ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

ESS 411/511 Geophysical Continuum Mechanics Class #26

For next class, please Read (*https://courses.washington.edu/ess511/NOTES/*)

- Ed's note on volume elements
- Ed's note on conservation laws
- Ed's note on constitutive relations
- **Raymond notes on stress and moments**
- Turcotte and Schubert Section 3.9

- ESS 511 60-second project updates on Friday

Class-prep: Strain Compatibility (Break-out rooms)

For small strains, the strain is defined (Equation 4.62) as

$$2\epsilon_{ij} = \frac{\partial u_i}{\partial X_A}\delta_{Aj} + \frac{\partial u_j}{\partial X_B}\delta_{Bi} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = u_{i,j} + u_{j,i}$$

where $u_i = x_i - X_A\, d_{iA}$ is displacement, the difference between current and initial positions. (The Kroenecker delta just makes it clear how to relate the "*A*" subscripts in the initial coordinate system to the "i" subscripts in the current coordinate system.)

Section 4.8 describes strain compatibility, and the 81 strain-compatibility equations that relate the various second derivatives.

$$\epsilon_{ij,km} + \epsilon_{km,ij} - \epsilon_{ik,jm} - \epsilon_{jm,ik} = 0$$

Assignment

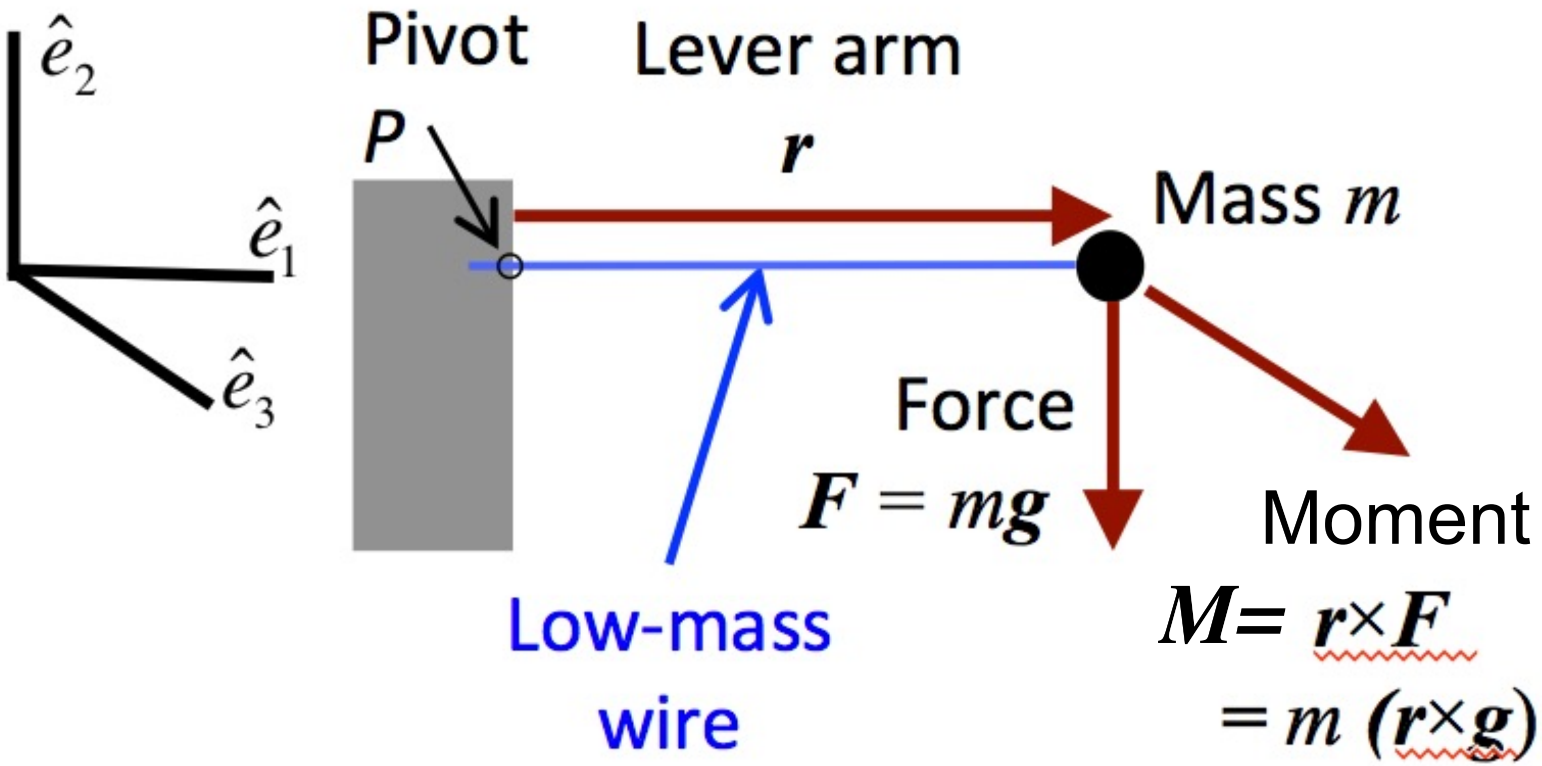
81 equations– that's a lot of equations!

- Why are only 6 of them needed in practice?
- Compatibility with what? What would *strain incompatibility* look like?

On page 129 – "*It may be shown that the compatibility equations, either Eq 4.90 or Eq 4.91, are both necessary and sufficient for a single-valued displacement fi*eld of a body occupying a simply connected domain."

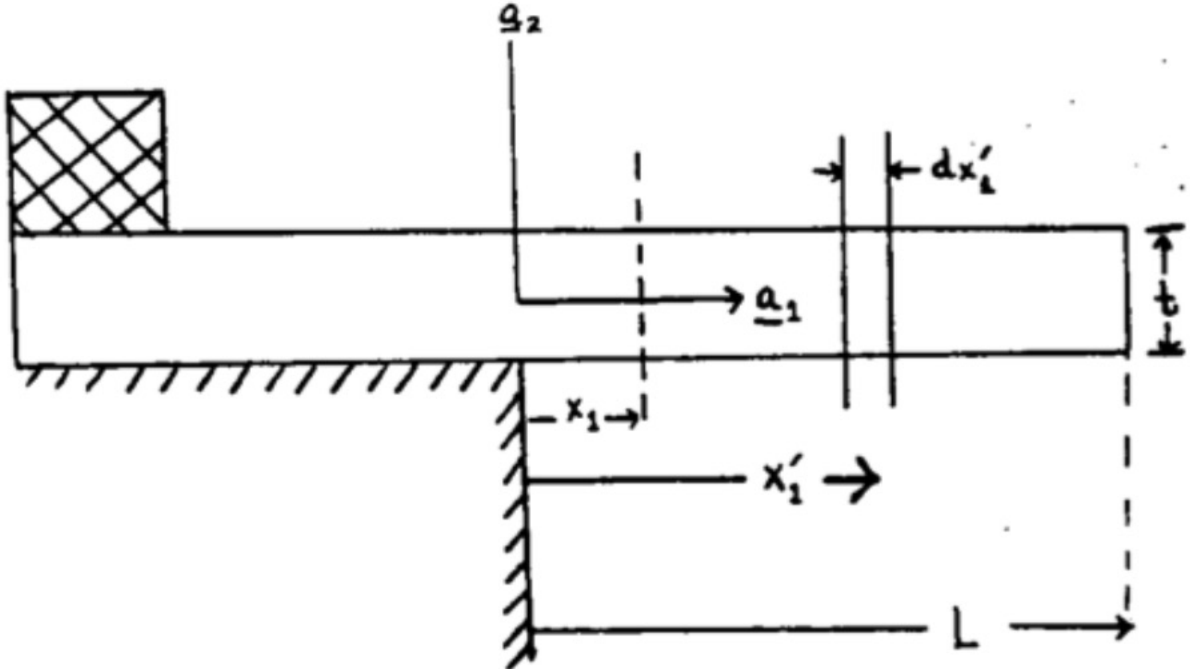
- How would you paraphrase this sentence into simple concepts?

Please give me a moment …

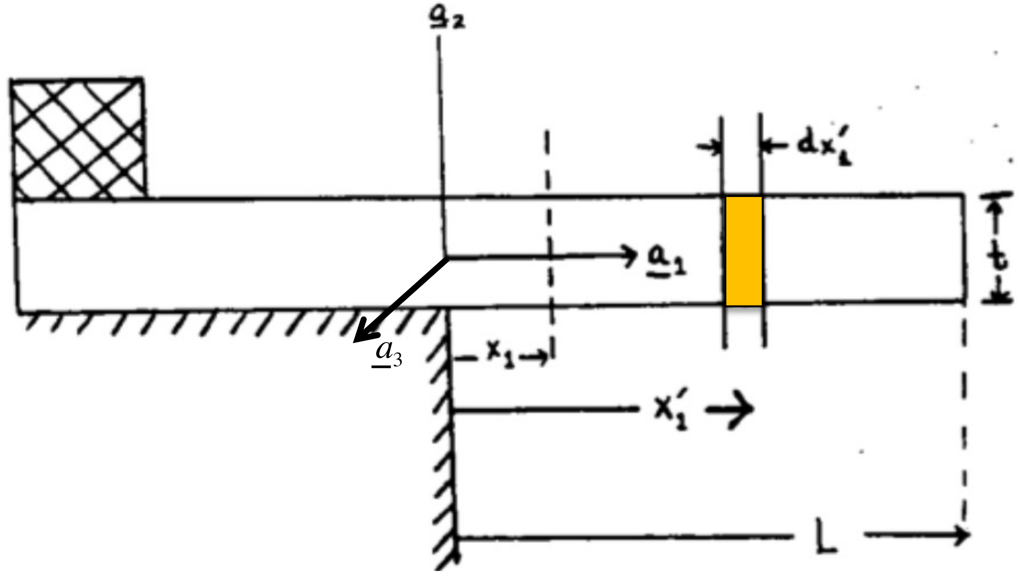


- This figure illustrates the moment exerted around the point P by a point mass *m* on the end of a stiff low-weight wire and subjected to gravity.
- "Moment" and torque" are often used interchangeably, although technically a moment is a static concept, and torque applies to a rotation motion.
- Both involve a vector cross product between a force and a lever arm (or *moment arm*), so the units are Newton meters (N m).

A hanging plate



Average stress across the beam (per unit width)



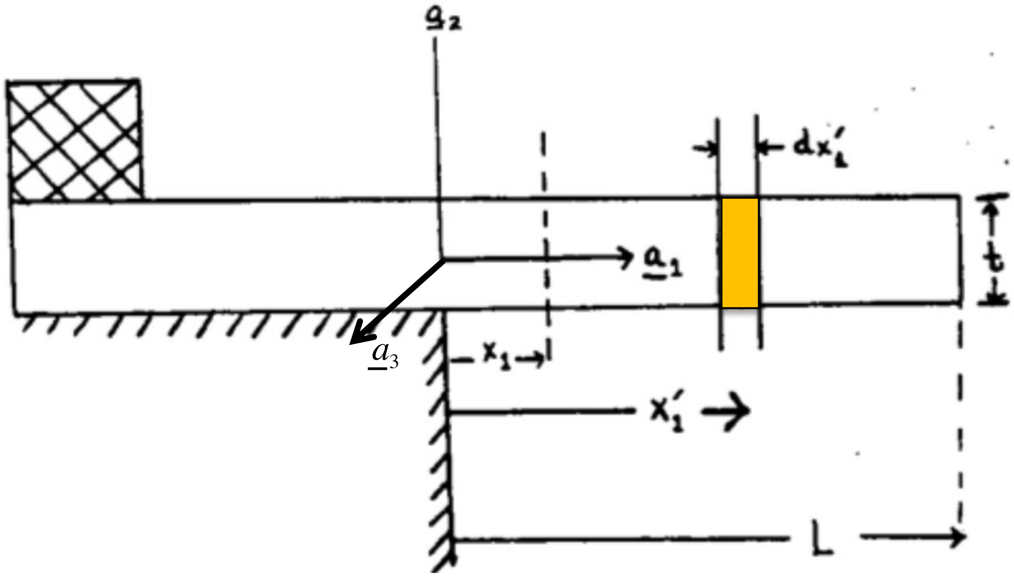
$<\sigma_{11}>t =$?

$<\sigma_{12}>t =$?

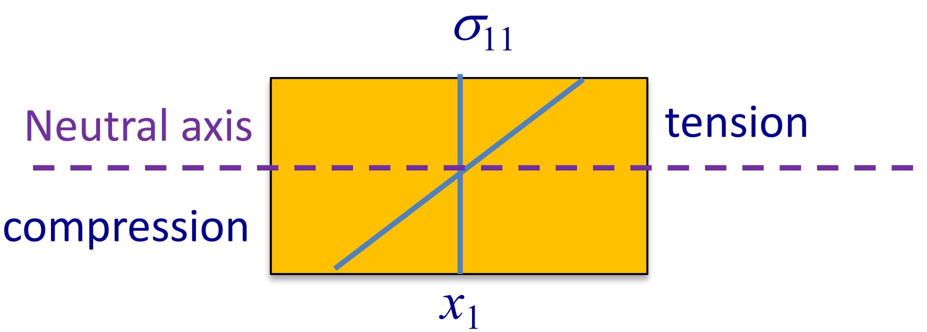
$<\sigma_{13}>t =$?

There are many ways to analyze this.
Let's examine force balance on a small
Element of the beam.

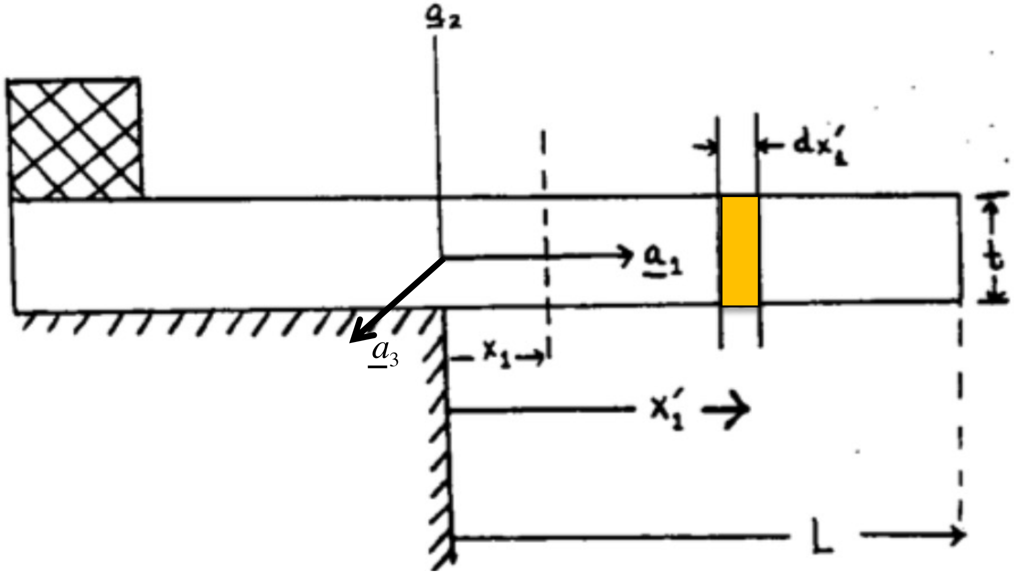
Average σ_{11} in the beam



Although $\langle\sigma_{11}\rangle=0$, there must be:
- tension ($\sigma_{11}>0$) in the upper part and
- compression ($\sigma_{11}<0$) in the lower part, in order to prevent the material to the right from falling down.

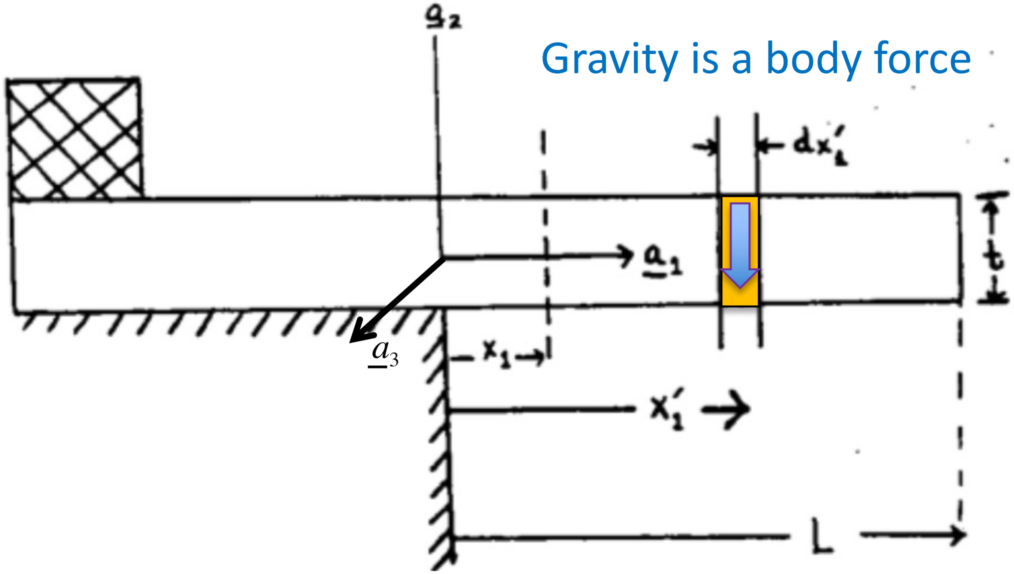


Average σ_{22} in the beam



Can σ_{22} support any load? Hint: What are the boundary conditions at the top and bottom of the small volume element? Another Hint: The fundamental theorem of calculus is $\int_a^b f(t)dt = F(b) - F(a)$
Generalizations are dealt with in the notes.

Average σ_{12} in the beam

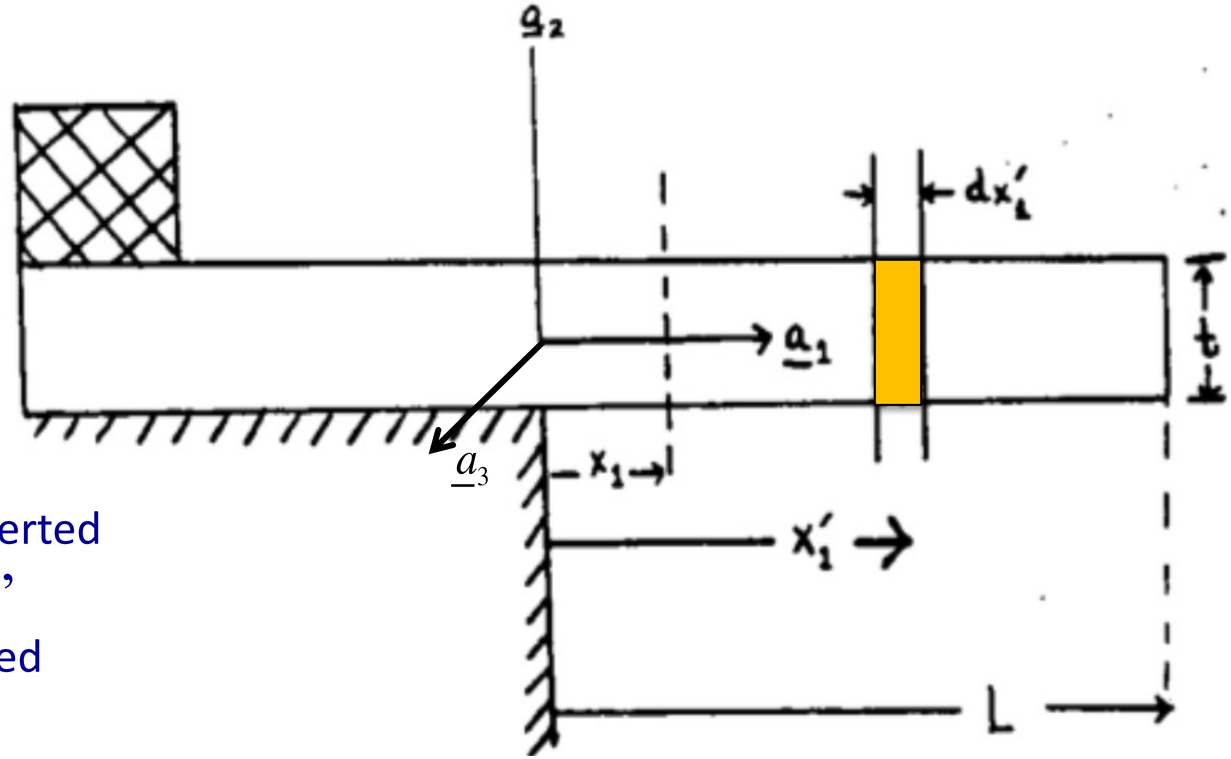


$\Sigma\ F_2 = \langle\sigma_{12}\rangle$ + Sum of Body Forces = ?

What is the sum of the body forces?

Answer: $\langle\sigma_{12}\rangle = (L\text{-}x)\ \rho\ g\ t$ (units of Force/length-into-board)

Incremental moment at x_1 due to outboard weight



Incremental moment at x_1 exerted by thin slice dx_1' of slab at x_1'
e.g. $\langle\sigma_{12}\rangle t$ is vertically directed force per unit width in x_3

$$dM_g = -\rho g t \, dx_1'(x_1' - x_1)$$

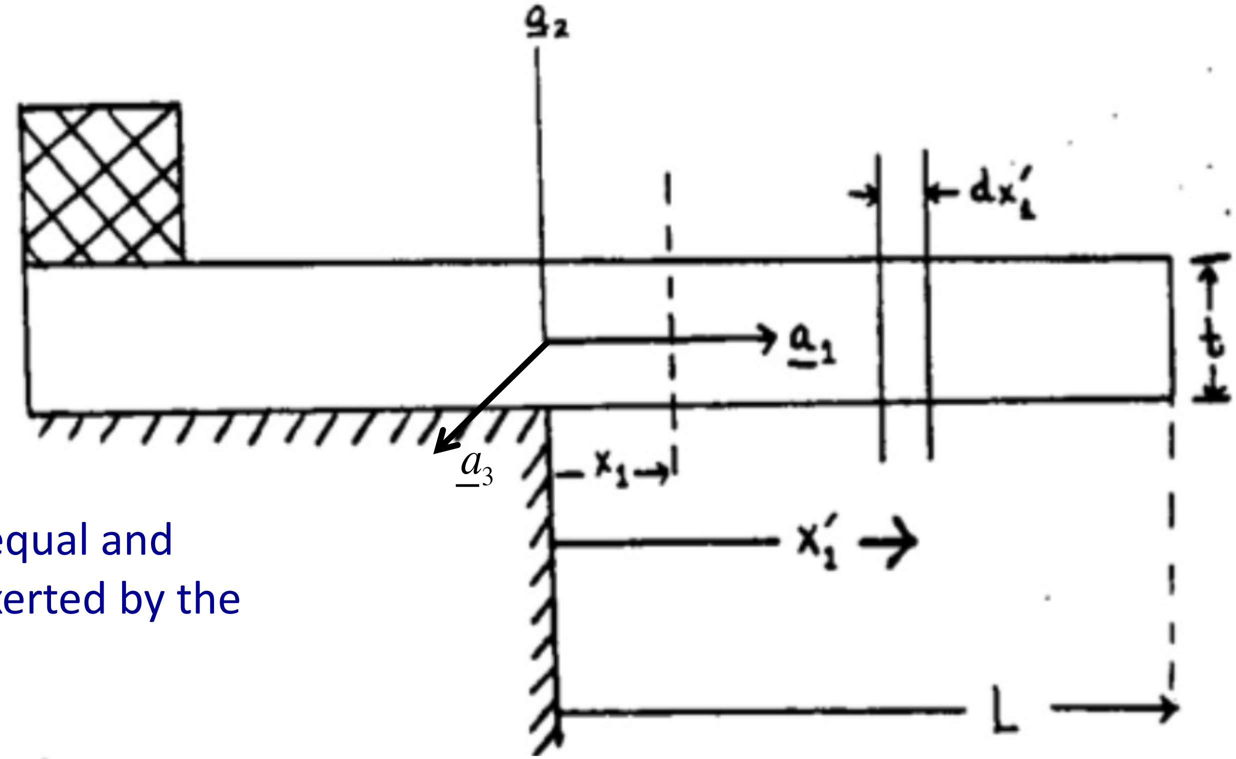
Now add up the incremental moments at x_1 due to *all* thin slices dx_1' to the right of x_1

$$M_g = \int_{x_1}^{L} \rho g t (x_1 - x_1') dx_1' = -\frac{1}{2}\rho g t (L - x_1)^2$$

The force per unit width: $\rho g \, t \, dx_1'$

The lever arm: $(x_1 - x_1')$

Incremental moment at x_1 due to stress in the beam

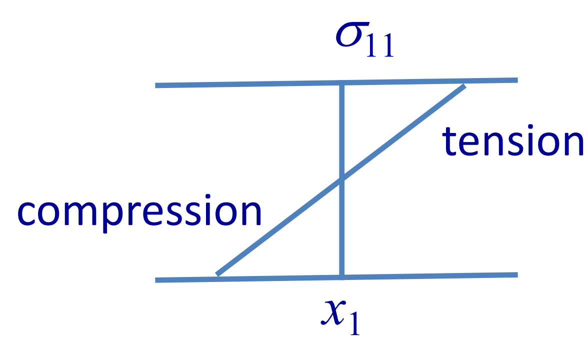


M_g must be balanced by an equal and opposite moment M_t at x_1 exerted by the stress state there.

$$M_t = + \int_{-t/2}^{t/2} \sigma_{11}(x_1, x_2')\, x_2'\, dx_2'$$

The force per unit width: $\sigma_{11}(x_1, x_2')\, dx_2'$

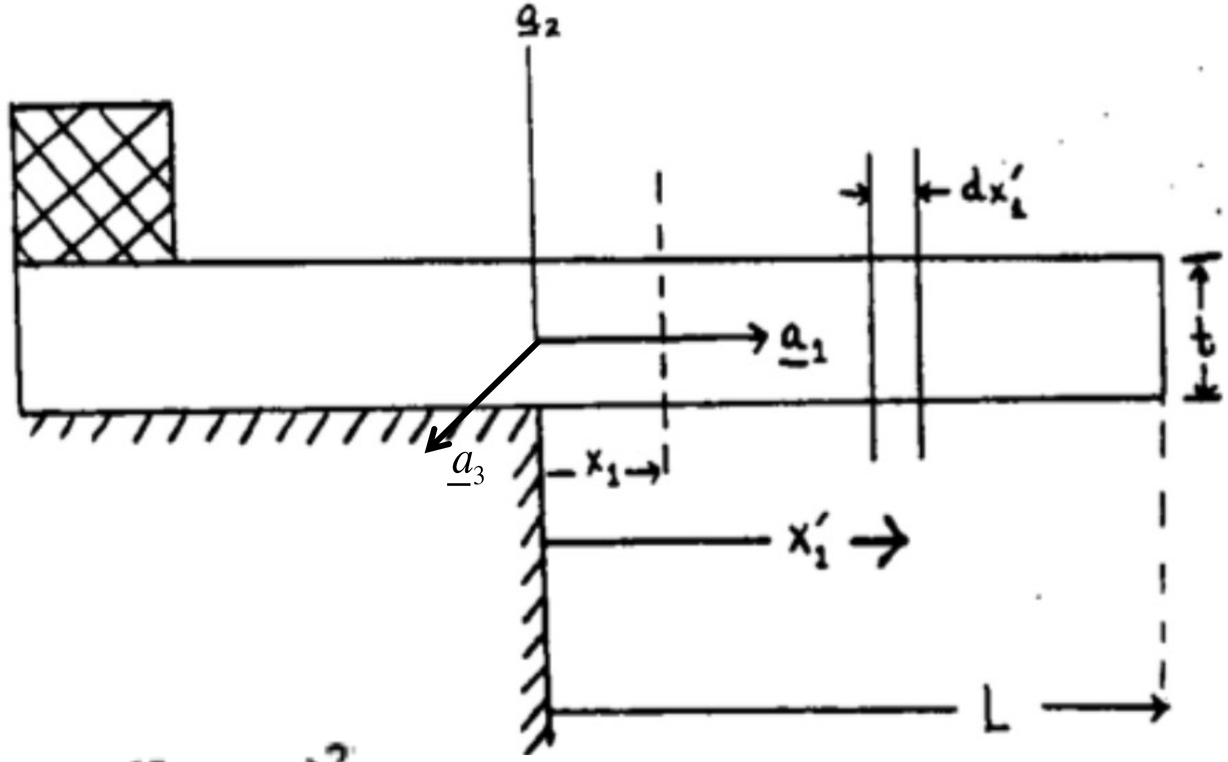
The lever arm: x_2'



Shear stresses at x_1 don't contribute because they have no moment arm around x_1

σ_{11} is assumed to be linear, but it is a very good assumption.

Putting it together -



$M_t + M_g = 0$

$$\int_{-t/2}^{t/2} \sigma_{11}(x_1, x_2')x_2' dx_2' = \frac{1}{2}\rho g t\,(L - x_1)^2$$

$$M_g = \int_{x_1}^{L} \rho g t(x_1 - x_1')dx_1' = -\frac{1}{2}\rho g t(L - x_1)^2$$

Moments and Tensors and Moments Tensors (oh my!)?

What is the tensor rank of (***r*** x ***F***)?

Why do seismologists speak of a moment tensor?

Answer: because slip on a fault is due to force *double couples* rather than *individual forces*, ”We imply more directional quantities [than in traditional rotational mechanics” (Aki and Richards)

Chapter 3 / REPRESENTATION OF SEISMIC SOURCES

(Aki and Richards, 2001)

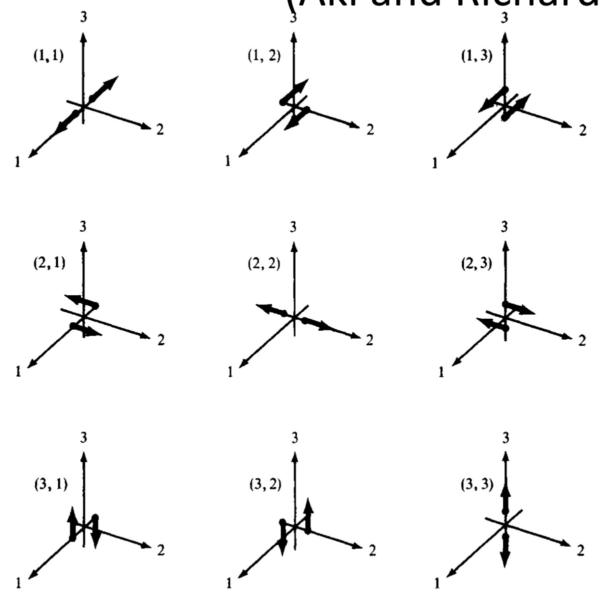
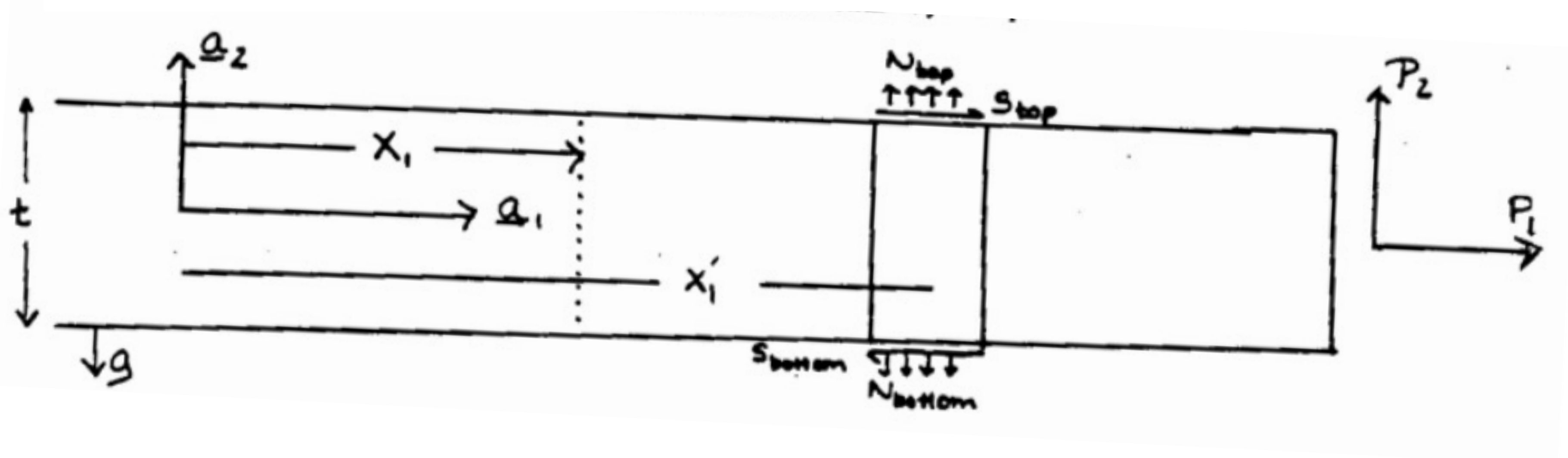


FIGURE 3.7
The nine possible couples that are required to obtain equivalent forces for a generally oriented displacement discontinuity in anisotropic media.

$$M_{pq} = \iint_{\Sigma} m_{pq} \, d\Sigma = \iint_{\Sigma} [u_i] \, \nu_j c_{ijpq} \, d\Sigma,$$

Now include tractions on the top and bottom

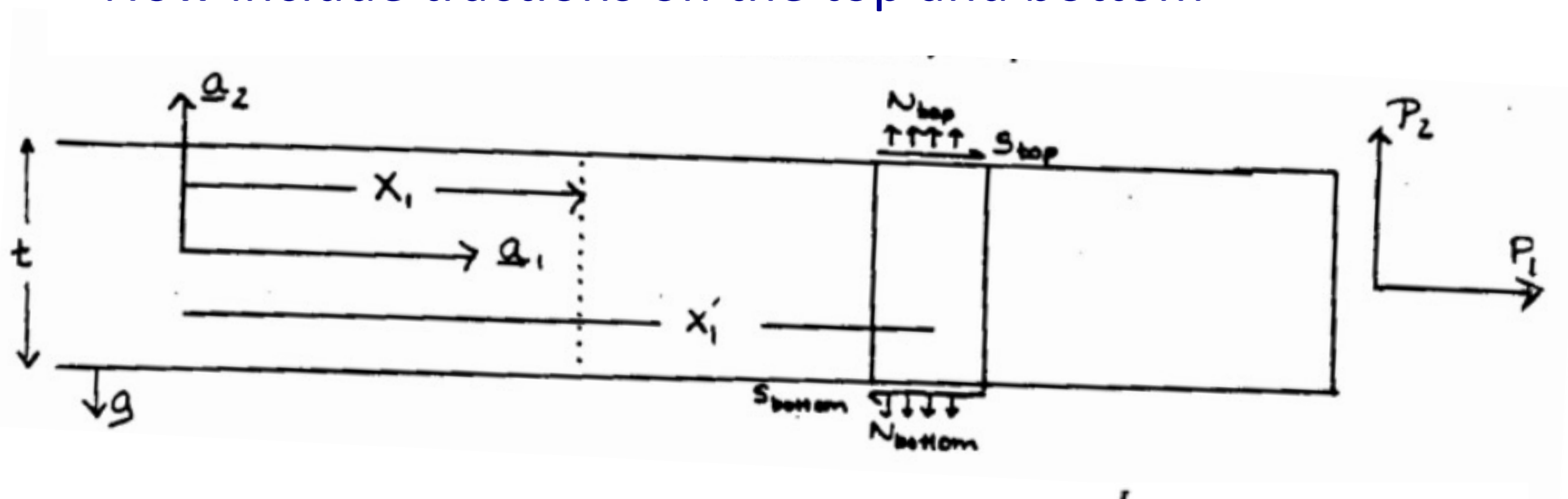


$$F_2(x_1') = -\rho g t + N_{top} - N_{bottom}$$

$$F_1(x_1') = S_{top} - S_{bottom}$$

Putting it together

Now include tractions on the top and bottom



$$F_2(x_1') = -\rho g t + N_{top} - N_{bottom}$$

$$F_1(x_1') = S_{top} - S_{bottom}$$

$$<\sigma_{11}>t + \int_{x_1}^{L} F_1(x_1')dx_1' + P_1 = 0$$

$$<\sigma_{11}> = \frac{1}{t}\{P_1 + \int^{L} F_1(x_1')dx_1'\}$$

Putting it together

$$<\sigma_{12}>t + \int_{x_1}^{L} F_2(x_1')dx_1' + P_2 = 0$$

$$<\sigma_{12}> = \frac{1}{t}\{P_2 + \int_{x_1}^{L} F_2(x_1')dx_1'\}$$

$$M_L = \int_{x_1}^{L} F_2(x_1')(x_1' - x_1)dx_1' + P_2(L - x_1) - P_1(u_2(L) - u_2(x_1))$$

$$M_t = + \int_{-t/2}^{t/2} \sigma_{11}(x_1, x_2')x_2'dx_2'$$

$$M_t + M_L = 0$$

$$M_t(x_1) = -\int_{x_1}^{L} F_2(x_1')(x_1' - x_1)dx_1' - P_2(L - x_1) + P_1(u_2(L) - u_2(x_1))$$