ESS 411/511 Geophysical Continuum Mechanics Class #29

Highlights from Class #28 – Shaan Haque
Today's highlights on Friday – Jason Ott

Today

- Elastic waves

For Friday please read

- Ed's notes on kinematic waves
  (on right sidebar at)
  https://courses.washington.edu/ess511/NOTES/)

Problem Sets

Problem Set #6 is partly graded

- I hope to have results for you by problem session tomorrow

Problem Set #7

- Due today

For Problems Lab on Thursday

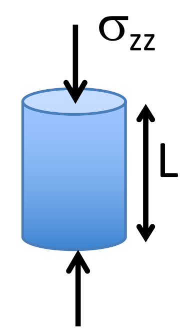
- Study Questions for take-at-home Final exam.

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Elastic solid

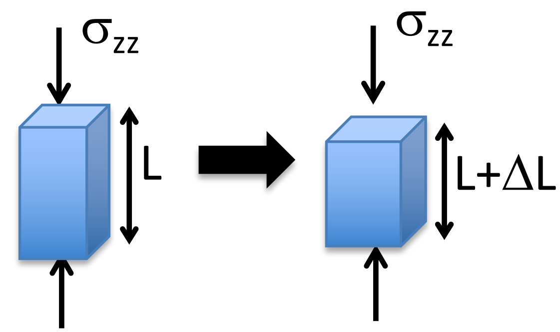


Stress in terms of strain $$\boxed{\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}}$$

Strain in terms of stress $$\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu \delta_{ij} \sigma_{kk}\right]$$

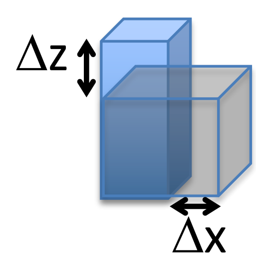
Young's modulus $$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

(Relates normal stresses and strains)



Poisson's ratio $$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

(Ratio of strains in direction of stress and normal to stress)



Other combinations of elastic parameters are possible

$$\boxed{\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}} \qquad \epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu \delta_{ij} \sigma_{kk}\right]$$

Young's modulus E = stress/strain
(or slope of stress-strain relation)

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$ (units?)

Shear modulus $G = \frac{E}{2(1+\nu)} = \mu$ (units?)

(Relates shear stress to shear strain)

Poisson's ratio
ν = - transverse strain/axial strain

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$ (units?)

Bulk modulus $K = \frac{E}{3(1-2\nu)}$ (units?)

(Ratio of volumetric stress to volumetric strain)

Elastic waves

Equations of motion $$\rho \ddot{u}_l = \sigma_{lm,m} + b_l$$

ρ is density
u_l is displacement
b_l is body force

(units of all terms?)

- How is this related to Newton's 2nd law?

Stress $$\sigma_{lm} = \lambda\,\delta_{lm}\,\epsilon_{nn} + 2\mu\,\epsilon_{lm}$$

Strain $$\epsilon_{lm} = \frac{1}{2}\left(u_{l,m} + u_{m,l}\right)$$

Putting it together

$$\begin{aligned}
\rho \ddot{u}_l &= \left[\lambda\,\delta_{lm}\epsilon_{nn} + 2\mu\,\epsilon_{lm}\right]_{,m} + b_l \\
&= \left[\lambda\,\delta_{lm}u_{n,n} + \mu\left(u_{l,m} + u_{m,l}\right)\right]_{,m} + b_l \\
&= \lambda\,u_{n,nl} + \mu\,u_{l,mm} + \mu\,u_{m,lm} + b_l \\
&= (\lambda + \mu)\,u_{n,nl} + \mu\,u_{l,nn} + b_l
\end{aligned}$$

Elastic waves

Putting it together

$$\begin{aligned}\rho \ddot{u}_l &= \left[\lambda\, \delta_{lm}\epsilon_{nn} + 2\mu\, \epsilon_{lm}\right]_{,m} + b_l \\ &= \left[\lambda\, \delta_{lm}u_{n,n} + \mu\,(u_{l,m} + u_{m,l})\right]_{,m} + b_l \\ &= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\ &= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l\end{aligned}$$

Let's look for plane-wave solutions

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A} \exp\left(i\, \mathbf{k} \cdot \mathbf{x} - i\, \omega t\right) \qquad (1)$$

$\mathbf{x}$ is the position vector $\left(i = \sqrt{-1}\right)$

$\mathbf{A}$ is a polarization vector

direction and magnitude of displacements $\mathbf{u}$ during wave motion

$\mathbf{k}$ is a wavenumber

direction of wave propagation and the spatial periodicity of the waves.

In index notation, (1) is written as

$$u_l(\mathbf{x}, t) = A_l \exp\left(i\, k_m x_m - i\, \omega t\right) \qquad (2)$$

Elastic waves

Equations of motion

$$
\begin{aligned}
\rho \ddot{u}_l &= [\lambda\, \delta_{lm} \epsilon_{nn} + 2\mu\, \epsilon_{lm}]_{,m} + b_l \\
&= [\lambda\, \delta_{lm} u_{n,n} + \mu\, (u_{l,m} + u_{m,l})]_{,m} + b_l \\
&= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\
&= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l
\end{aligned}
$$

In absence of body forces, set $b_l = 0$

Put (2) into equations of motion

$$u_l(\mathbf{x}, t) = A_l \exp\left(i\, k_m x_m - i\, \omega t\right) \tag{2}$$

After a bunch of algebra (in the notes),

$$\rho\, \omega^2 A_l = (\lambda + \mu)\, A_n k_n k_l + \mu\, A_l k_n k_n$$

When ***A*** and ***k*** are parallel, we get compressional waves (p waves)
When ***A*** and ***k*** are orthogonal, we get shear waves (s waves)

When ***A*** and ***k*** are parallel, we get compressional waves (primary, or p waves)

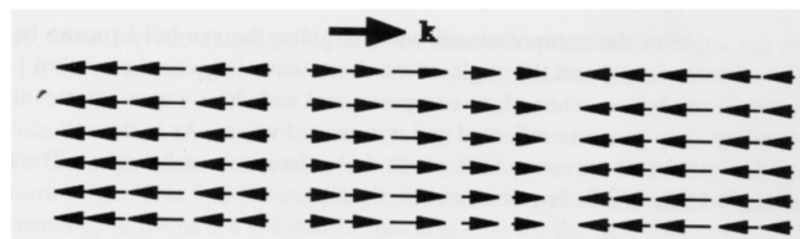
$$\mathbf{A} = c\mathbf{k} \qquad (*)$$

$$\rho\,\omega^2 A_l = (\lambda + \mu)\,A_n k_n k_l + \mu\,A_l k_n k_n$$

Use (*): $$\rho\,\omega^2 c\,k_l = (\lambda + \mu)\,c\,k_n k_n k_l + \mu\,c\,k_l k_n k_n$$

Cancel $c\,k_l$ (*): $$\begin{aligned}\rho\,\omega^2 &= (\lambda + \mu)\,k_n k_n + \mu\,k_n k_n \\ &= (\lambda + \mu)\,k_n k_n\end{aligned}$$

$$\frac{\omega^2}{|\mathbf{k}|^2} = \left(\frac{\lambda + 2\mu}{\rho}\right)$$



Recall

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A}\exp\left(i\,\mathbf{k}\cdot\mathbf{x} - i\,\omega t\right)$$

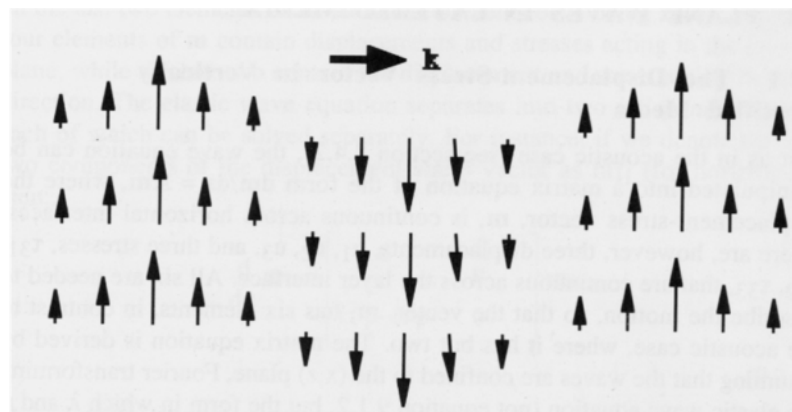
Phase velocity of primary wave (p wave) $$\alpha = \frac{\omega}{|\mathbf{k}|} = \sqrt{\left(\frac{\lambda + 2\mu}{\rho}\right)}$$

When ***A*** and ***k*** are orthogonal, we get shear waves (secondary, or s waves)

$$A_n k_n = 0 \qquad (**)$$

$$\rho\,\omega^2 A_l = (\lambda + \mu)\, A_n k_n k_l + \mu\, A_l k_n k_n$$

Use (**), then cancel A_l (*)



$$\rho\,\omega^2 = \mu\, k_n k_n$$

$$\frac{\omega^2}{|\mathbf{k}|^2} = \frac{\mu}{\rho}$$

Phase velocity of secondary wave (s wave)

$$\beta = \frac{\omega}{|\mathbf{k}|} = \sqrt{\left(\frac{\mu}{\rho}\right)}$$

Other complications

Then it gets more messy …

- Finite domains need boundary conditions
- Reflections and refraction at boundaries
- Energy dissipation and temperature effects
- Finite wave packets (each Fourier component is a sine wave)
- Spatial gradients and anisotropy in elastic coefficients
- Others?   (consult a seismologist ☺)

Class prep – Dynamic waves (e.g. elastic seismic waves) are described by a momentum conservation equation, whereas kinematic waves are described by a mass conservation or continuity equation.

Assignment

- What is the order of the differential equation describing a dynamic wave?
- What is the order of the differential equation describing a kinematic wave?
- How does that difference affect the possible waves?

In 1-D, a continuity equation for density $\rho(x,t)$ is written as

$$\frac{\partial \rho}{\partial t} + c(x,t)\frac{\partial \rho}{\partial x} = 0$$

where the kinematic wave velocity $c(x,t)$ is related to material velocity v through

$$c(x,t) = \frac{\partial q}{\partial \rho} = \frac{\partial\,(\rho\, v)}{\partial \rho} = \left(v + \rho\frac{\partial v}{\partial \rho}\right)$$

- In a river or tidal estuary, ρ is water depth.
- Is $\partial v/\partial \rho$ positive or negative?
- Does the wave (e.g. tidal bore) move faster or slower than the water?

On a highway, ρ is vehicles per km.
Is $\partial v/\partial \rho$ positive or negative?
Does the wave in traffic move faster or slower than the vehicles?

Faculty peer teaching reviews

In addition to your student course evaluations, we also depend on faculty peer teaching reviews to assess the effectiveness of our teaching, and how to improve it.

- This is particularly important for new faculty members who are developing their teaching methods and classes and are being reviewed for promotion or tenure.
- Brad has asked me to write a peer teaching review of his contributions in this course.
- In addition to my own observations, I am also interested in your feedback for my report. Any comments will (a) be anonymous, and (b) will not be seen by Brad until after course grades have been submitted. ☺
- What is working well? What have you liked about Brad's teaching and course contributions?
- What had been effective? (The UW values effective teachers.) What has not?
- Are there things that he could improve in future years? (The UW values Faculty who continually work to improve their teaching.)