ESS 533/ATMS 512 Dynamics of Ice Masses
Homework on temperatures in ice and snow

1. Ice is described by
   - density of ρ_i = 900 kg m^{-3},
   - thermal conductivity of k_i = 2.5 W m^{-1} K^{-1},
   - heat capacity of c = 2000 J kg^{-1} K^{-1}.

Suppose that the surface temperature on an ice sheet can be represented as

$$T(z,t) = T_{av} + T_{cycle} \sin\left(2\pi\omega t\right)$$

where ω is frequency (***not*** angular frequency, e.g. ω = 1 yr^{-1} is the annual cycle) , t=0 on March 21 in the northern hemisphere, and snowfall rate is "small" (more on that later).

(i) If firn has a density of ρ=500 kg m^{-3}, how do you expect its conductivity to differ from the conductivity of ice, and why? What characteristics of firn should control heat transfer?
Clearly density is an imperfect proxy for thermal conductivity, but often density is all we have. Estimate its conductivity k_f by taking the average of the values from the Schwerdtfeger formula (upper limit) and the van Dusen formula (lower limit). [Both formulae are given in the temperature chapters in Cuffey and Paterson (2010) and in Paterson (1994).]

(ii) Use the analytical solution from class (or Cuffey and Paterson (2010), Chapter 9) to find the depth where the amplitude of the seasonal cycle is reduced to e^{-1} of its surface value.

(iii) Find the depth where the cycle amplitude is reduced to e^{-2} of its surface value T_{cycle}.

(iv) How fast does the temperature maximum move downward, and how long does it take to reach the depths in (ii) and (iii)?

(v) Now let's revisit the question of "small" in the definition of this question. How large would the accumulation rate need to be before you would be concerned about downward advection quantitatively impacting your results above? What dimensionless number would you use to assess this question?

(vi) Comment on the relationship between your answers to parts (iv) and (v).

(vii) If T_{cycle}=10°C, how deep would you have to drill a hole in order to measure the mean annual temperature T_{av} to within 0.2°C (±0.1°C)?

2. The Péclet number expresses the relative importance of advection and diffusion Show how the Péclet number arises naturally when you nondimensionalize the 1-D advective-diffusion equation for heat flow with constant thermal parameters,

$$\rho c\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}\right)=k\frac{\partial^2 T}{\partial x^2}$$

3. The analytical solution* for a step change to T_{final} at the surface (x=0) of a half-space initially at T_{init} everywhere

$$T(x,t)=T_{final}+\left(T_{init}-T_{final}\right)\operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right) \qquad (**)$$

where $\kappa = k/\rho c$ is thermal diffusivity, and erf ( ) is the error function,

$$\operatorname{erf}(y)=\frac{2}{\sqrt{\pi}}\int_0^y \exp\left(-s^2\right)\mathrm{d}s$$

* Carslaw and Jaeger (1946). *Conduction of heat in solids*.

(i) Suppose that the temperature profile at a site in East Antarctica was uniform with depth at the Last Glacial Maximum (LGM) at 10 ka (low accumulation, very slow flow, and virtually zero geothermal flux). We can view the end of the last ice age into the Holocene 10,000 years ago, as a step change in surface temperature from T_{LGM} to T_{HOL}. Use the equation (**) above to find the depth today where we could expect the temperature to have warmed by half of the surface step.

(ii) Confirm your result in (i), by making a plot of temperature profiles at 1000-year intervals. Use T_{LGM}= -50°C, T_{HOL}= -40°C. Use Equation (**) above.

(iii) Now plot temperature solutions over the same depths and times for an ice sheet that had the same T_LGM but a linear temperature profile at LGM, due to geothermal flux q_{geo} = 50 mW m^{-2}. Use Fourier's Law and Equation (**) above.