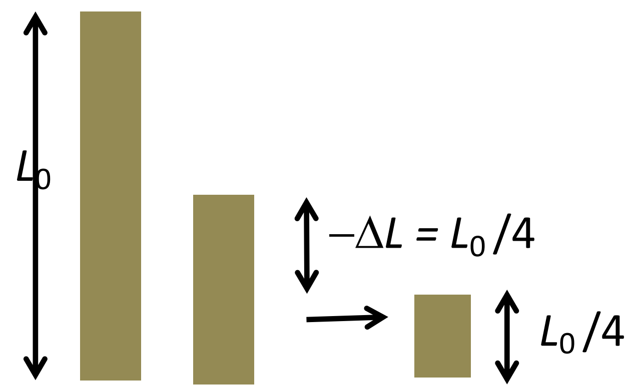
But this approach is not accurate for large strains e.g. when $-\Delta L$ is large fraction of L_0. For example, break a big change into 2 smaller changes:

Step 1



$$\varepsilon = \frac{\Delta L}{L_0} = \frac{-L_0/2}{L_0} = -\tfrac{1}{2}$$

Step 2



$$\varepsilon = \frac{\Delta L}{L_0} = \frac{-L_0/4}{L_0} = -\tfrac{1}{4}$$

But in both steps the height was halved, so the incremental strain should be the same …

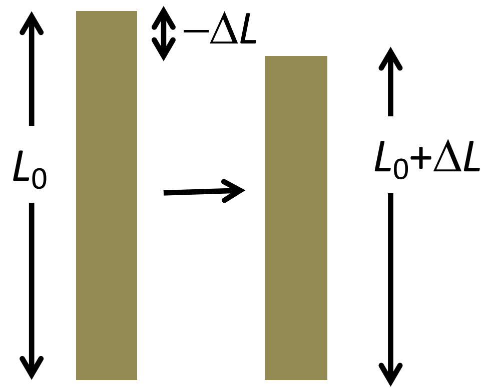
Finite Strain in 1-D

when $-\Delta L$ is large fraction of L_0, and L is the final length, we can address large strains by adding up a whole series of small strains.

- We have to reset L_0 at each step and call it l

$$\varepsilon = \frac{\Delta L}{L_0}$$

?



$$\varepsilon = \int_{L_0}^{L} \frac{dl}{l} = \ln(L) - \ln(L_o) = \ln\left(L/L_o \right)$$

$$\varepsilon = \ln\left(L/L_o \right) = \ln\left(\frac{L_0 + \Delta L}{L_0} \right) = \ln\left(1 + \frac{\Delta L}{L_0} \right)$$

(remembering that $\Delta L<0$)

Now recall the Taylor series for $\ln(1+x)$ when $-1< x < 1$

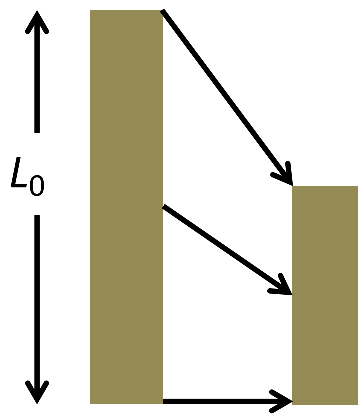
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

With $x= (\Delta L / L_O)$,

$$\varepsilon = \ln\left(1 + \frac{\Delta L}{L_0} \right)$$

$$= \frac{\Delta L}{L_0} - \frac{1}{2}\left(\frac{\Delta L}{L_0} \right)^2 + \text{some small stuff}$$

- The first term looks familiar …
- The second term corrects for the changing l as strain proceeds



A bar is shortened uniformly in the x_1 direction

$x_1 = kX_1$, e.g. $k = 1/2$

$$k = \left(x_1 \big/ X_1 \right) = \left(\frac{L_0 + \Delta L}{L_0} \right)$$

$$= \left(1 + \frac{\Delta L}{L_0} \right)$$

Eulerian Finite Strain Tensor

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{A,i} u_{A,j} \quad (**)$$

Displacement field
(i=j=A=1)

$$u_1 = x_1 - X_1 = x_1 - \left(x_1 \big/ k \right)$$

$$= \left(1 - 1 \big/ k \right) x_1$$

(particles move down so $u_1 < 0$)

Strain field

$$2e_{11} = 2\frac{\partial u_1}{\partial x_1} - \left(\frac{\partial u_1}{\partial x_1}\right)^2 \qquad \text{from } (**)$$

$$= 2(1 - {}^{1}/_{k}) - (1 - {}^{1}/_{k})^2$$

and

$$\left(1 - 1 \big/ k \right) = \left(1 - \frac{L_0}{L_0 + \Delta L} \right) = \frac{\Delta L}{L_0 + \Delta L}$$

so

$$2e_{11} = 2\left(\frac{\Delta L}{\Delta L + L_0}\right) - \left(\frac{\Delta L}{\Delta L + L_0}\right)^2$$

Let’s compare …

Traditional 1-D logarithmic finite strain

$$\varepsilon = \ln\left(1 + \frac{\Delta L}{L_0}\right)$$

$$= \frac{\Delta L}{L_0} - \frac{1}{2}\left(\frac{\Delta L}{L_0}\right)^2 + \text{some small stuff}$$

Eulerian finite strain tensor in 1-D

$$e_{11} = \left(\frac{\Delta L}{\Delta L + L_0}\right) - \frac{1}{2}\left(\frac{\Delta L}{\Delta L + L_0}\right)^2$$

- They are clearly very similar for small ΔL
- Both show a quadratic correction for larger strains

Example of a strain – is it small?

When an s-wave from a major Kamchatka quake arrives at Seattle, the ground is strained by the passing wave. To estimate the strain, we need to estimate

- Displacement amplitude
- Wavelength (think about frequency of vibration and typical wave propagation speed)

- What is your estimate for the strain?
- Would you need finite strain theory to analyze this?

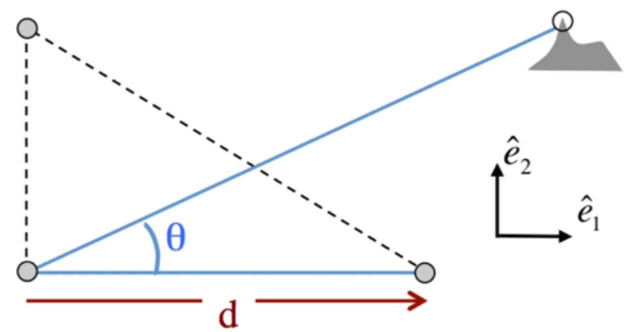
Class-prep: Moving Magma (Break-outs)

For small displacements and strains, the strain tensor can be written as (Eq. 4.72)

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \epsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \epsilon_{33} \end{bmatrix}$$

Magma is on the move at Mt Baker ski area

- You are a surveyor with a "total station" (theodolite and EDM) to measure angles and distances.
- Stress τ_{ij} is related to strain ε_{ij} by $\tau_{ij} = k\, \varepsilon_{ij}$ (k is an elastic-strength parameter)
- you have 3 benchmarks (survey stations) arranged as a right-angled triangle.



Assignment

(1) Understanding the strain tensor

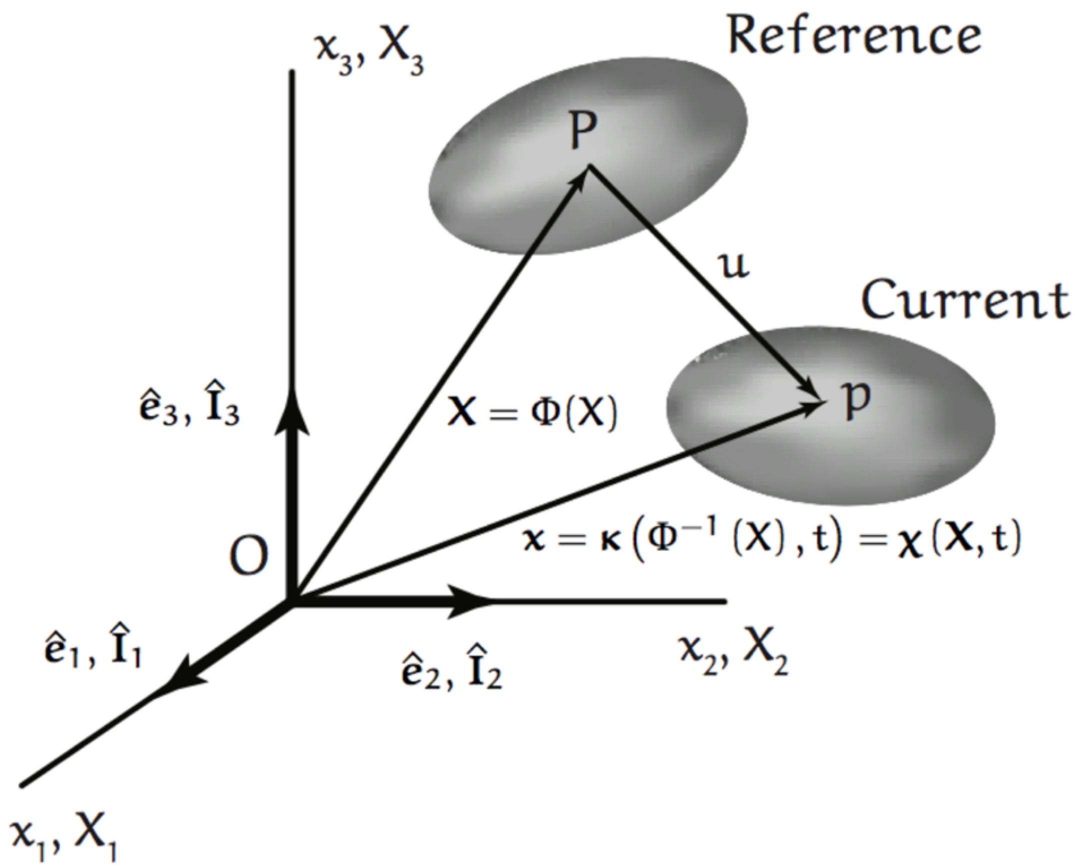
- Physical meaning of the diagonal entries?
- Physical meaning of the off-diagonal entries?

(2) The strain tensor and the stress tensor

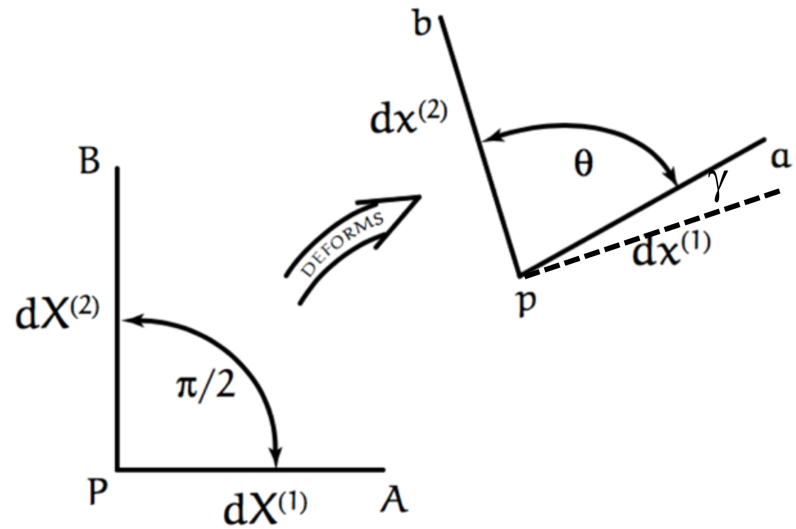
You have 1 month to determine the strain rate and stress in the ground from multiple surveys using your 3 survey benchmarks and the mountain peak.

- What measurements will you plan to make over the one-month period, to derive the strain tensor from your data?
- How will you calculate the stress tensor?
- How will you determine the orientation of a potential fissure?

Initial and Final Configurations



Small strain entries



$$(dx)^2 - (dX)^2 = 2\epsilon_{ij} dX_i dX_j$$

$$\frac{dx - dX}{dX} \cdot \frac{dx + dX}{dX} = 2\epsilon_{ij} \frac{dX_i}{dX} \frac{dX_j}{dX}$$

$$dX_i / dX = N_i$$

a unit vector in the direction of d**X**

$$(dx + dX)/dX \approx 2$$

$$\frac{dx - dX}{dX} = \epsilon_{ij} N_i N_j$$

change in length per unit original length for the element in the direction of **N**, called longitudinal strain.

If $N = \hat{I}_1$, then $\hat{\mathbf{I}}_1 \cdot \boldsymbol{\epsilon} \cdot \hat{\mathbf{I}}_1 = \epsilon_{11}$

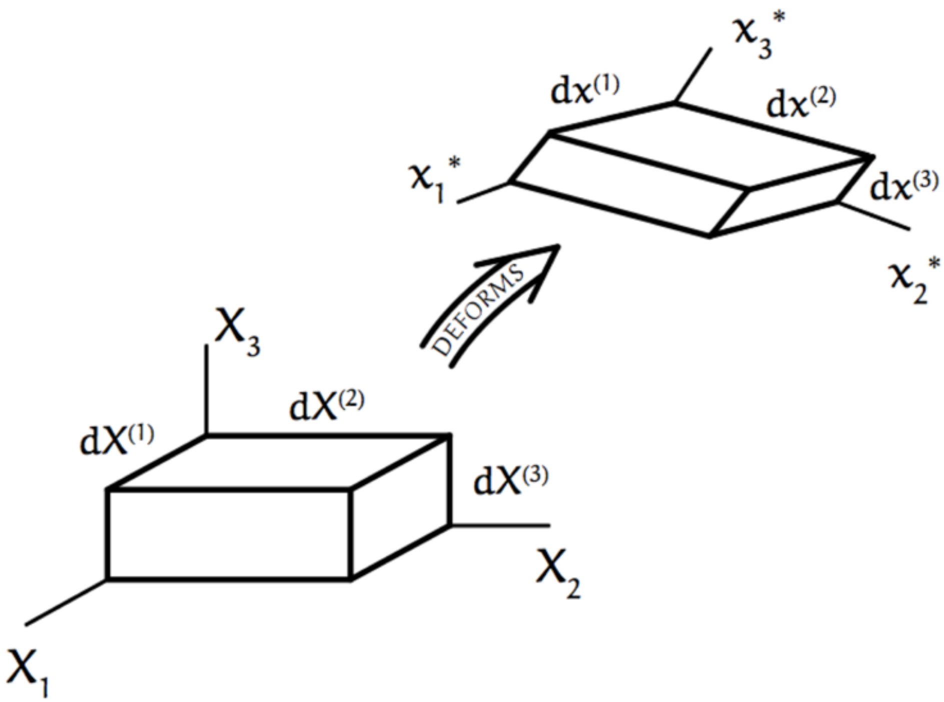
$$\cos\theta = \cos\left(\frac{\pi}{2} - \gamma\right) = \sin\gamma \approx \gamma$$

$$d\mathbf{x}^{(1)} \approx d\mathbf{X}^{(1)} \text{ and } d\mathbf{x}^{(2)} \approx d\mathbf{X}^{(2)}$$

$$\gamma \approx \cos\theta = \frac{d\mathbf{X}^{(1)}}{dx^{(1)}} \cdot 2\boldsymbol{\epsilon} \cdot \frac{d\mathbf{X}^{(2)}}{dx^{(2)}} \approx \hat{\mathbf{N}}_{(1)} \cdot 2\boldsymbol{\epsilon} \cdot \hat{\mathbf{N}}_{(2)}$$

If we set $\hat{\mathbf{N}}_{(1)} = \hat{\mathbf{I}}_1$ and $\hat{\mathbf{N}}_{(2)} = \hat{\mathbf{I}}_2$

$$\gamma_{12} = 2\,[1,0,0] \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2\epsilon_{12}$$


x_3^*
$dx^{(1)}$
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x_1^*
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DEFORMS
X_3
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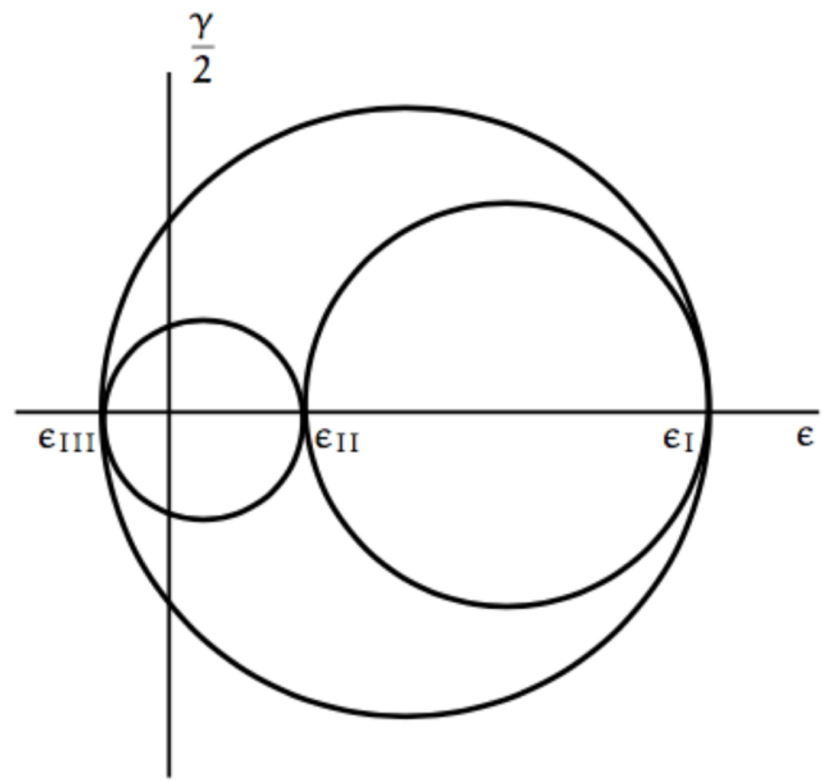
$\frac{\gamma}{2}$
ϵ_{III}
ϵ_{II}
ϵ_{I}
ϵ

Deviatoric stress

Mean stress … $$\sigma_M = \frac{1}{3}(t_{11} + t_{22} + t_{33}) = \frac{1}{3}t_{ii}$$

(How is this related to pressure?)

Spherical stress … $$[t_{ij}] = \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix}$$

(Where might you find a state of stress like this?)

Decomposition … $$t_{ij} = S_{ij} + \delta_{ij}\sigma_M = S_{ij} + \frac{1}{3}\delta_{ij}t_{kk}$$

Deviatoric stress … $$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} = \begin{bmatrix} t_{11} - \sigma_M & t_{12} & t_{13} \\ t_{12} & t_{22} - \sigma_M & t_{23} \\ t_{13} & t_{23} & t_{33} - \sigma_M \end{bmatrix}$$

What sort of materials would best be described by deviatoric stress?

Deviatoric strain

Just as with deviatoric stress and mean stress …

we can define a mean normal strain $\varepsilon_M = \frac{1}{3}\varepsilon_{ii}$

and an infinitesimal deviatoric strain tensor η_{ij}

$$\eta_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk} = \epsilon_{ij} - \delta_{ij}\epsilon_M \ ,$$

and in matrix form

$$\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} - \epsilon_M & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} - \epsilon_M & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} - \epsilon_{33} \end{bmatrix}$$

Where would deviatoric stress be a useful concept?