ESS 411/511 Geophysical Continuum Mechanics Class #27

Highlights from Class #26 – Madie Mamer
Today's highlights on Monday – Abigail Thienes

Today
- ESS 511 60-second project updates today

For Monday please read
- Ed's note on constitutive relations (on right sidebar at) *https://courses.washington.edu/ess511/NOTES/*)

For Problems Lab next week
- Study Questions for take-at-home Final exam will be posted this weekend.

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Warm-up (break-out rooms)

- What exactly are conservation laws?
- Why are they useful?
- What are some examples?

Some key ideas about rates of change

Velocity is rate of change of position or displacement $v_i = \dfrac{D(x_i)}{Dt}$ for particle X_A

Spatial velocity-gradient tensor $L_{ij} = \dfrac{\partial v_i}{\partial x_j}$

Symmetric and skew-symmetric decomposition $L_{ij} = d_{ij} + w_{ij}$

Deformation-rate tensor or strain-rate tensor

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

Vorticity or spin tensor

$$w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

Rate of change of line elements MSM Section 4.12

A line element

$$\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}$$

$$F_{iA} = \frac{\partial x_i}{\partial X_A}$$ is the deformation-gradient tensor

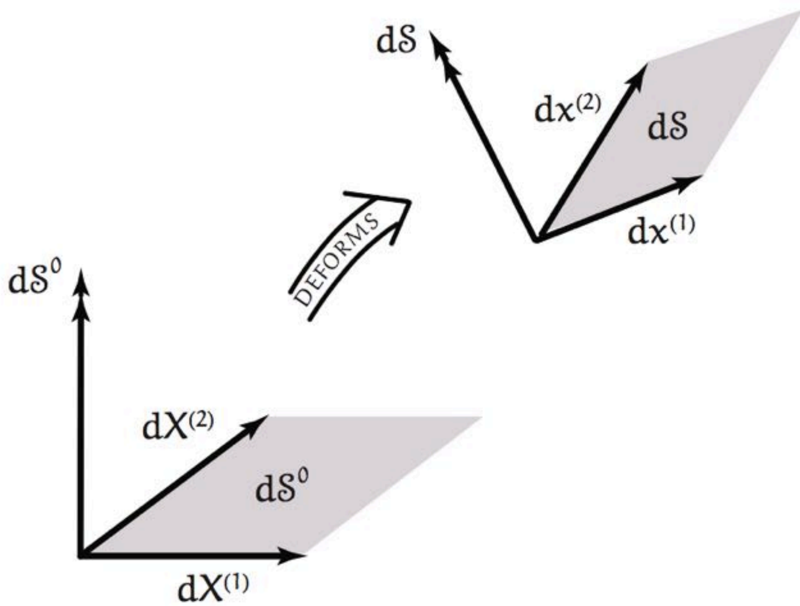
Rate of change

$$\overline{\mathrm{d}\mathbf{x}}^{\,\cdot} = \dot{\mathbf{F}} \cdot \mathrm{d}\mathbf{X} = \mathbf{L} \cdot \mathbf{F} \cdot \mathrm{d}\mathbf{X} = \mathbf{L} \cdot \mathrm{d}\mathbf{x}$$

or $$\overline{\mathrm{d}x_i}^{\,\cdot} = v_{i,j}\,\mathrm{d}x_j$$

Rate of change of area elements MSM Section 4.12

$$d\mathcal{S}^0_A = \varepsilon_{ABC} dX^{(1)}_B dX^{(2)}_C$$



$$d\mathcal{S}_i = \varepsilon_{ijk} dx^{(1)}_j dx^{(2)}_k$$
$$= \varepsilon_{ijk} x_{j,B} dX^{(1)}_B x_{k,C} dX^{(2)}_C$$

After using $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$.

$$d\boldsymbol{\mathcal{S}} \cdot \mathbf{F} = J d\boldsymbol{\mathcal{S}}^0$$
$$J = \left| \frac{\partial x_i}{\partial X_A} \right| = \det(F_{iA})$$

Now take time derivative

$$d\dot{\boldsymbol{\mathcal{S}}} = (\operatorname{tr} \mathbf{L})\, d\boldsymbol{\mathcal{S}} - d\boldsymbol{\mathcal{S}} \cdot \mathbf{L}$$

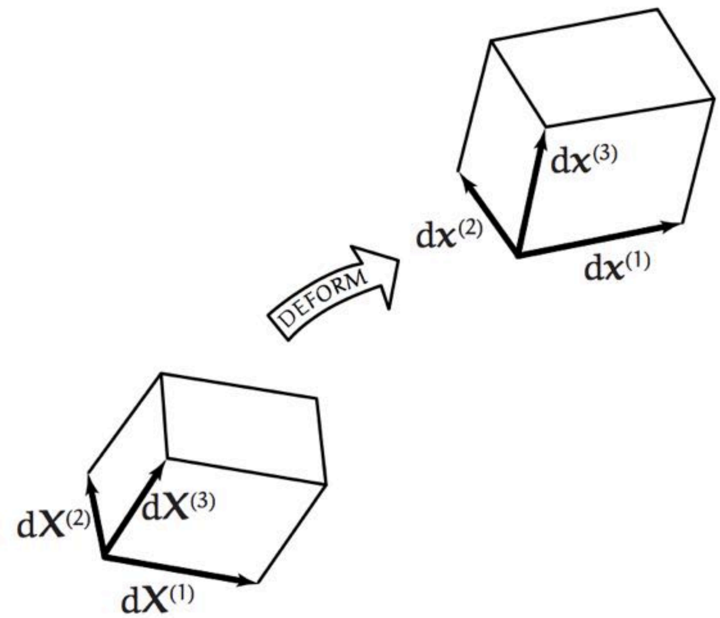
or $$d\dot{\mathcal{S}}_i = v_{k,k} d\mathcal{S}_i - d\mathcal{S}_j v_{j,i}$$

Rate of change of volume elements MSM Section 4.12

$$d\mathcal{V} = d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)}$$
$$= \varepsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)}$$
$$= \left[d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)} \right]$$

$$d\mathcal{V} = \left[\mathbf{F} \cdot d\mathbf{X}^{(1)}, \mathbf{F} \cdot d\mathbf{X}^{(2)}, \mathbf{F} \cdot d\mathbf{X}^{(3)} \right]$$
$$= \varepsilon_{ijk} x_{i,A} x_{j,B} x_{k,C} dX_A^{(1)} dX_B^{(2)} dX_C^{(3)}$$
$$= \det(\mathbf{F}) \left[d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}, d\mathbf{X}^{(3)} \right]$$

$$d\mathcal{V} = J d\mathcal{V}^0$$



Now take time derivative

$$\dot{\overline{d\mathcal{V}}} = \dot{J} d\mathcal{V}^0 = J \,\mathrm{tr}\,(\mathbf{L}) d\mathcal{V}^0$$
$$= J v_{i,i} d\mathcal{V}^0 = v_{i,i} d\mathcal{V}$$

Class-prep: Conservation Laws

Please read

- MSM Section 5.1 *Material Derivatives of Line, Surface and Volume Integrals*

Also

- Ed's notes on volume elements
- Ed's notes on Conservation Laws

Both are on the class web site at

https://courses.washington.edu/ess511/NOTES/notes.html

Assignment

- Please identify each term or factor in the equation (MSM 5.3).
- Explain in words what the equation does, and why it is useful.

$$\dot{P}_{ij\ldots}(t) = \int_{\mathcal{V}} \frac{\partial P^*_{ij\ldots}}{\partial t}\, d\mathcal{V} + \int_{\mathcal{S}} v_k P^*_{ij\ldots} n_k \, d\mathcal{S}$$

Discrete *vs* Continuum Conservation Laws

	Point Masses	Continuum
Mass	$\frac{d}{dt}\left(\sum_i m_i\right) = 0$	$\frac{d}{dt}\left(\int_V \rho \, dV\right) = 0$
Momentum	$\frac{d}{dt}\left(\sum_i m_i \vec{v}_i\right) = \sum_j \vec{F}_j$	$\frac{d}{dt}\left(\int_V \rho \vec{v} \, dV\right) = \int_V \rho \vec{b} \, dV + \int_S \vec{t}^{(n)} ds$
Energy	$\frac{d}{dt}\left(\sum_i E_i\right) = \sum_j \vec{F}_j \cdot \vec{u}_j + \ldots$	$\frac{d}{dt}\left(\int_V \rho e \, dV\right) = ?$

How do integrated quantities change through time?

$$P_{ij\ldots}(t) = \int_V P^*_{ij\ldots}(\vec{x},t)\,dV$$

$$\dot{P}_{ij\ldots}(t) = \frac{d}{dt}\left(\int_V P^*_{ij\ldots}(\vec{x},t)\,dV\right) \quad (1)$$

$$= \frac{d}{dt}\left(\int_{V^0} P^*_{ij\ldots}(\vec{x}(\vec{X},t),t)\,J\,dV^0\right) \quad (2)$$

Because V° is fixed, we can take $\frac{d}{dt}$ inside integral in (2)

$$\dot{P}_{ij\ldots}(t) = \int_{V^\circ} \left[P^*_{ij\ldots}(\vec{x},t)\, J\right]^{\circ} dV^\circ$$

$$= \int_{V^\circ} \left[\dot{P}^*_{ij\ldots}\, J + P^*_{ij\ldots}\, \dot{J}\right] dV^\circ$$

$$= \int_{V^\circ} \left[\dot{P}^*_{ij\ldots} + v_{k,k} P^*_{ij\ldots}\right] J\, dV^\circ$$

$$\boxed{\dot{P}_{ij\ldots}(t) = \int_V \left(\dot{P}^*_{ij\ldots} + v_{k,k} P^*_{ij\ldots}\right) dV} \qquad (A)$$

$$\dot{P}_{ij\ldots}(t) = \int_V \left(\frac{\partial P^*_{ij\ldots}}{\partial t} + v_k \frac{\partial P^*_{ij\ldots}}{\partial x_k} + v_{k,k} P^*_{ij\ldots} \right) dV$$

$$= \int_V \left(\frac{\partial P^*_{ij}}{\partial t} + \left(v_k P^*_{ij\ldots} \right)_{,k} \right) dV$$

$$\dot{P}_{ij\ldots}(t) = \int_V \left(\frac{\partial P^*_{ij\ldots}}{\partial t} \right) dV + \int_S \left(v_k P^*_{ij\ldots} n_k \right) dS$$

Conservation of Mass

$$\dot{m} = \frac{d}{dt}\left(\int_V \rho(\vec{x},t)\, dV \right) = 0$$

$$\dot{P}_{ij\ldots} = \rho$$

$$\dot{m} = \int_V \left(\dot{\rho} + \rho v_{k,k}\right) dV = 0$$

V is arbitrary

$$\boxed{\dot{\rho} + \rho v_{k,k} = 0}$$

$$\frac{\partial \rho}{\partial t} + v_k \frac{\partial \rho}{\partial x_k} + \rho v_{k,k} = 0$$

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0$$

Because $\dot{m} = 0$.

$$\int_V \rho(\vec{x},t)\,dV = \int_{V^\circ} \rho_\circ(\vec{X})\,dV^\circ$$

$$\int_{V^\circ} \Big(\rho(\vec{X},t)\,J - \rho_\circ(\vec{X})\Big)\,dV^\circ = 0$$

$$\rho J = \rho_\circ$$

$$\boxed{(\rho J)^{\bullet} = 0}$$

If $P^*_{ij\ldots}(\vec{x},t)$ (a property per unit volume)

is rewritten as $\rho A^*_{ij\ldots}(\vec{x},t)$ then $A^*_{ij\ldots}(\vec{x},t)$

is the same property per unit mass

$$\dot{P}_{ij\ldots}(t) = \frac{d}{dt}\left(\int_V A^*_{ij\ldots}(\vec{x},t)\, \rho\, dV \right)$$

$$= \frac{d}{dt}\left[\int_{V^\circ} A^*_{ij\ldots}(\vec{X},t)\, \rho\, J\, dV^\circ \right)$$

$$\dot{P}_{ij\ldots}(t) = \int_{V^\circ} \left[\dot{A}^*_{ij\ldots}\, \rho J + A^*_{ij\ldots} \underbrace{(\rho J)^\circ}_{?} \right] dV^\circ$$

$$\boxed{\dot{P}_{ij\ldots}(t) = \int_V \dot{A}_{ij\ldots}\, \rho\, dV} \quad \text{B}$$

Conservation of Linear Momentum

$$p_i(t) = \int_V \rho v_i \, dV$$

$$\frac{dp_i}{dt} = \int_V \rho \dot{v}_i \, dV$$

$$= \int_V \rho b_i \, dV + \int_S t_i^{(n)} \, dS$$

$$= \int_V \rho b_i \, dV + \int_S \sigma_{ij} n_j \, dS$$

$$\int_V \rho \dot{v}_i \, dV = \int_V \rho b_i \, dV + \int_V \sigma_{ij,j} \, dV$$

$$\int_V (\rho \dot{v}_i - \sigma_{ij,j} - \rho b_i) \, dV = 0$$

$$(\rho \dot{v}_i - \sigma_{ij,j} - \rho b_i) = 0$$

Conservation of Energy

Total energy of a system is $E_{total} = E_k + E_{int}$

Kinetic $E_k = \int_V \frac{1}{2} \rho v_i v_i \, dV$ (1) $\quad v_i v_i = \frac{\text{momentum}}{\text{unit mass}}$

Internal $E_{int} = \int_V \rho u \, dV$ (2) $\quad u = \frac{\text{internal energy}}{\text{unit mass}}$

[enthalpy e is a common form
Strain energy $(\sigma_{ij} e_{ij})$ another]

$$\dot{E}_{total} = \int_V \frac{\rho}{2} (v_i v_i)^{\cdot} \, dV + \int_V \rho \dot{u} \, dV$$

$$= \int_V (\rho v_i \dot{v}_i + \rho \dot{u}) \, dV \quad (3)$$

First law of thermodynamics says

Rate of change of total energy = Rate of work done on system + rate of heat input

$$\dot{E} = P + \dot{Q} \qquad (4)$$

$$P = \underbrace{\int_S v_i t_i \, dS}_{\text{rate of work by surface tractions}} + \underbrace{\int_V \rho b_i v_i \, dV}_{\text{rate of work by body forces}} = \boxed{A} + \boxed{B} \qquad (5)$$

$$\boxed{A} = \int_S v_i \sigma_{ij} n_j \, dS = \underbrace{\int_V (v_i \sigma_{ij})_{,j} \, dV}_{\text{Divergence theorem}} = \int_V \underbrace{(v_{i,j} \sigma_{ij} + v_i \sigma_{ij,j})}_{\text{product rule}} dV$$

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$$\boxed{A} = \int_V \Big[L_{ij}\sigma_{ij} + v_i \underbrace{(\rho \dot{v}_i - \rho b_i)}_{\text{conservation of momentum}} \Big] dV$$

$L_{ij} = v_{i,j}$

$L_{ij} = \underbrace{D_{ij}}_{\text{Strain rate}} + \underbrace{W_{ij}}_{\text{Vorticity}}$

$$= \int_V \Big[D_{ij}\sigma_{ij} + \underbrace{W_{ij}\sigma_{ij}}_{=0 \text{ (symmetric} \times \text{antisymmetric)}} + \rho v_i \dot{v}_i - \rho v_i b_i \Big] dV$$

$$= \int_V \Big[D_{ij}\sigma_{ij} + \rho v_i \dot{v}_i - \rho v_i b_i \Big] dV \qquad (6)$$

$$P = \boxed{A} + \boxed{B} \text{ from } (5)$$

$$= \int_V \left[D_{ij}\sigma_{ij} + \rho v_i \dot{v}_i \right] dV \qquad (7)$$

Work done by surface tractions & body forces is converted into strain energy and kinetic energy

$$\dot{Q} = -\int_S \underbrace{q_i n_i \, dS}_{\text{heat flux across boundaries}} + \int_V \underbrace{\rho r \, dV}_{\text{heat sources inside volume}} \qquad (8)$$

$$= \int_V \left[-q_{i,i} + \rho r \right] dV \qquad (9)$$

So first Law in 4 $(\dot{E} = P + \dot{Q})$

$$\Rightarrow \quad \dot{E} = \int_V \left[D_{ij}\sigma_{ij} + \rho v_i \dot{v}_i - q_{i,i} + \rho r \right] dV \qquad (10)$$

Now equate (3) and (10)

$$\int_V (\rho v_i \dot{v}_i + \rho \dot{u})\, dV = \int_V [D_{ij}\sigma_{ij} + \rho v_i \dot{v}_i - q_{i,i} + \rho r]\, dV$$

The kinetic energy change cancels

$$\int_V [\rho \dot{u} - D_{ij}\sigma_{ij} + q_{i,i} - \rho r]\, dV = 0$$

or $$\boxed{\rho \dot{u} = D_{ij}\sigma_{ij} - q_{i,i} + \rho r} \quad (11)$$

[Internal energy changes] due to [work of deformation] [flow of heat] [internal sources]

If internal energy is just enthalpy $e(T)$

$$e(T) = \int_0^T c(\theta)\,d\theta \quad (12)$$

c = specific heat capacity
T = temperature

$$q_i = -k_{ij} T_{,j} \quad (13)$$

Fourier's Law

Then

$$\rho \frac{d}{dt}\left[\int_0^T c(\theta)\,d\theta\right] = D_{ij}\sigma_{ij} + \left(k_{ij}T_{,j}\right)_{,i} + \rho r \quad (14)$$

giving an energy conservation equation in terms of temperature T.