Earth and Space Sciences 411/511

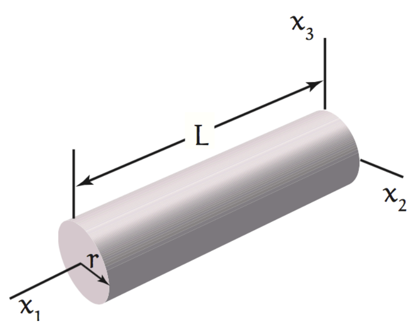
Geophysical Continuum Mechanics
Autumn 2021 Midterm Study Questions

The Midterm exam will be a take-at-home test with a time limit of 2.5 hours. It should be taken without access to notes, or books, or the web, or other people. If you need more time, please draw and annotate a line on your answer sheets after 2.5 hours to indicate how far you got in 2.5 hours, then continue.
The exam will consist of 4 questions, which will be selected from the questions below, perhaps after slight modifications of numerical constants. We strongly encourage you to try all the questions before you take the test. If you try to write the test without solving the questions ahead, you will run out of time.
Be sure to explain **in words** what you are doing at each step, and read the **Tips for Writing an Exam** on the class web site.

Problem 1. Strength of the crust
Assuming the crust has numerous preexisting faults in all directions, use Mohr circles to examine the strength of the crust, which is defined by the maximum σ_1 - σ_3 that the crust can support before faulting occurs to relieve stress. Assume that σ_1 and σ_3 are both negative, with $\sigma_1 > \sigma_3$ (the mathematical/engineering convention in Mase, Smelser, and Mase) and $\sigma_2 = \sigma_3$. Draw the sliding line $\sigma_S = -\mu\, \sigma_N$ with coefficient of sliding friction $\mu = \tan\alpha$ and angle of sliding friction $\alpha = 35°$. (Note faults have negligible cohesion.) Mark the lithostatic stress (vertical normal stress) associated with a depth of 6 km, that is, ρgz, with $\rho = 2800$ kg/m^3 and $g = 9.8$ m/s^2 (ignore possible pore pressure). Near Earth's surface, one of the principal stresses must be vertical; the others will be horizontal. Draw the Mohr circle for σ_1 vertical and strength (σ_1-σ_3) as large as possible. Then draw the Mohr circle for σ_3 vertical and strength (σ_1-σ_3) as large as possible. Carefully estimate the ratio of the strength of the crust in compression to its strength in extension. Sketch the strength diagram down to 6 km for this sliding line.

Problem 2. (M&M and MSM 3.9)
The stress components in a circular cylinder of length L and radius r are given by

$$[t_{ij}] = \begin{bmatrix} Ax_2 + Bx_3 & Cx_3 & -Cx_2 \\ Cx_3 & 0 & 0 \\ -Cx_2 & 0 & 0 \end{bmatrix}$$



(a) Verify that in the absence of body forces, the equilibrium equations (Mase, Smelser, & Mase, Eq. 3.27) are satisfied.
(b) Show that the stress vector vanishes at all points on the curved surface of the cylinder.

Hint: You can write a unit normal vector to the curved surface of the cylinder as
$\boldsymbol{n} = (x_2/r)\, \boldsymbol{e_2} + (x_3/r)\, \boldsymbol{e_3}$ *or* $\boldsymbol{n} = 1/r\,(0, x_2, x_3)$.

Problem 3. Mohr's circles for stress

Explain in prose (use of neat diagrams to illustrate your prose is good):
(a) what Mohr's circles for stress represent about a stress state,
(b) how they can be constructed, and
(c) why they are useful for investigating fault failures.

Problem 4. Mohr's circles and fracturing

Assume that σ_1 and σ_3 are both negative, σ_2 is halfway between σ_1 and σ_3, $\sigma_1 \geq \sigma_3$, (mathematical/engineering convention in Mase, Smelser, and Mase). Internal friction exists (i.e., $n = \tan\phi > 0$). Label all relevant axes, lines, points, angles etc.

(a) Use Mohr circle diagrams to show why

- rocks at depth do not fracture under lithostatic pressure alone and
- the deviatoric stress needed for fracture increases at greater depth.

(b) Suppose a rock under compression in all directions (e.g. deep in the Earth) is stressed close to its brittle limit. Show graphically which change is more likely to make the rock fracture:

- decreasing the ***magnitude*** of σ_1 or
- increasing the ***magnitude*** of σ_3 by the same amount.
- Please comment – is it intuitive that reducing the applied stress on a rock can make it break?

Problem 5. (M&M or MSM 3.29)

In a continuum, the stress field relative to axes $Ox_1x_2x_3$ is given by

$$[t_{ij}] = \begin{bmatrix} x_1^2 x_2 & x_1(1-x_2^2) & 0 \\ x_1(1-x_2^2) & \frac{1}{3}(x_2^3 - 3x_2) & 0 \\ 0 & 0 & 2x_3^2 \end{bmatrix}$$

Determine

(a) the body force distribution if the equilibrium equations are satisfied through-out the field,
(b) the principal stresses at point P = $(a, 0, 2a^{1/2})$,
(c) the maximum shear stress at P,
(d) the principal deviator stresses S_{ij} at P.

Note typo in 3rd edition; σ_{22} *should be* $\frac{1}{3}(x_2^3 - 3x_2)$.

Answer (which would ***not*** be provided on actual test.)

(α) $b_1 = b_2 = 0$, $b_3 = -4x_3/\rho$.
(b) $\sigma_I = 8a$, $\sigma_{II} = a$, $\sigma_{III} = -a$
(c) $(\sigma_s)_{max} = \pm 4.5\, a$
(d) $S_I = \frac{16}{3}a, S_{II} = -\frac{5}{3}a, S_{III} = -\frac{11}{3}a$

Problem 6. (M&M or MSM 3.22)
The state of stress referred to axes $Px_1x_2x_3$ is given in MPa by the matrix

$$[t_{ij}] = \begin{bmatrix} 9 & 12 & 0 \\ 12 & -9 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Determine

(a) the normal and shear components, σ_N and σ_S, respectively, on the plane at P whose unit normal is $\hat{n} = \frac{1}{5}(4\hat{e}_1 + 3\hat{e}_2)$

(b) Verify the result determined in (a) by a Mohr's circle construction similar to that shown in the figure from Example 3.5 (and in HW 4).

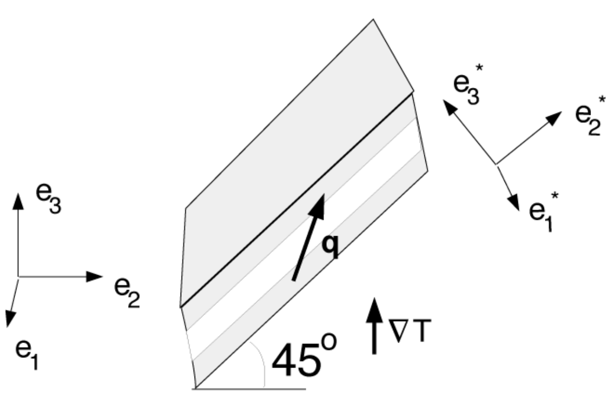
Answer (which would ***not*** be provided on actual test.)

$\sigma_N = 14.04$ MPa, $\sigma_S = 5.28$ MPa

*Note that the Mohr circle construction is simplified by the fact that **n** is perpendicular to x_3 and hence **n*** is perpendicular to the principal direction associated with principal stress value 5. So it's going to plot where? You still need to determine the a_{ij} matrix, but it's a fairly simple one.*

Problem 7: **Heat flow in anisotropic rocks**
Heat flow $\boldsymbol{q}$ (in W m^{-2}) is given by $q_i = -k_{ij}T_{,j}$ where k_{ij} is the thermal-conductivity tensor (W m^{-1} deg^{-1}), and $T_{,j}$ is the temperature gradient $\partial T/\partial x_j$. The bulk thermal conductivity can be written as a tensor when the rocks are composed of a stack of layers of different rock types, and when we view the rock on scales larger than the layer thicknesses. For example, in a stack of rocks with alternating high and low intrinsic conductivities, it is easier for heat to flow parallel to the layers than across the layers, because of the difficulty in getting through the low-conductivity layers. Suppose that the principal directions of the conductivity tensor are oriented normal to and parallel to the rock layering. Suppose further that the bulk conductivity normal to layering is half the bulk conductivity in the direction of the layering, and that there is no preferred direction in the plane of the layering.

- If the rocks are not lying flat, but are tilted at 45º to horizontal, and if the temperature gradient is aligned with vertical, find the angle from vertical at which heat is actually flowing through the rocks on the scale at which the continuum approximation is valid.



Problem 8: The momentum flux tensor

Let ρ and **v** denote the density and velocity of a fluid (e.g. a mud flow) at a given point P in space at a given time *t*. Flux describes the *rate* of transfer of something across a unit area. If **Π** is a plane with normal **n** at point P, then the momentum flux across **Π** at point P and at time *t* is defined to be $\rho\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n}) = \rho\,\mathbf{vv}\,\mathbf{n}$, where **vv** is a dyad. (A dyad **ab** or a_ib_j is special tensor formed by an outer product of two vectors **a** and **b** – see Mase, Smelser and Mase, Section 2.2.4).

As indicated in the figure below,

$\rho\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}A\,\mathrm{d}t$

is the momentum at point P and time *t* that is carried across an oriented differential surface element of area **n**dA with normal vector **n** in time d*t*; and ρ**vv** is called the momentum flux tensor at P and *t*. Consider its units and its vectorial nature.

(a) If $\mathbf{v} = (v_1, v_2, v_3)$, determine the Cartesian components of **vv**. What is the symmetric part of **vv**?

(b) Work out the determinant of **vv**. Note that **vv** sends all directions **n** into the direction of **v**. This is a very strong condition because 3 dimensions get "squashed down" to 1. Think about the eigenvectors of **vv**, especially the two perpendicular to **v**. What must their eigenvalues be? Does **vv** have an inverse? Use the information above and the invariance of the trace to determine all three eigenvalues and eigenvectors of **vv**. Work out the determinant of **vu**, and consider the fact that $\mathbf{vu}\,\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}$ for any **w** so it too sends all directions into the direction of **v.** Does **vu** have an inverse? (Bottom line is, dyads are pretty limited tensors. But, any arbitrary tensor can be constructed as the sum of three dyads.)

c) If $\mathbf{v} = (3, 2, -1)$ m/s and $\rho = 2000$ kg/m^3 at a given point and time, determine the momentum flux across a plane with normal $\mathbf{n} = 1/5(3, 4, 0)$. Use SI units and consider that the coordinate system is (east, north, and up).

d) Consider a mudflow traveling with velocity $\mathbf{v} = (3, 3, -1)$ m/s through a 10m high by 20m wide channel oriented due northeast. In 5 sec how much momentum passes through a hula hoop with radius 1m suspended in the middle of the channel? The surface encircled by this hoop has a normal that is horizontal but points at an angle of 20° east of north. For God's sake, draw a diagram!! This large amount of momentum quantifies the greater ability of mudflows to move dense objects compared to, for example, high winds.