ESS 533/ATMS 512
Dynamics of Snow and Ice
Snow Dynamics Problem Set

1. Snow Creep

Snow on a sloping surface deforms according to a constitutive relationship between deformation rate $\dot{\varepsilon}_{ij}$ and stress σ_{ij} described by bulk viscosity k and shear viscosity η. Equivalently, this constitutive relation can also be written in terms of two different constitutive parameters, ν and η_E, where ν is a viscous analogue of Poisson's ratio, and η_E is a constant with units of viscosity (viscous analogue of Young's modulus).

$$\dot{\varepsilon}_{ij} = \frac{1}{\eta_E}\left((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right) \tag{1}$$

With the origin at the snow surface, the x_1 axis points down-slope along the surface, which has a slope of α, and the x_2 direction is downward and normal to the sloping surface. The snow does not glide along the ground. The surface slope α, the snow thickness h, and the snow density $\rho(x_2)$ do not vary with x_1 and x_3. The average density $\bar{\rho}(x_2)$ above depth x_2 can be written as

$$\bar{\rho}(x_2) = \frac{1}{x_2}\int_0^{x_2} \rho(x_2')dx_2' \tag{2}$$

a) Find all components of the stress tensor σ_{ij} and the deformation rate tensor $\dot{\varepsilon}_{ij}$ as functions of surface-normal depth x_2. (It is OK to neglect atmospheric pressure, because pores in the snow are presumably inter-connected and ice crystals themselves are incompressible. Therefore, air pressure gradients cannot contribute to compaction.)

b) Find the down-slope creep velocity $u_1(x_2)$ and the settling velocity $u_2(x_2)$ in terms of η_E, ν, $\bar{\rho}(x_2)$ and surface slope α. You can assume that ν is independent of stress and density.

c) Find the angle θ of the velocity vector with respect to the surface (the x_1 axis) in terms of Poisson's ratio ν. This angle is called "the creep angle" for snow, and it can be helpful for understanding the impacts of snow on buried or partially buried structures. What is the theoretical range of θ, based on the known range of ν (*if* Equation (1) correctly represents the constitutive behavior of the snow).

2. Avalanche!

At time $t=t_0$, relatively homogeneous snow starts accumulating rapidly at the rate of $b(t)$. on surfaces with uniform slope α. During a storm cycle in maritime climates, the failure surface for avalanches is usually the base of the new snow, where the slope-parallel shear stress increases with the integrated amount of snowfall, i.e.

$$\sigma_{xz}(t) = g\int_{t_0}^{t} b(t')\cos(\alpha)\sin(\alpha)dt'$$

Meanwhile, the fracture shear strength $\sigma_{fz}(t)$ of the new snow increases with time t, as stronger bonds develop between individual snow grains. Bonds develop faster when the normal stress $\sigma_{zz}(t)$ is higher, because the grains are pushed together. Bonds can also develop faster in denser snow, where grains are closer together. However, to use the latter factor, you must be able estimate the rate of densification of the snow. We are not going to go there in this question.

Instead, suppose that, as a really simple first model for snow-slope stability, you decide to try a fracture-strength rule for the snow at the base having the form

$$\sigma_{fz}(t) = C_0 + C_1\sigma_{zz}(t)\,(t - t_0)$$

where C_0 and C_1 are positive constants. (It would be really amazing if this were the best equation to describe the snow strength. It is simply a first guess that should show the right qualitative tendencies. However, it does incorporate strengthening with age(t-t_0), and strengthening with normal load σ_{zz}. The Conway&Wilbour SNOSS model uses a more complicated rule.)

The stability index is a parameter that is used widely in the avalanche community. It is defined by

$$\bar{\Sigma}_z(t) = \frac{\sigma_{fz}(t)}{\sigma_{xz}(t)}$$

and the expected time to failure when $\bar{\Sigma}_z$ drops to unity, is

$$t_f(t) = \frac{(\bar{\Sigma}_z(t) - 1)}{d\bar{\Sigma}_z(t)/dt}$$

You will use the values:

Slope angle	α=45°
Accumulation rate	b =1.3 gm cm^{-2} h^{-1} (measured on a horizontal surface)
Storm duration	6 hours
Initial snow strength	C_0=120 Pa
Strength rate factor	C_1=0.1 h^{-1}

a) To answer this question efficiently, first check out the SNOSS paper; Conway, H. and C. Wilbour (1999) Evolution of snow slope stability. *Cold Regions Science and Technology* 30, 67-77.
b) Show how you would derive the expression for shear stress $\sigma_{xz}(t)$ at the base of the new snow (be sure give attention to the cos(α) factor).
c) Using the definitions above, explain in words the meaning of the stability parameter $\sigma_{xz}(t)$ and the time-to-failure parameter $t_z(t)$.
d) Graphical solutions, e.g. using MATLAB, are probably the easiest way to address this problem. Find and plot the histories of $\sigma_{xz}(t)$, $\sigma_{fz}(t)$, $\bar{\Sigma}_z(t)$ and $t_f(t)$ through the storm cycle if no avalanches occur. Since $\bar{\Sigma}_z(t)$ can be very large when snow starts to accumulate, you might consider log (base 10) plots.
e) If avalanches did not get triggered, when might the slopes be (somewhat) stable again after the storm?
f) If avalanches are triggered as soon as the slopes become unstable, how many times might avalanches occur during this storm, and when might they occur? Show your plots for this case.
g) How low would the accumulation rate have to be, before you would predict no instability on this slope? Show your plots.
h) What are some weaknesses and limitations of this model?