ESS 411/511 Geophysical Continuum Mechanics Class #30

Highlights from Class #29 – Chloe Mcburney
Today's highlights on Monday – Alexandria Vasquez-Hernandez
(at ESS 511 term-project reports)

Today
- Elastic plane waves
- Kinematic waves

Based on notes on right sidebar at
https://courses.washington.edu/ess511/NOTES/

- Ed's notes on elastic waves
- Ed's notes on kinematic waves

ESS 411/511 Geophysical Continuum Mechanics Class #30

ESS 511 term-project reports

- Monday 8:30-10:20 a.m. (which would have been our final-exam slot)
- 5 reports
- 10 minutes to present
- 5 minutes for questions, discussions, and transition

- Do we need to start at 8:30?

Problem Sets

Problem Set #7

- Due on Sunday …

I am preparing some brief notes about issues that you encountered on the Mid-term and on Problem Sets #4, #5, and #6

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Adapting to classes on Zoom

Warm-up –
(break-out rooms)

Roll waves on a Seattle sidewalk
(Yes, it occasionally rains here … ☺)

- Are these kinematic waves, or dynamic waves?
- Where is the water the deepest?
- Which is moving faster, the water or the wave?

Elastic waves

ρ is density
u_l is displacement
b_l is body force

Equations of motion $$\rho \ddot{u}_l = \sigma_{lm,m} + b_l$$

Stress $$\sigma_{lm} = \lambda\, \delta_{lm}\, \epsilon_{nn} + 2\mu\, \epsilon_{lm}$$

Strain $$\epsilon_{lm} = \frac{1}{2}\left(u_{l,m} + u_{m,l}\right)$$

Putting it together

$$\begin{aligned}
\rho \ddot{u}_l &= \left[\lambda\, \delta_{lm}\epsilon_{nn} + 2\mu\, \epsilon_{lm}\right]_{,m} + b_l \\
&= \left[\lambda\, \delta_{lm} u_{n,n} + \mu \left(u_{l,m} + u_{m,l}\right)\right]_{,m} + b_l \\
&= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\
&= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l
\end{aligned}$$

Putting it together

Elastic waves

$$
\begin{aligned}
\rho \ddot{u}_l &= [\lambda\, \delta_{lm}\epsilon_{nn} + 2\mu\, \epsilon_{lm}]_{,m} + b_l \\
&= [\lambda\, \delta_{lm} u_{n,n} + \mu\, (u_{l,m} + u_{m,l})]_{,m} + b_l \\
&= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\
&= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l
\end{aligned}
$$

Lets look for plane-wave solutions

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A} \exp\left(i\, \mathbf{k} \cdot \mathbf{x} - i\, \omega t\right) \qquad (1)$$

$\mathbf{x}$ is the position vector

$\mathbf{A}$ is a polarization vector

direction and magnitude of displacements $\mathbf{u}$ during wave motion

$\mathbf{k}$ is a wavenumber

direction of wave propagation and the spatial periodicity of the waves.

In index notation, (1) is written as

$$u_l(\mathbf{x}, t) = A_l \exp\left(i\, k_m x_m - i\, \omega t\right) \qquad (2)$$

Elastic waves

Equations of motion

$$\begin{aligned}\rho \ddot{u}_l &= \left[\lambda\, \delta_{lm}\epsilon_{nn} + 2\mu\, \epsilon_{lm}\right]_{,m} + b_l \\ &= \left[\lambda\, \delta_{lm}u_{n,n} + \mu\,(u_{l,m} + u_{m,l})\right]_{,m} + b_l \\ &= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\ &= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l\end{aligned}$$

In absence of body forces, set b_l=0

Put (2) into equations of motion

$$u_l(\mathbf{x}, t) = A_l \exp\left(i\, k_m x_m - i\, \omega t\right) \qquad (2)$$

After a bunch of algebra (in the notes),

$$\rho\, \omega^2 A_l = (\lambda + \mu)\, A_n k_n k_l + \mu\, A_l k_n k_n$$

When ***A*** and ***k*** are parallel, we get compressional waves (p waves)
When ***A*** and ***k*** are orthogonal, we get shear waves (s waves)

Elastic waves

Equations of motion

$$\begin{aligned} \rho\ddot{u}_l &= \left[\lambda\,\delta_{lm}\epsilon_{nn} + 2\mu\,\epsilon_{lm}\right]_{,m} + b_l \\ &= \left[\lambda\,\delta_{lm}u_{n,n} + \mu\,(u_{l,m} + u_{m,l})\right]_{,m} + b_l \\ &= \lambda\,u_{n,nl} + \mu\,u_{l,mm} + \mu\,u_{m,lm} + b_l \\ &= (\lambda + \mu)\,u_{n,nl} + \mu\,u_{l,nn} + b_l \end{aligned}$$

In absence of body forces, set b_l=0

Put (2) into equations of motion

$$u_l(\mathbf{x}, t) = A_l \exp\left(i\,k_m x_m - i\,\omega t\right) \qquad (2)$$

After a bunch of algebra (in the notes),

$$\rho\,\omega^2 A_l = (\lambda + \mu)\,A_n k_n k_l + \mu\,A_l k_n k_n$$

When ***A*** and ***k*** are parallel, we get compressional waves (p waves)
When ***A*** and ***k*** are orthogonal, we get shear waves (s waves)

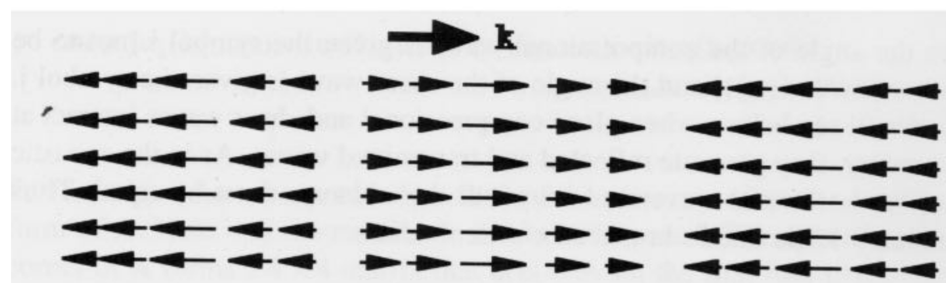
When ***A*** and ***k*** are parallel, we get compressional waves (p waves)

$$\mathbf{A} = c\mathbf{k}$$

$$\rho\,\omega^2 c\,k_l = (\lambda+\mu)\,c\,k_n k_n k_l + \mu\,c\,k_l k_n k_n$$

$$\begin{aligned}\rho\,\omega^2 &= (\lambda+\mu)\,k_n k_n + \mu\,k_n k_n \\ &= (\lambda+\mu)\,k_n k_n\end{aligned}$$

$$\frac{\omega^2}{|\mathbf{k}|^2} = \left(\frac{\lambda+2\mu}{\rho}\right)$$



Phase velocity
of p wave

$$\alpha = \frac{\omega}{|\mathbf{k}|} = \sqrt{\left(\frac{\lambda+2\mu}{\rho}\right)}$$

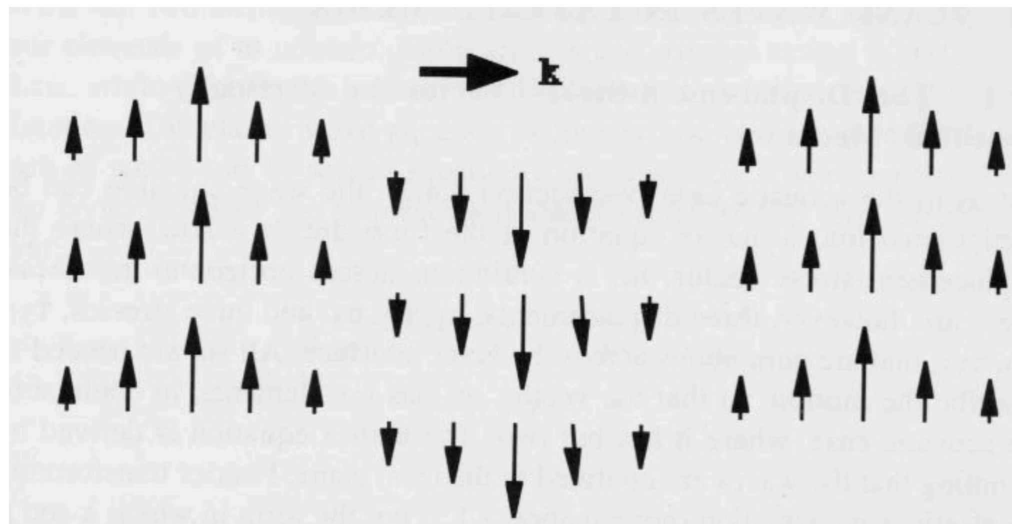
When ***A*** and ***k*** are orthogonal, we get shear waves (s waves)

$$A_n k_n = 0$$

$$\rho\,\omega^2 = \mu\, k_n k_n$$

$$\frac{\omega^2}{|\mathbf{k}|^2} = \frac{\mu}{\rho}$$

$$\beta = \frac{\omega}{|\mathbf{k}|} = \sqrt{\left(\frac{\mu}{\rho}\right)}$$



Let’s look at some waves

https://courses.washington.edu/ess511/NOTES/EDS_NOTES/roll_waves_2019_12_20_FV_Ave_E.mp4

Dynamic and kinematic waves in 1-D

Dynamic wave in 1-D

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

c is a wave velocity

Initial condition

- $u(x,0) = u_0(x)$

Boundary conditions

- infinite domain

Solutions

$$u(x,t) = u_0(x - ct)$$

$$u(x,t) = u_0(x + ct)$$

Waves can travel in both directions at velocity *c*

Kinematic wave in 1-D

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

c is a wave velocity

Initial condition

- $u(x,0) = u_0(x)$

Boundary conditions

- infinite domain

Solutions

$$u(x,t) = u_0(x - ct)$$

Waves can travel in only one direction at velocity *c*

Kinematic waves – from a continuity equation

Continuity equation $$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0$$

ρ is density

q is flux

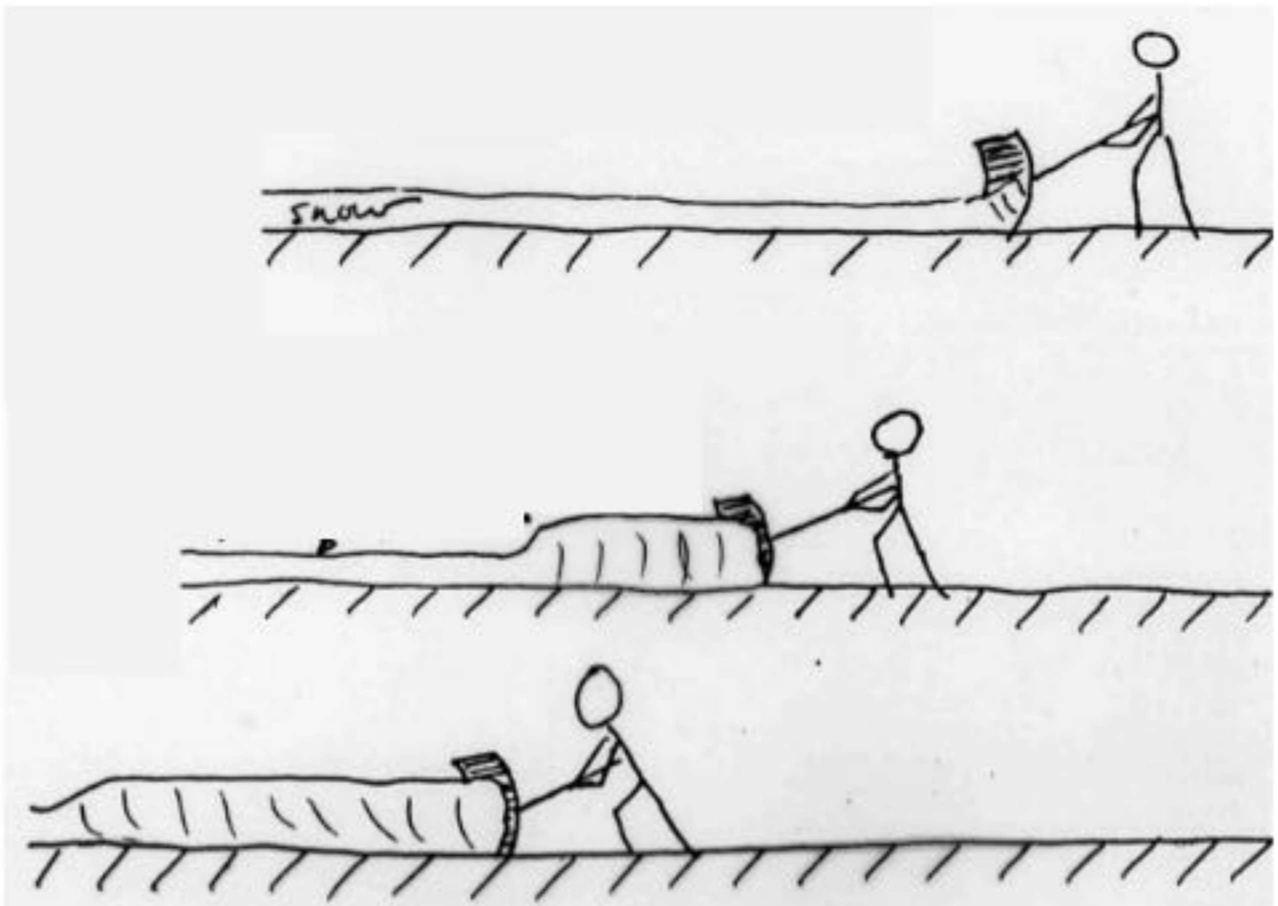
Flux gradient $$\frac{\partial q(\rho)}{\partial x} = \frac{\partial q}{\partial \rho}\frac{\partial \rho}{\partial x}$$

Wave speed c $$\frac{\partial \rho}{\partial t} + c(x,t)\frac{\partial \rho}{\partial x} = 0$$

$$c(x,t) = \frac{\partial q}{\partial \rho} = \frac{\partial\left(\rho\, v\right)}{\partial \rho} = \left(v + \rho\frac{\partial v}{\partial \rho}\right)$$

Key question how does v depend on ρ?

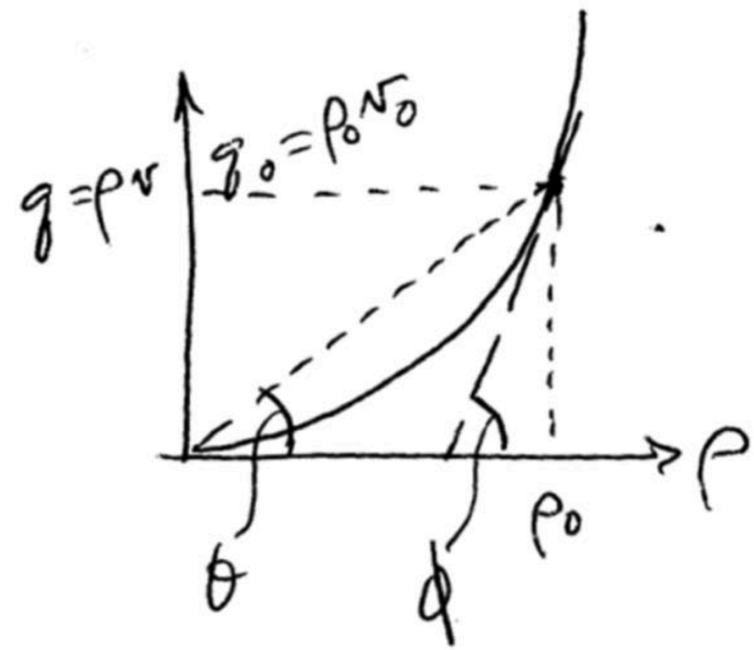
dv/dρ >0



Flood waves

$$v(\rho) = K\rho^{\alpha}$$

$$q(\rho) = K\rho^{1+\alpha}$$



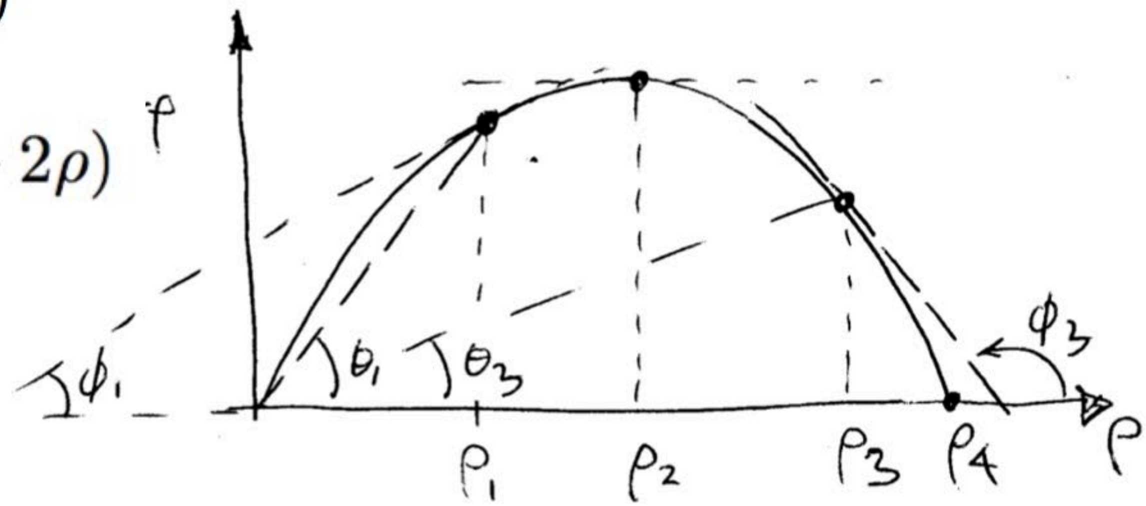
Traffic waves

$$v(\rho) = K(\rho_{\max} - \rho)$$

$$c(\rho) = \frac{\partial q}{\partial \rho} = K(\rho_{\max} - 2\rho)$$

$$q(\rho) = K\rho(\rho_{\max} - \rho)$$

$$v_{\max} = K\rho_{\max}$$



$$c_1 = \left[\frac{\partial q}{\partial \rho}\right]_1 = \tan(\phi_1)$$

$$v_1 = q_1/\rho_1 = \tan(\theta_1)$$