Earth and Space Sciences 411/511
Geophysical Continuum Mechanics
Fall 2021

Study Questions for Final Exam

The exam is closed-book. You are welcome to use a calculator and a sheet of notes.
The time for the exam will be 2.5 hours. If you need more time, please draw and annotate a line on your answer sheets after 2.5 hours to indicate how far you got in 2.5 hours, then continue.
The actual exam will be composed of 4 of these questions, or closely related questions.
Be sure to explain **in words** what you are doing at each step, and read the **Tips for Writing an Exam** on the class web site.
We strongly encourage you to try all the questions before you take the test. If you try to write the test without solving the questions ahead, you will run out of time.

1. Material and spatial coordinates

(a) Particles originally at $\boldsymbol{X}=(X_1,X_2,X_3)$, are subsequently found at time t at positions given by

$x_1=X_1 + X_3\,(\exp(-t) - 1)$
$x_2=X_2\exp(-t)$
$x_3=X_3\exp(t)$

- Find the particle that is at $\boldsymbol{x} = (x_1, x_2, x_3)$ at time t.
- Express the velocity field in spatial coordinates.
- Express the displacement $\boldsymbol{u}$ in both material and spatial form.

(b) The temperature field $T(\boldsymbol{x},t)$ in the body follows

$T = \exp(-t)\,(3x_1 + x_2 - 2x_3)$

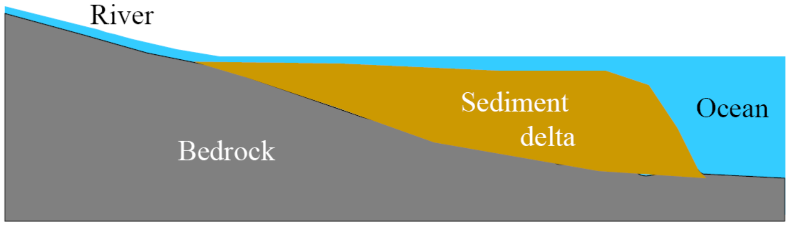
- Express the rate of change of temperature in both material and spatial form.

2. Troubles at Mardi Gras

According to an article in ***Eos*** [November 07, 2006, 87(45)], coastal Louisiana and the Mississippi delta are sinking. Who would have thought there might be consequences? See also T.H. Dixon (2015) ***Eos*** [August 27, 2015]. See website for more papers about New Orleans.

The 2006 article identifies 3 different mechanisms of sinking, quite apart from the challenge of rising sea level. These sinking mechanisms include:

1. Compaction of Holocene sediments (the Holocene is the most recent 10,000 years, since the end of the last Ice Age). The sediments have high porosity and high water content when deposited, and the water is squeezed out over time, due to weight of additional sediment deposited on top.
2. Isostatic sinking of Earth's lithosphere due to the sediment load deposited in the river delta.
3. Normal faulting of the frontal parts of the delta, as blocks of sediment slip off into deeper water. (See figure, and recall normal faulting of the Tibetan Plateau as seen on the World Stress Map.)



Vertically exaggerated cartoon of a delta built from sediments carried by a river.

Concerned that tourist brochures printed in future years will have to be modified to advise visitors to bring scuba gear, the Greater New Orleans ***Mardi Gras Finance, Food, and Partying Advisory Committee*** wants to hire an acknowledged world expert in *Geophysical Continuum Mechanics* to advise them on the future of relative sea level (RSL) in and around their fair city.
Congratulations, you have just won the contract!

- Describe in prose, in approximately 1 page (maximum 2 pages), the physical processes that your model includes for each of the 3 mechanisms, how you describe the mechanism in your model, and the physical parameters that your model needs in order to compute strain and strain rates. You are welcome to include a few diagrams and equations, but don't try to include derivations. (Politicians are not interested in derivations.)

3. Tensors

a) *Strain compatibility*

A strain tensor ε_{ij} for small displacements u_i can be written as $2\varepsilon_{ij} = u_{i,j} + u_{j,i}$. This tensor has 9 elements, but due to symmetry, at most 6 of those can take distinct values.

- Which elements are distinct, and which ones are not?
- Can those 6 elements of ε_{ij} be assigned to be independent functions of position x_i, and still represent deformation of a continuum expressed by a 3-component vector u_i without overlaps or gaps? Why or why not?

Consider the symmetric matrix

$$\begin{bmatrix} x_1 & x_1 x_2 & 0 \\ x_1 x_2 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where (x_1, x_2, x_3) is the position vector.

- Prove by direct manipulation of the matrix elements whether this matrix corresponds to a continuous single-valued strain field u_i (or not).
- Verify your result with the strain compatibility equations.

b) *Stress conventions*

In continuum mechanics, stress σ_{ij} is the force per unit area acting in direction $\mathbf{e}_j$ on the coordinate face whose normal is $\mathbf{e}_i$. With this definition, extensile principal stresses are positive, and compressive principal stresses are negative.

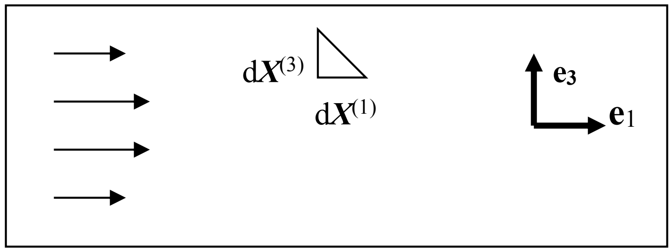
However, in a convention often used in earth sciences, shear stresses σ_{ij} are given by the force per unit area acting in direction $\mathbf{e}_j$ on coordinate face whose normal is $\mathbf{e}_i$, but compressive normal stresses (i=j) are positive and extensile normal stresses are negative.

- Show that a matrix σ_{ij} whose elements follow the geological convention is not a consistent representation of the physics. (Hint – it is enough to find one counter-example.)

4. Strain rate and strain on a lava flow

Viscous lava is flowing in a long uniform rectilinear channel with width $2W$. Coordinate direction $\mathbf{e}_1$ is aligned with the channel and $\mathbf{e}_3$ is the cross-channel direction. The origin is at the channel center. An intrepid volcanologist suspects that the velocity field of the lava at the surface has the form

$$v_1 = v_0\left(1-\left(\frac{x_3}{W}\right)^2\right)$$
$$v_2 = 0$$
$$v_3 = 0$$



Her team of volcanologists wants to monitor strains on the lava crust as it moves. Using a helicopter, the team has placed 3 (very heat-resistant!) markers forming a small right isosceles triangle on the lava surface, such that the 2 equal sides $d\boldsymbol{X}^{(1)}$ and $d\boldsymbol{X}^{(3)}$ were initially aligned with the $\mathbf{e}_1$ and $\mathbf{e}_3$ axes at time $t=0$. They were unable to place the markers on the centerline of the flow.

a) Find the deformation-gradient tensor $\mathbf{F}_{iA}$ at time t.
b) Using the deformation-gradient tensor, find the lengths and orientations of the 3 sides of the triangle at all subsequent times t.
c) Find the Lagrangian finite-strain tensor that describes the deformation field as a function of X_1, X_3, and t.
d) Find the Eulerian finite-strain tensor that describes the deformation field as a function of x_1, x_3, and t.
e) Discuss the difference between your answers in c) and d).

5. Conservation laws

Explain in prose and diagrams:

- The basic concepts underlying the development of conservation laws for mass, momentum, and energy in a continuum. (While the concepts should be explained clearly, there is no need for lengthy derivations.)
- The assumptions that have to be made about the material.
- What are constitutive equations and why are they useful or necessary?

6. Stress in rock towers

In 3 dimensions, Hooke's law for an isotropic medium is $\sigma_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij}$ and λ and μ are the Lamé parameters. Strain can be described by $e_{ij} = (1/E)(\,(1+\nu)\,\sigma_{ij} - \nu\,\delta_{ij}\,\sigma_{kk})$. Young's modulus E and Poisson's ratio ν can also be written in terms of λ and μ.

In isotropic media, the principal axes of stress and strain coincide, so in the principal coordinate system, $e_{(q)} = (1/E)\,((1+\nu)\,\sigma_{(q)} - \nu(\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)})\,)$ where q takes the values 1, 2, or 3. (Also see Turcotte and Schubert, beginning of Chap 3.)

Devil's Tower in Wyoming, like other tall and thin volcanic necks around the world, can be approximated by a rectangular column of rock of height h and footprint $w\times w$ that is free to expand or contract in the horizontal directions. Some geophysicists are also rock climbers, and as they ascend Devil's Tower, they almost all wonder how much the tower has shortened due to gravity (thereby bringing the summit closer to the surrounding plains, and shortening their climb). Coincidently, they also wonder how much the tower has expanded in the horizontal directions because of gravity (shortening their walk from the parking lot to the base of the tower).

(a) Horizontal stresses are 0 ($\sigma_{(2)}=\sigma_{(3)}=0$). Why?
(b) What is the vertical stress at a distance z from the top of the column?
(c) What is the vertical strain distribution $e_{(1)}(z)$? Use the equations above.
(d) Now, the rock above any given depth has also contracted due to the mass above it. So to get how much the entire length of the column contracts in the vertical at $z=h$, integrate the strain $e_{(1)}(z)$ dz from 0 to h.
(e) How much did it strain horizontally, and how much did it expand horizontally? Use the equations above.
(f) Now obtain numerical values for (d) and (e) for h = 500 m, w = 200 m, ρ = 2700 kg m^{-3}, E = 70 GPa, ν = 0.25.

7. Stress in sedimentary basins

In 3 dimensions, Hooke's law for an isotropic medium is $\sigma_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij}$ and λ and μ are the Lamé parameters. Strain can be described by $e_{ij} = (1/E)(\,(1+\nu)\,\sigma_{ij} - \nu\,\delta_{ij}\,\sigma_{kk})$. Young's modulus E and Poisson's ratio ν can also be written in terms of λ and μ.

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Sometimes a geologist needs to estimate the state of stress in the crust or under a sedimentary basin. On time scales long enough that the region behaves like a fluid, the stress tensor is $-P\,\mathbf{I}$ where $P = \rho g z$.

However, on times scales at which the material behaves elastically, σ_{xx} and σ_{zz} are not equal, and the stress tensor depends on the boundary conditions for the problem (i.e., how much support there is from the sides). For example, sedimentary basins in Nevada are grabens bounded by very steep normal faults, and fed by sediments eroded off nearby mountains. These basins could be approximated by rectilinear troughs with rigid vertical sides. The rigidity of the sides implies no horizontal strain i.e., $e_2=e_3=0$.

Let's investigate the stress state at an originally-unstressed surface that has been rapidly covered with sediments of density ρ = 2500 kg/m^3 to a depth of 5 km in the recent past.

a) What is the expression for vertical stress at depth z in the sediments? At the buried former surface?
b) What are the expressions for the horizontal normal stresses there? (They are also compressive, but smaller than the vertical stress.) Find their values, assuming that ν = 0.25 and E=70GPa.
c) Write the complete stress tensor for a point at depth z in the center of the basin.
d) What is the relationship between σ_{xx}, σ_{yy} and σ_{zz}?
e) If all the sediments are removed rapidly on a time scale too short for the crust to behave like a fluid, the surface will return to its original state of stress prior to sedimentation. However, over

very long times, crustal materials flow viscously and the unequal principal stresses at depth approach lithostatic. Write the stress tensor at depth z after such a long time has passed.

f) If 2 km of material is then rapidly eroded, what will the vertical stress be at the newly-exposed surface?

g) If the material remained confined during the rapid erosion (i.e., $e_2 = e_3 = 0$), what will the horizontal compressive stresses be at the newly-exposed surface? Because we can assume that the sedimentary material is elastic on this short time scale, the equations above can also be used to relate changes $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\sigma_3$.
[Note that Turcotte and Schubert state "This mechanism is one explanation for the wide-spread occurrence of near-surface compressive stresses in the continents."]

8. A Moving water blob

The temperature in an ocean is uniform at T_0, except for a cylindrical blob of warm water with radius R=R_0, whose temperature structure is given in the material description ($R^2 = X^2 + Y^2$) by

$$T(R,t) = T_0 + \Delta T(t)\cos\left[\frac{\pi R}{2R_0}\right] \qquad \text{for } R \le R_0.$$

$$T(R,t) = T_0 \qquad \text{for } R > R_0. \qquad (1)$$

The temperature does not vary with depth. The blob is moving at velocity $\mathbf{v}(t)$, whose magnitude can vary through time. Its peak temperature anomaly $\Delta T(t)$ may also vary through time, but the blob maintains its spatial shape (half cycle of cosine). It also moves in a straight line. Let's align the x axis with the velocity vector $\mathbf{v}(t)$ of the blob, so that

$$\mathbf{v}(t) = [u(t), v(t), w(t)] = [u(t), 0, 0] \qquad (2)$$

a) Suppose the blob maintains its temperature, i.e. $\Delta T(t)$=ΔT_0 in Equation (1).

- What is the material derivative of temperature for the water parcel at the center of the blob, as it moves along the x axis?

b) Also suppose that the velocity is constant, i.e. $\mathbf{v}$=[u_0, 0, 0]. At time t=0, the center of the blob passes a moored buoy at $\mathbf{x}$=[0, 0, 0].
[**Hints**: The independent variable in your solution will need to be a propagating function like a wave, i.e. with a form like (x-u_0t). You also need to account for the limited spatial extent of the blob, as in Equation (1) above.]

i. Sketch the temperature profile along the x axis at times t= -R_0/u_0, 0, and R_0/u_0.
ii. Write down the equations that describe the temperature $T(x,t)$ in spatial coordinates along the x axis (y=z=0) for all times t.
iii. Find the spatial gradient of temperature $\partial T(x,t)/\partial x$ along the x axis at all times.
iv. Find the spatial description of the rate of change of temperature $\partial T(x,t)/\partial t$ at all points along the x axis.

c) Now, what happens at the moored buoy at $\mathbf{x}$=[0, 0, 0]?

i. Find the temperature history recorded by the moored buoy.

ii. Find the spatial gradient of temperature at the buoy at x=0 at all times.

iii. Find the rate of change of temperature specifically at the moored buoy at x=0.

d) Now find the rate-of-change history $\partial T(0,t)/\partial t$ at the moored buoy at x=0 using the material-derivative equation in spatial coordinates, using your answers to a) and c) ii., and Equation (2)

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathrm{v} \bullet \nabla T = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Does your result agree with your answer in part c) iii. above?