

ESS 411/511 Geophysical Continuum Mechanics Class #23

Highlights from Class #22 – McKenzie Carlson

Today's highlights on next Monday – Peter Lindquist

Problem Set #6

- is posted.
- Since Thursday is a holiday, we will use the class period on Wednesday as a group collaborative session to work on HW #6.

Midterm

- Has been (mostly) marked, and I expect to return annotated versions to you later today or tomorrow.

ESS 411/511 Geophysical Continuum Mechanics Class #23

For Problems Lab on Wednesday

- Be sure to be familiar with MSM Chapter 4
- For classes next week
- Read (<https://courses.washington.edu/ess511/NOTES/>)
 - Ed's note on volume elements
 - Ed's note on conservation laws
 - Ed's note on constitutive relations

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

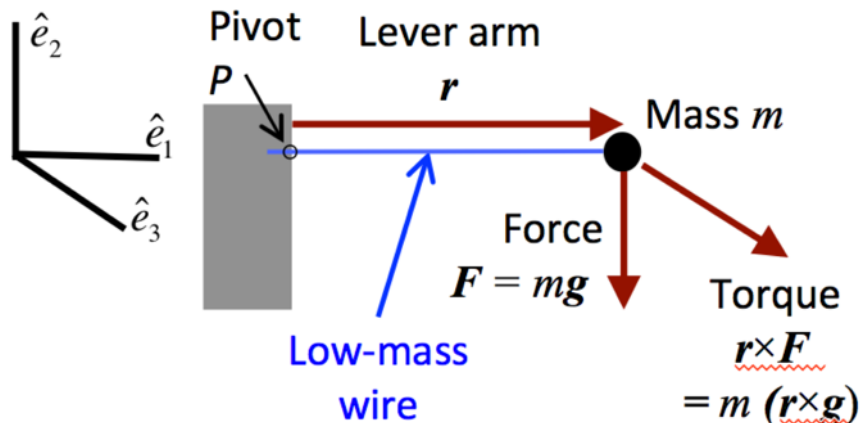
- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Class-prep: Please give me a *moment* ...

Please Read *Raymond notes on Stress and Moments* on class web site

https://courses.washington.edu/ess511/NOTES/CFR_CHAPTERS/CFR_stress_notes.pdf

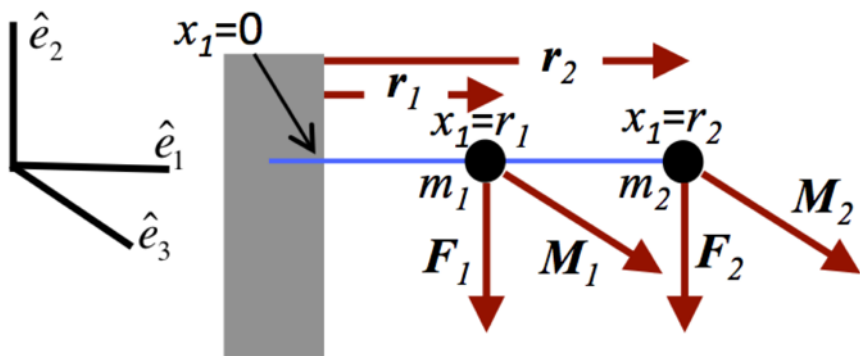
“Moment” and torque” are often used interchangeably, although technically a moment is a static concept, and torque applies to a rotation motion. Both involve a vector cross product between a force and a lever arm (or *moment arm*), so the units are Newton meters (N m). The first figure illustrates the moment exerted around the point P by a point mass m on the end of a stiff low-weight wire and subjected to gravity.



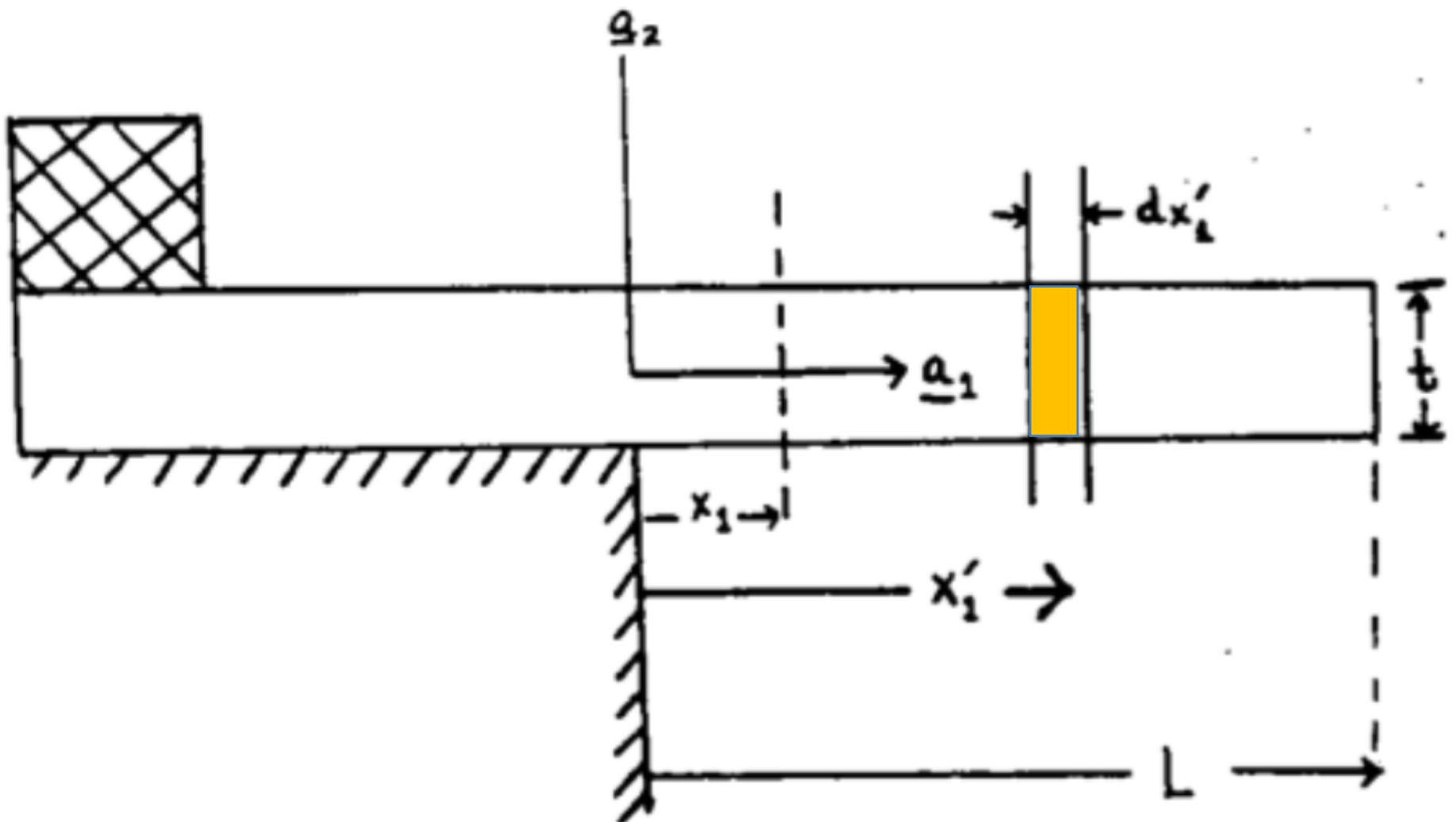
Now let's put a second mass on the wire, and see how the moments work.

Assignment

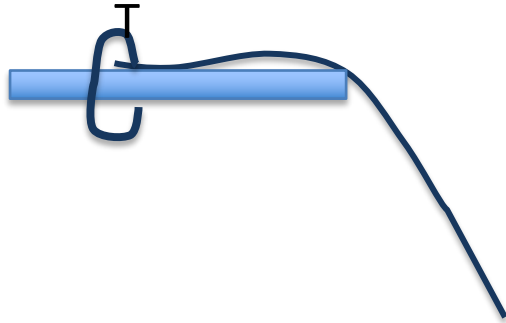
- What is the moment exerted by mass m_1 at point $x_1=0$?
- What is the moment exerted by mass m_2 at point $x_1=0$?
- What is the total moment exerted by both masses m_1 and m_2 at point $x_1=0$?
- What is the moment exerted by mass m_2 at point $x_1=r_1$?
- Suppose that we keep adding more discrete masses on the wire until we effectively have a solid steel bar. In a couple of sentences, explain how you could still find the net moment exerted at any particular point, such as $x_1=0$. (Consider that integral calculus could help.)



A hanging plate



Where is bending the greatest?



You can try this at home with a sheet of paper hanging over the edge of a table ...
(it isn't dangerous 😊).

Where do you think *bending moment* is the greatest?

But first, to simplify the math, we will assume for now that the deflections are very small.

Average stress across the beam (per unit width in x_3)

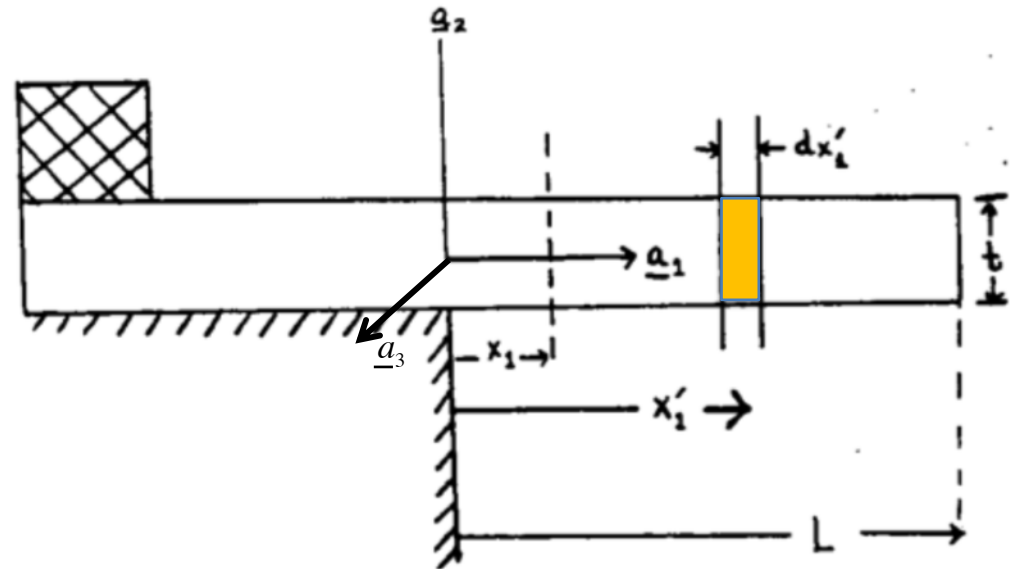
Vertical plane at x_1 must support the weight of all material to the right.

e.g. $\langle \sigma_{12} \rangle t$ is vertically directed force per unit width in x_3

$$\langle \sigma_{11} \rangle t = \rho g_1 (L - x_1) t = 0$$

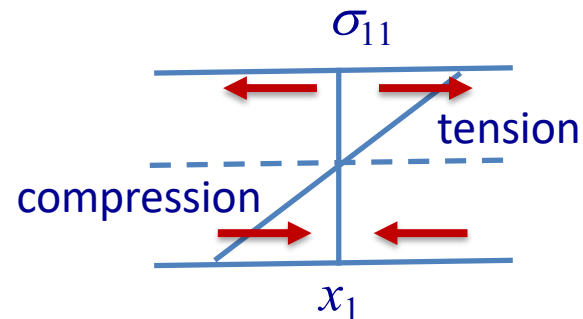
$$\langle \sigma_{12} \rangle t = \rho g_2 (L - x_1) t$$

$$\langle \sigma_{13} \rangle t = \rho g_3 (L - x_1) t = 0$$

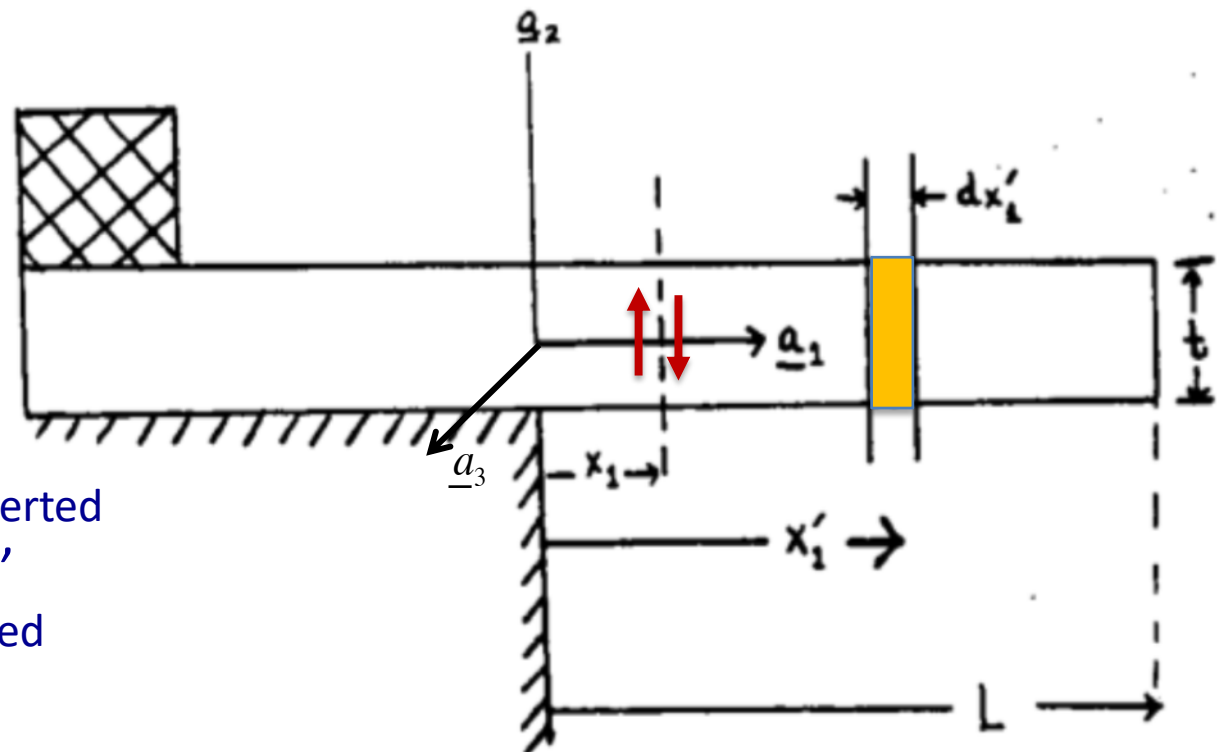


Although $\langle \sigma_{11} \rangle = 0$, there must be

- tension ($\sigma_{11} > 0$) in the upper part and
- compression ($\sigma_{11} < 0$) in the lower part, in order to prevent the material to the right from falling down.



Incremental moment
at x_1 due to outboard
weight



Incremental moment at x_1 exerted
by thin slice dx_1' of slab at x_1'
e.g. $\langle \sigma_{12} \rangle t$ is vertically directed
force per unit width in x_3

$$dM_g = - \rho g t \, dx_1' (x_1' - x_1)$$

Now add up the incremental moments at x_1 due to *all* thin slices dx_1' to the right of x_1

$$M_g = \int_{x_1}^L \rho g t (x_1 - x_1') dx_1' = - \frac{1}{2} \rho g t (L - x_1)^2$$

The force per unit width: $\rho g t \, dx_1'$

The lever arm: $(x_1 - x_1')$

Incremental moment
at x_1 due to stress in
the beam

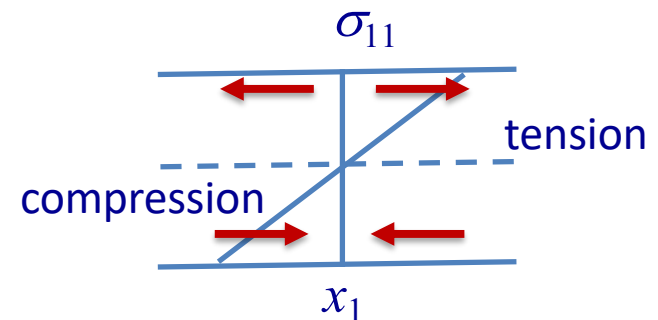
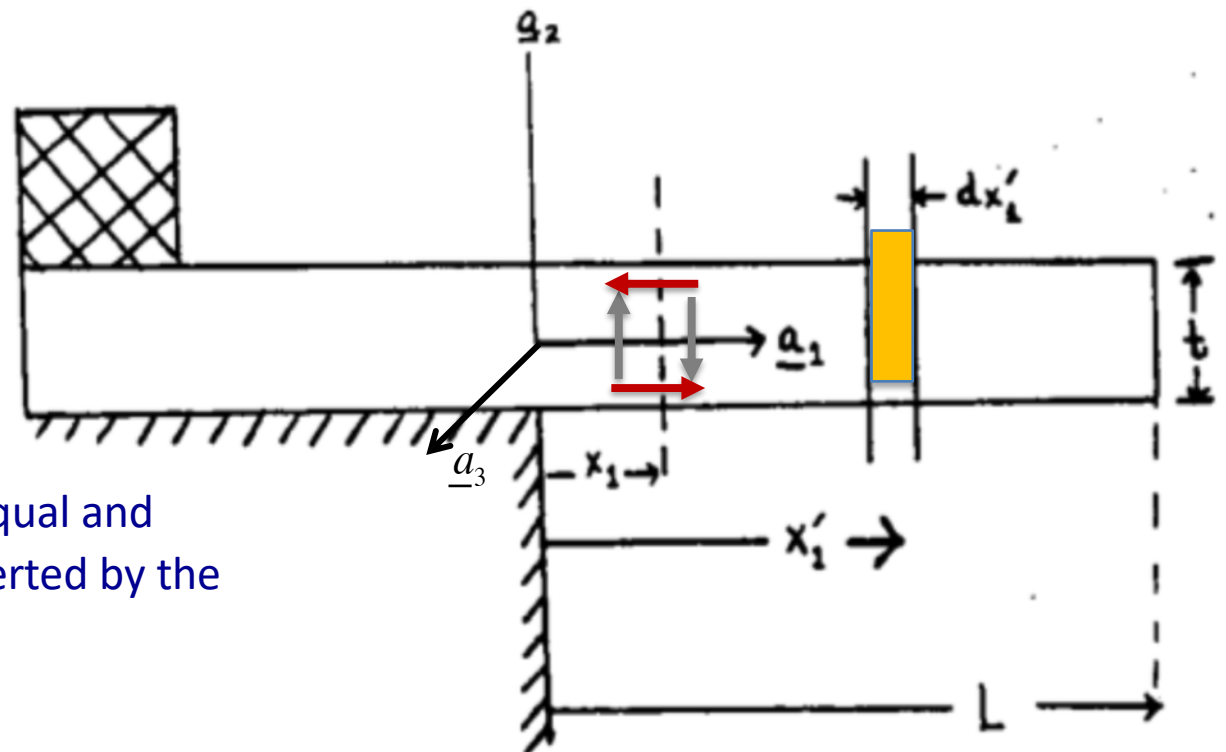
M_g must be balanced by an equal and
opposite moment M_t at x_1 exerted by the
stress state there.

$$M_t = + \int_{-t/2}^{t/2} \sigma_{11}(x_1, x_2') x_2' a dx_2'$$

The force per unit width: $\sigma_{11}(x_1, x_2') dx_2'$

The lever arm: x_2'

Shear stresses σ_{12} and σ_{13} at x_1 don't
contribute because they have no moment
arm around x_1 ($x_1 - x_1 = 0$)



σ_{11} is assumed to be linear, but
it is a very good assumption.

Incremental moment at x_1 due to stress in the beam

M_g must be balanced by an equal and opposite moment M_t at x_1 exerted by the stress state there.

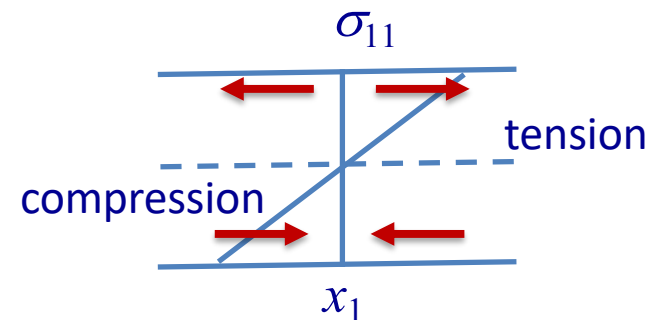
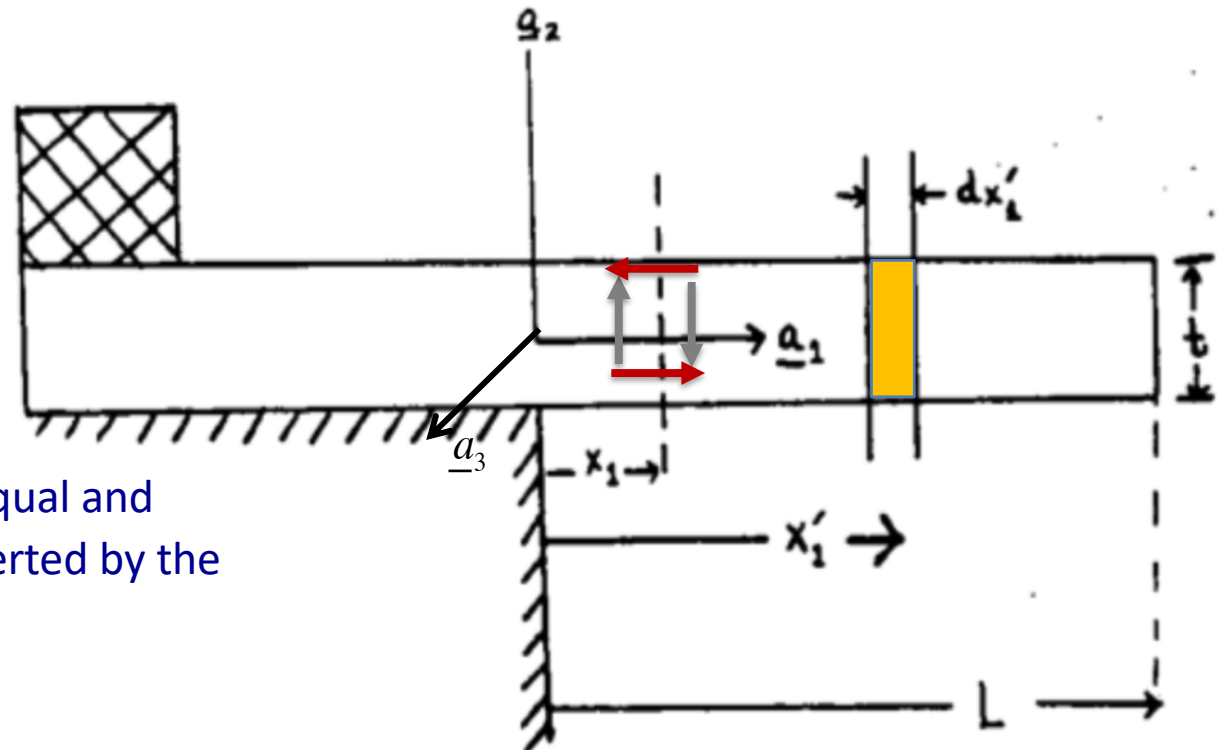
$$M_t = + \int_{-t/2}^{t/2} \sigma_{11}(x_1, x_2') x_2' dx_2'$$

The force per unit width: $\sigma_{11}(x_1, x_2') dx_2'$

The lever arm: x_2'

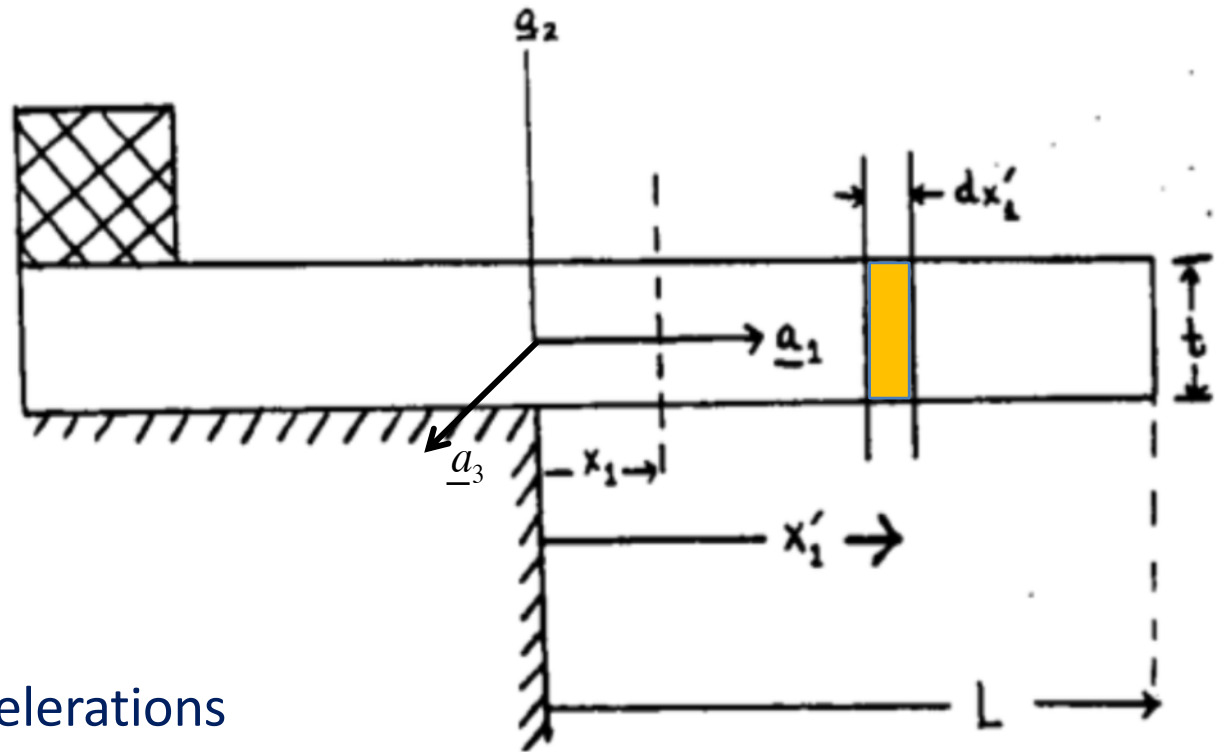
Shear stresses σ_{12} and σ_{13} at x_1 don't contribute because they have no moment arm around x_1 ($x_1 - x_1 = 0$)

Or don't change sign across the neutral plane at $x_2 = 0$



σ_{11} is assumed to be linear, but it is a very good assumption.

Putting it together -

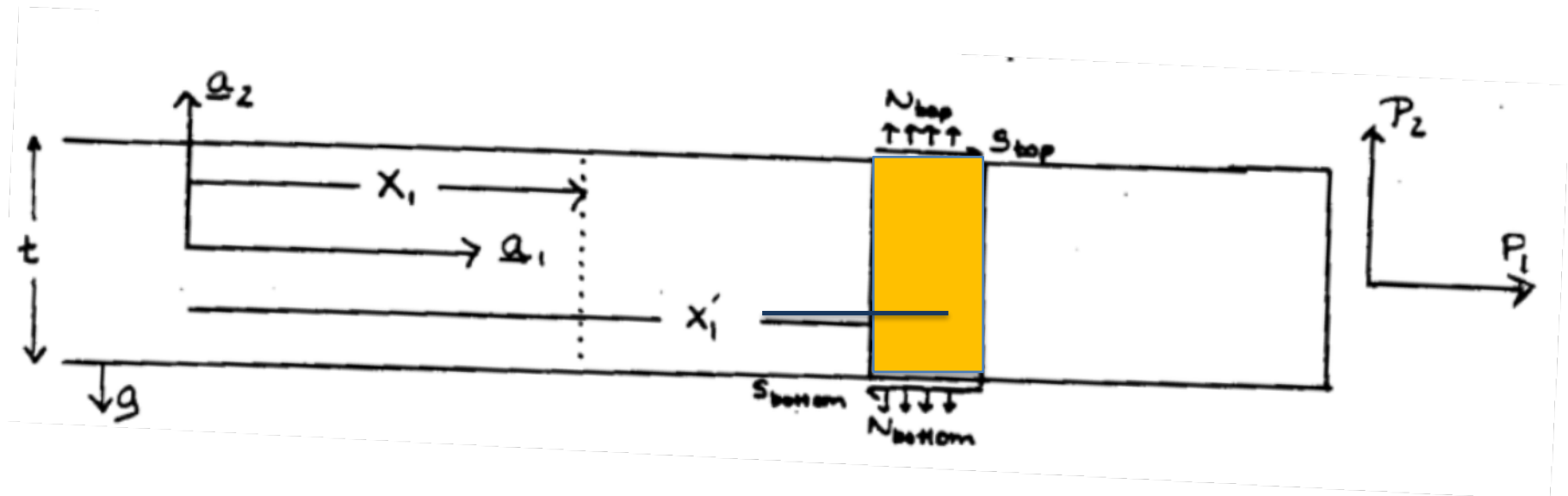


To avoid angular accelerations

$$M_t + M_g = 0$$

$$\int_{-t/2}^{t/2} \sigma_{11}(x_1, x'_2) x'_2 dx'_2 = \frac{1}{2} \rho g t (L - x_1)^2$$

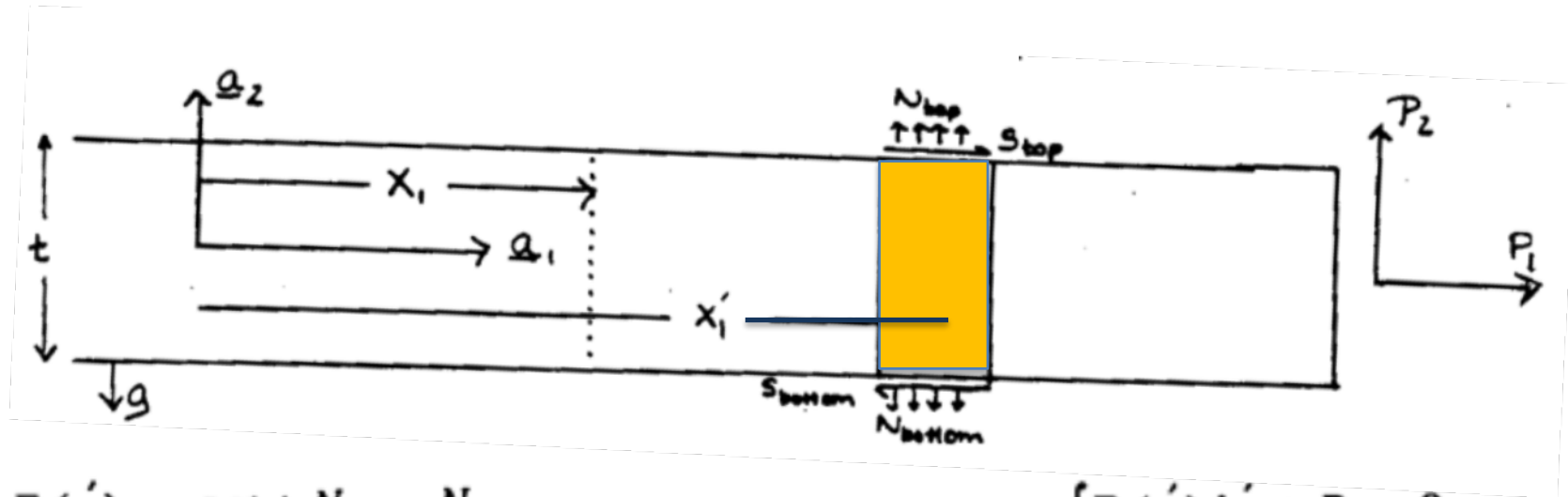
Now include tractions on the top and bottom



$$F_2(x_1') = -\rho g t + N_{top} - N_{bottom}$$

$$F_1(x_1') = S_{top} - S_{bottom}$$

Now include tractions on the top and bottom



$$F_2(x_1') = -\rho g t + N_{top} - N_{bottom}$$

$$F_1(x_1') = S_{top} - S_{bottom}$$

Putting it
together

$$\langle \sigma_{11} \rangle t + \int_{x_1}^L F_1(x_1') dx_1' + P_1 = 0$$

$$\langle \sigma_{11} \rangle = \frac{1}{L} \{ P_1 + \int_{x_1}^L F_1(x_1') dx_1' \}$$

$$\langle \sigma_{12} \rangle t + \int_{x_1}^L F_2(x_1') dx_1' + P_2 = 0$$

$$\langle \sigma_{12} \rangle = \frac{1}{t} \{ P_2 + \int_{x_1}^L F_2(x_1') dx_1' \}$$

Let's find the shape $u_2(x_1)$ of the beam

Outboard
moment

$$M_L = \int_{x_1}^L F_2(x'_1)(x'_1 - x_1)dx'_1 + P_2(L - x_1) - P_1(u_2(L) - u_2(x_1))$$

Internal-stress
moment

$$M_t = + \int_{-b/2}^{b/2} \sigma_{11}(x_1, x'_2)x'_2 dx'_2$$

Their sum

$$M_t + M_L = 0$$

gives

$$M_t(x_1) = - \int_{x_1}^L F_2(x'_1)(x'_1 - x_1)dx'_1 - P_2(L - x_1) + P_1(u_2(L) - u_2(x_1))$$

Let's find the shape $u_2(x_1)$ of the beam

$$M_I(x_1) = - \int_{x_1}^L F_2(x'_1)(x'_1 - x_1) dx'_1 - P_2(L - x_1) + P_1(u_2(L) - u_2(x_1))$$

But this is an integral equation for $u_2(x_1)$, and integral equations are hard ...

- Let's try to differentiate a couple of times.

Once -

$$\frac{dM_I(x_1)}{dx_1} = + \int_{x_1}^L F_2(x'_1) dx'_1 + F_2(x_1)(x_1 - x_1) + P_2 - P_1 \frac{\partial u_2}{\partial x_1} = \langle \sigma_{12} \rangle$$

Twice -

$$\frac{d^2 M_I(x_1)}{dx_1^2} = -F_2(x_1) - P_1 \frac{\partial^2 u_2}{\partial x_1^2}$$

Introduce flexural rigidity -

$$M_I = -D \frac{\partial^2 u_2}{\partial x_1^2} \quad \rightarrow$$

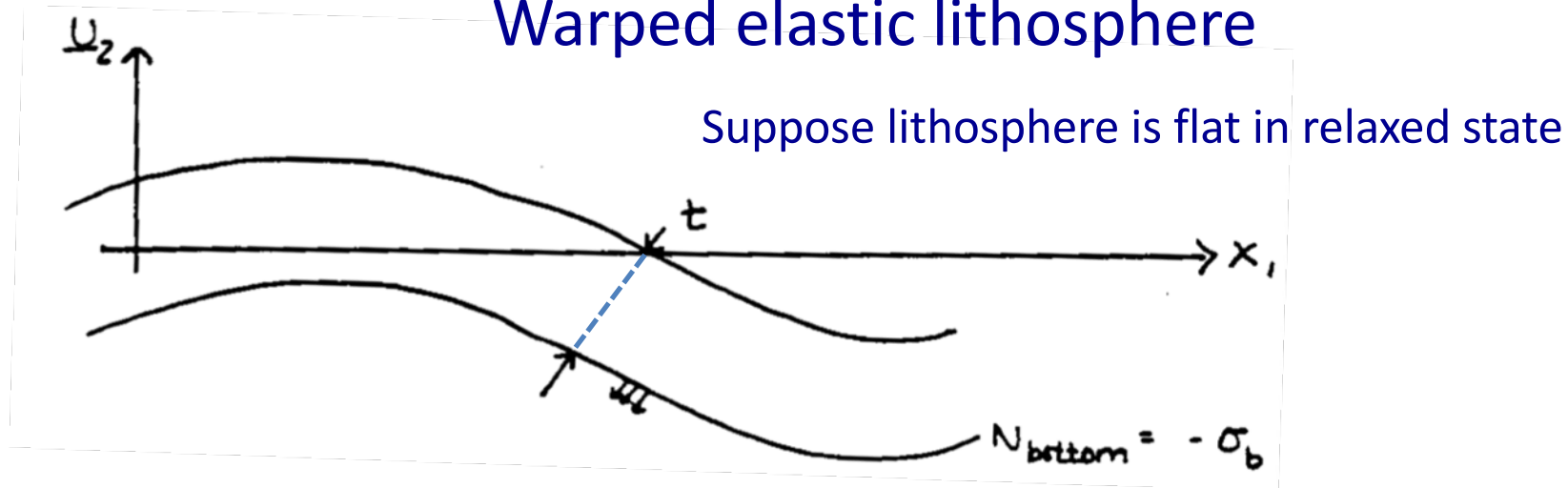
Elastic beam equation

$$\frac{\partial^4 u_2}{\partial x_1^4} = \frac{1}{D} \left[F_2(x_1) + P_1 \frac{\partial^2 u_2}{\partial x_1^2} \right]$$

Moments of interest in the Earth

- Rock overhangs
- Snow cornices
- Tidal flexure of ice shelves floating in the ocean
- Length scale of support by bending lithosphere
- Earthquake moment magnitude

Warped elastic lithosphere



Now lithosphere is warped into a sinusoid

- e.g. by loading with a big ice sheet

$$u_2(x_1) = a \sin kx_1 \quad \lambda = 2\pi/k$$

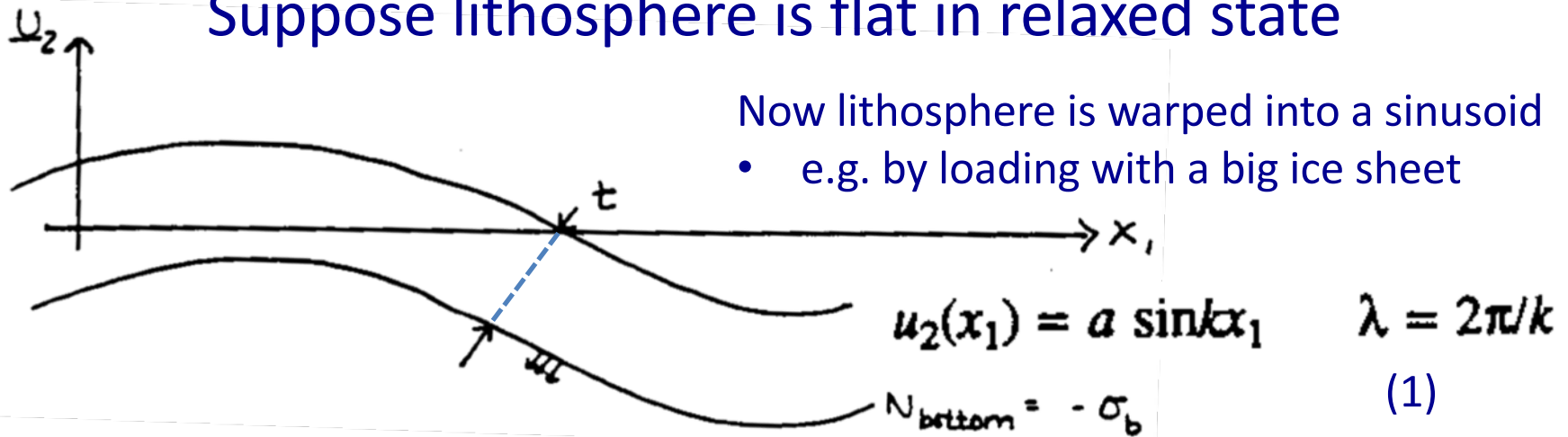
- Far from any edge, $P_1 = P_2 = 0$
- $N_{\text{top}} = 0$
- $N_{\text{bot}} = \sigma_b$

$$\frac{\partial^4 u_2}{\partial x_1^4} = k^4 a \sin kx_1 = \frac{1}{D} F_2(x_1)$$

σ_b results from 2 effects

- weight of the overlying lithosphere
- bending stresses in the elastic lithosphere, which tries to relax
 - pull the lithosphere up and away from the mantle in the hollows,
 - push down on the mantle under the crests.

Suppose lithosphere is flat in relaxed state



- Far from any edge, $P_1 = P_2 = 0$ Assume $S_{\text{top}} = S_{\text{bot}} = 0$ so $F_1(x_1) = 0$ (3)

- $N_{\text{top}} = 0$

- $N_{\text{bot}} = -\sigma_b$

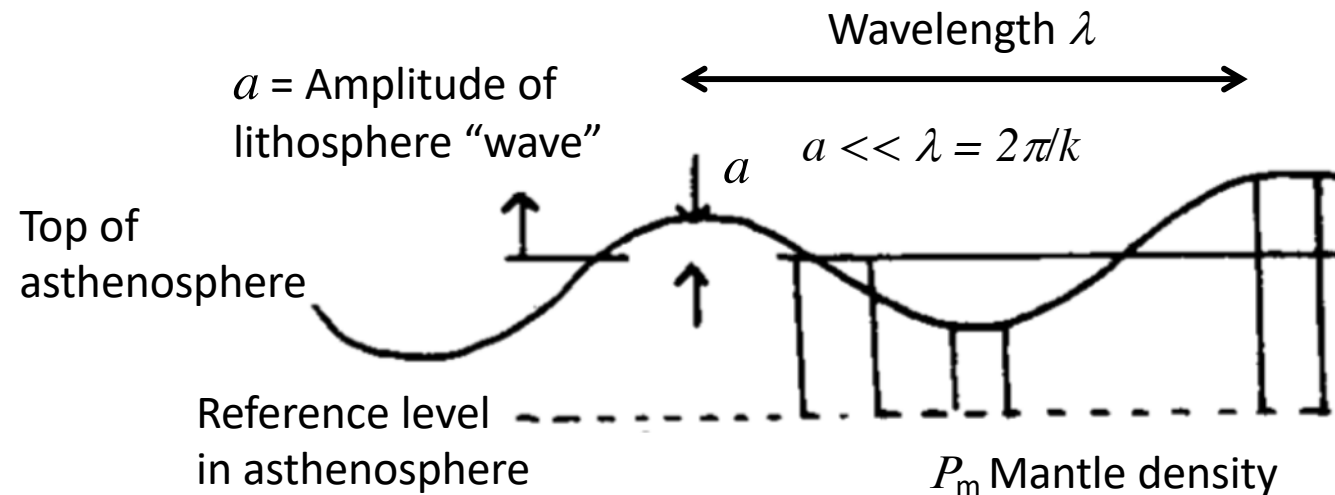
→ $F_2 = \sigma_b - \rho_c g t$ (2)

so $\frac{\partial^4 u_2}{\partial x_1^4} = k^4 a \sin kx_1 = \frac{1}{D} F_2(x_1)$ (4)

σ_b varies with x_1 and results from 2 effects

- weight of the overlying lithosphere
- bending stresses in the lithosphere, which tries to relax and flatten
 - pull the lithosphere up from the mantle in the hollows,
 - push down on the mantle under the crests.

From (2), (3), and (4) $\sigma_b = \rho_c g t + Dk^4 a \sin kx_1$ (5)



At the reference level, first account for

- weight of the mantle "bumps": $\rho_m g a \sin(kx_1)$
- weight of the lithosphere with thickness t : $\rho_c g t$

$$\sigma = +\rho_m g a \sin kx_1 + \rho_c g t + \text{const}$$

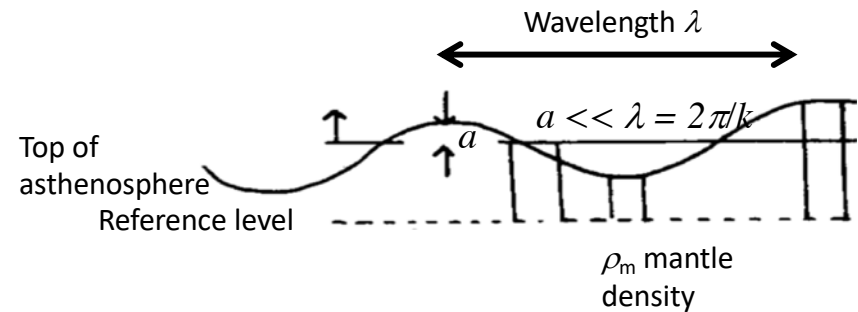
Now account for the bending moments trying to flatten the lithosphere

At the reference level, there are no horizontal stress gradients (why?)

$$\sigma = +\rho_m g a \sin kx_1 + \rho_c g t + Dk^4 a \sin kx_1 + \text{const}$$

$$= (\rho_m g + Dk^4) a \sin kx_1 + \rho_c g t + \text{const}$$

The flexural rigidity of the lithosphere adds a restoring force additional to the force coming from the topography



$$\sigma = +\rho_m g a \sin kx_1 + \rho_c g t + Dk^4 a \sin kx_1 + \text{const}$$

$$= (\rho_m g + Dk^4) a \sin kx_1 + \rho_c g t + \text{const}$$

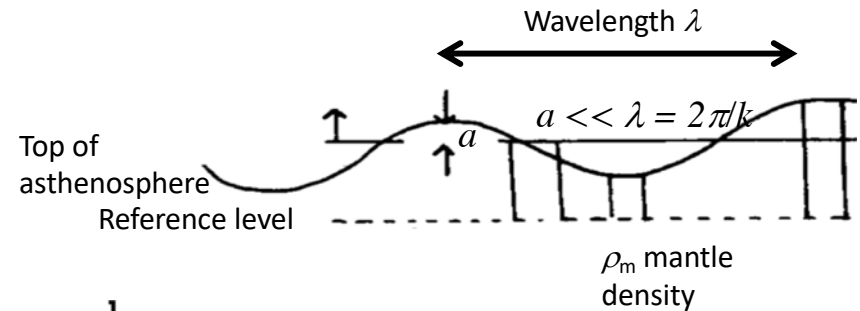
Bending stresses in the lithosphere become dominant $k^4 D > \rho_m g$ or $k > \left[\frac{\rho_m g}{D} \right]^{\frac{1}{4}}$

In terms of the wavelength λ , $\lambda = \frac{2\pi}{k} < 2\pi \left[\frac{D}{\rho_m g} \right]^{\frac{1}{4}}$

Short wavelengths can be supported elastically, but long wavelength waves just sag into the mantle based on their weight.

At the reference level in the asthenosphere, there are no horizontal stress gradients (why?)

Supportable wavelengths



Bending stresses in the lithosphere become dominant $k^4 D > \rho_m g$ or $k > \left[\frac{\rho_m g}{D} \right]^{1/4}$

Flexural rigidity $D = M t^3 / 12$

Elastic modulus $M = E / (1 - \nu^2) \approx 10^{10} \text{ Pa}$

$\rho_m g \sim 0.3 \times 10^5 \text{ Pa m}^{-1}$, and with $t = 100 \text{ km}$, $D = 10^{24} \text{ Pa m}^3$

So bending stresses become important for $\lambda < 500 \text{ km}$

At the reference level in the asthenosphere, there are no horizontal stress gradients (why?)

How to
measure
earthquake
size?

① - by amount of energy released

e.g. Richter Magnitude $m = C_1 \log E + C_2$ $E = \text{energy released}$

But how to measure the energy?

Energy goes into

- 1) elastic waves, acoustic waves.
 - speed of rupture matters
- 2) production of fault gouge (crushing rock)
 - rock type matters.
- 3) frictional heating of fault zone
 - normal and shear stresses matter
 - coefficient of friction matters
- 4) potential energy (uplift & subsidence)

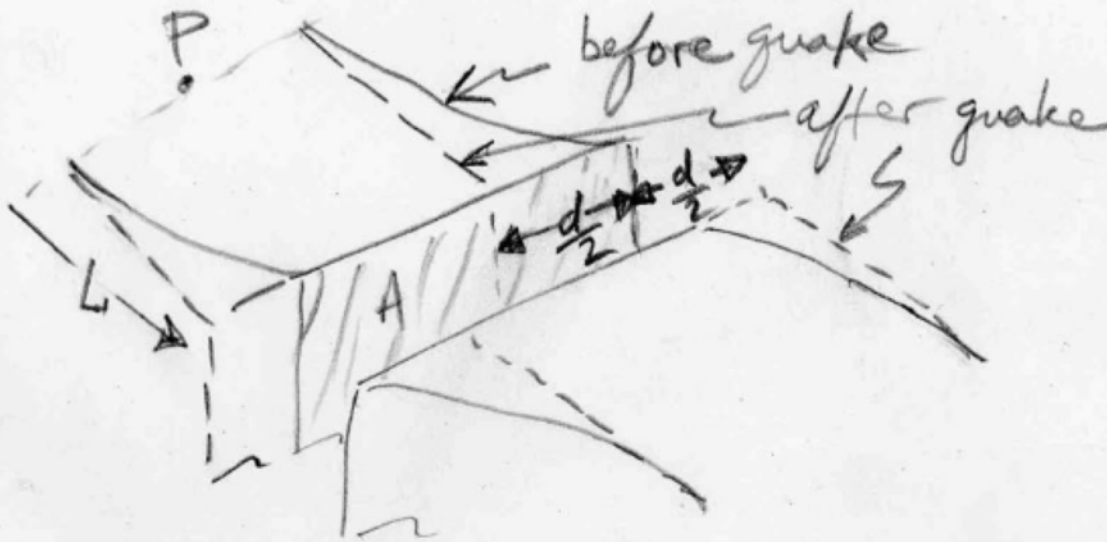
only 1) is readily available to seismologists
Maybe 4), after geodetic resurvey.

So getting a measure of total energy release
can be difficult.

Estimate strain energy released?

Maybe instead of measuring the energy directly, we can estimate the energy release based on the strain energy, which we can estimate from a few parameters of the rock and the fault.

Prior to the quake, elastic energy was stored in the rock around the fault.



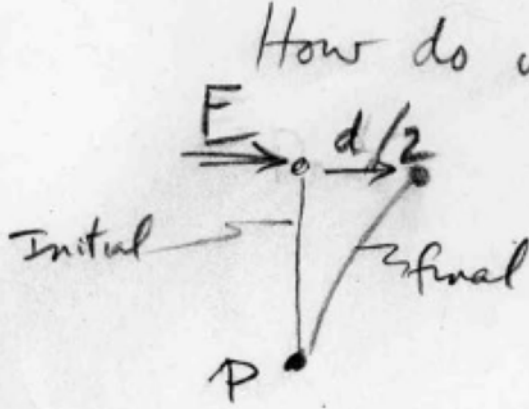
$\frac{d}{2}$ = slip during quake (one side of fault)

A = area of fault rupture.

L = distance away from fault where permanent offset is small.

Estimate strain energy in a spring?

How do we put energy into a "spring", e.g. a metal ruler?



Initial

Final

P

Force F applied to end of ruler causes displacement $d/2$

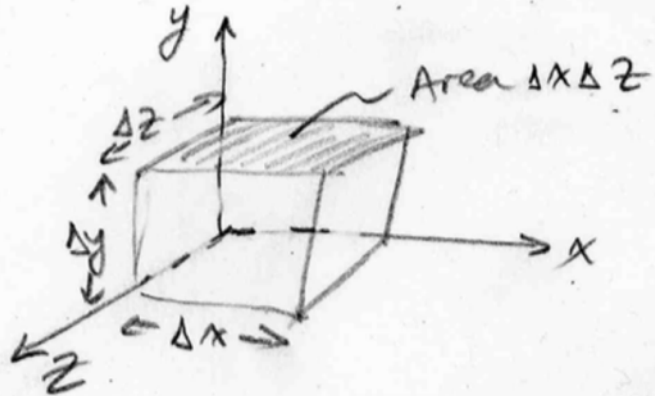
When force is released, spring returns to initial state.

(1 Spring represents one side of fault.)

Energy Stored in bent spring is $E = F \cdot d/2$

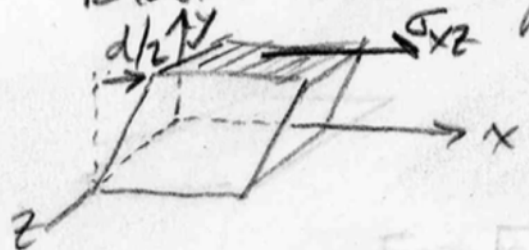
Energy released by a spring

Now let's use a block of a continuum as the spring.



Apply a force to the surface with area $\Delta x \Delta z$

Elastic block deforms with displacement $d/2$ (one side of a fault)



$$E = F \cdot d/2$$

The force is $|F| = \sigma_{xy} \Delta x \Delta z$ required to bring rock to failure limit.

The energy is $E = Fd = (\sigma_{xy} \Delta x \Delta z)d$

The shear strain is $\epsilon_{xy} \sim d/\Delta y$ inside the block.

Energy per unit volume

In general

$$\begin{aligned} \text{So } E &= (\sigma_{xy} \Delta x \Delta z) (\epsilon_{xy} \Delta y) \\ &= \sigma_{xy} \epsilon_{xy} (\Delta x \Delta y \Delta z) = \sigma_{xy} \epsilon_{xy} V \end{aligned}$$

Energy
Volume

$$E/V = \sigma_{xy} \epsilon_{xy}$$

Stored elastic strain energy in a volume is

$$E = \int_V \sigma_{xy} \epsilon_{xy} dV$$

Our fault has stored up elastic strain energy, and it releases it during a quake ($\sigma_{xy} \rightarrow 0$).

$$E = \int_V \sigma_{xy} \epsilon_{xy} dV$$

$$= \int_V \mu \epsilon_{xy}^2 dV$$

For elastic continuum

$$\sigma_{xy} \sim \mu \epsilon_{xy}$$

μ is an elastic
constitutive parameter

Simple, but not maybe simple enough ...

For the fault. $\epsilon_{xy} \sim \frac{d}{L}$ Strain
 $dV \sim AL$ volume strained.

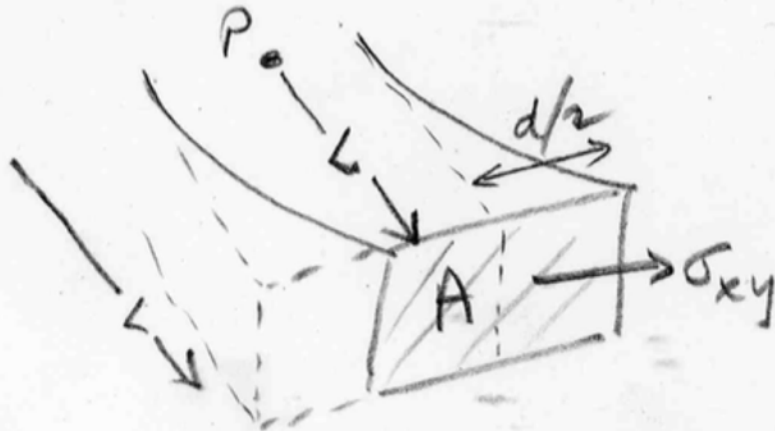
$$\text{so } E = \int_V \mu \epsilon_{xy}^2 dV \sim \mu \left(\frac{d}{L} \right)^2 AL \\ = \mu Ad \left(\frac{d}{L} \right)$$

We can estimate μ (rock elasticity)
 A (rupture area)
 d (fault motion)

But estimating L is tricky.
So let's look for a better way to estimate earthquake size.

(B) Moment Magnitude

Shear traction σ_{xy} brought fault to point of failure.
(σ_{xy} corresponds to P_2 in Raymond notes)



L is moment arm about P .

Moment created by σ_{xy} (and released during quake) is

$$M_1 = F L = (\sigma_{xy} A) L$$

$$\sigma_{xy} = \mu \epsilon_{xy} \quad \text{- constitutive relation of rock}$$

$$\epsilon_{xy} = \frac{d/2}{L} \quad \text{- strain}$$

So, on one side of fault
Moment release is

$$M_1 = (\mu \epsilon_{xy}) A L = \mu \frac{d A L}{2 L} = \mu \frac{d A}{2}$$

Both sides of the fault release moment

Moment release is

$$M = 2M_1 = 2\left(\mu \frac{Ad}{2}\right) = \mu Ad.$$

$$M = \boxed{\text{elastic modulus}} \times \boxed{\text{Fault area}} \times \boxed{\text{Slip}}$$

All are easily estimated.

$$\boxed{M = \mu Ad}$$

We don't need to know where P is located,