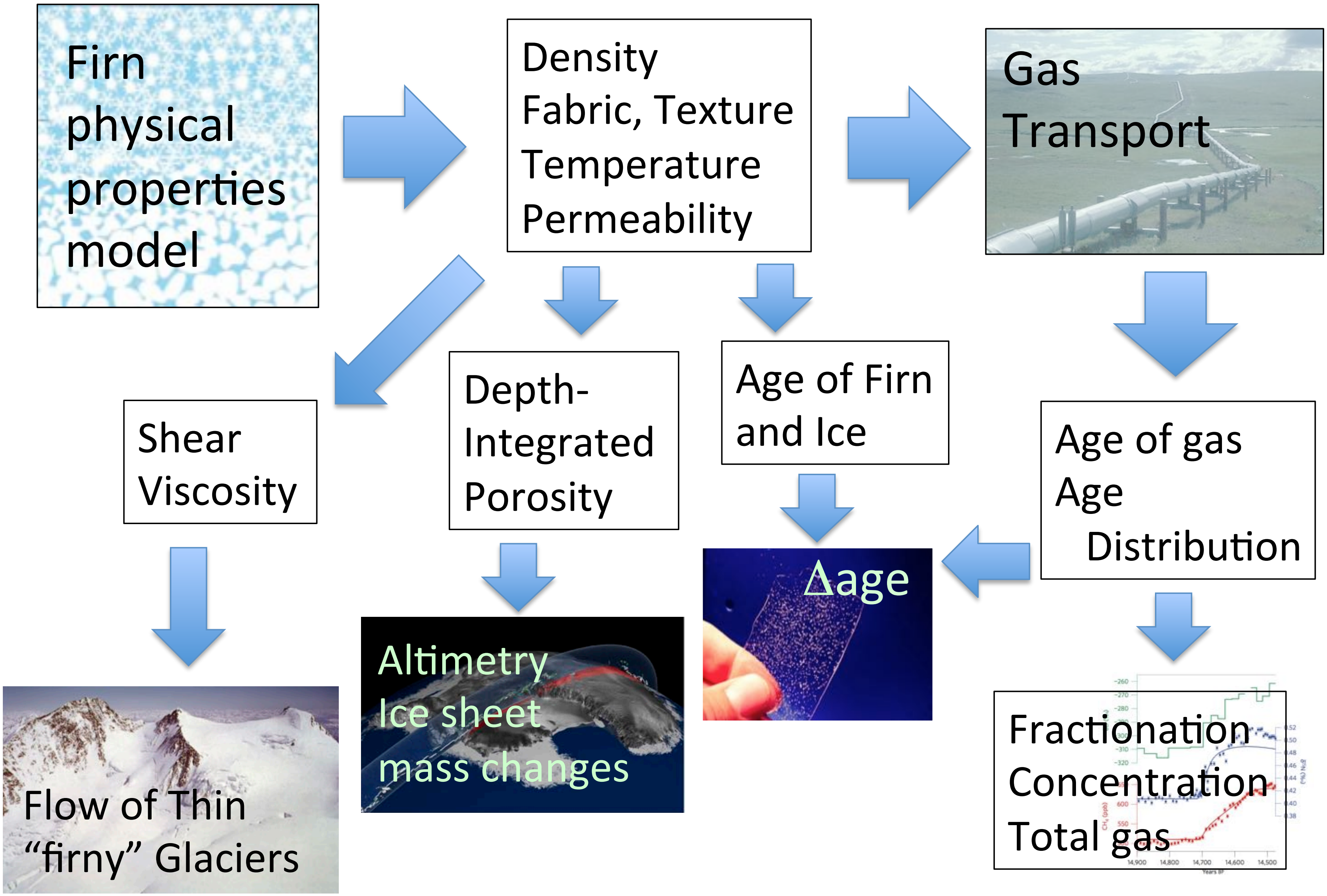
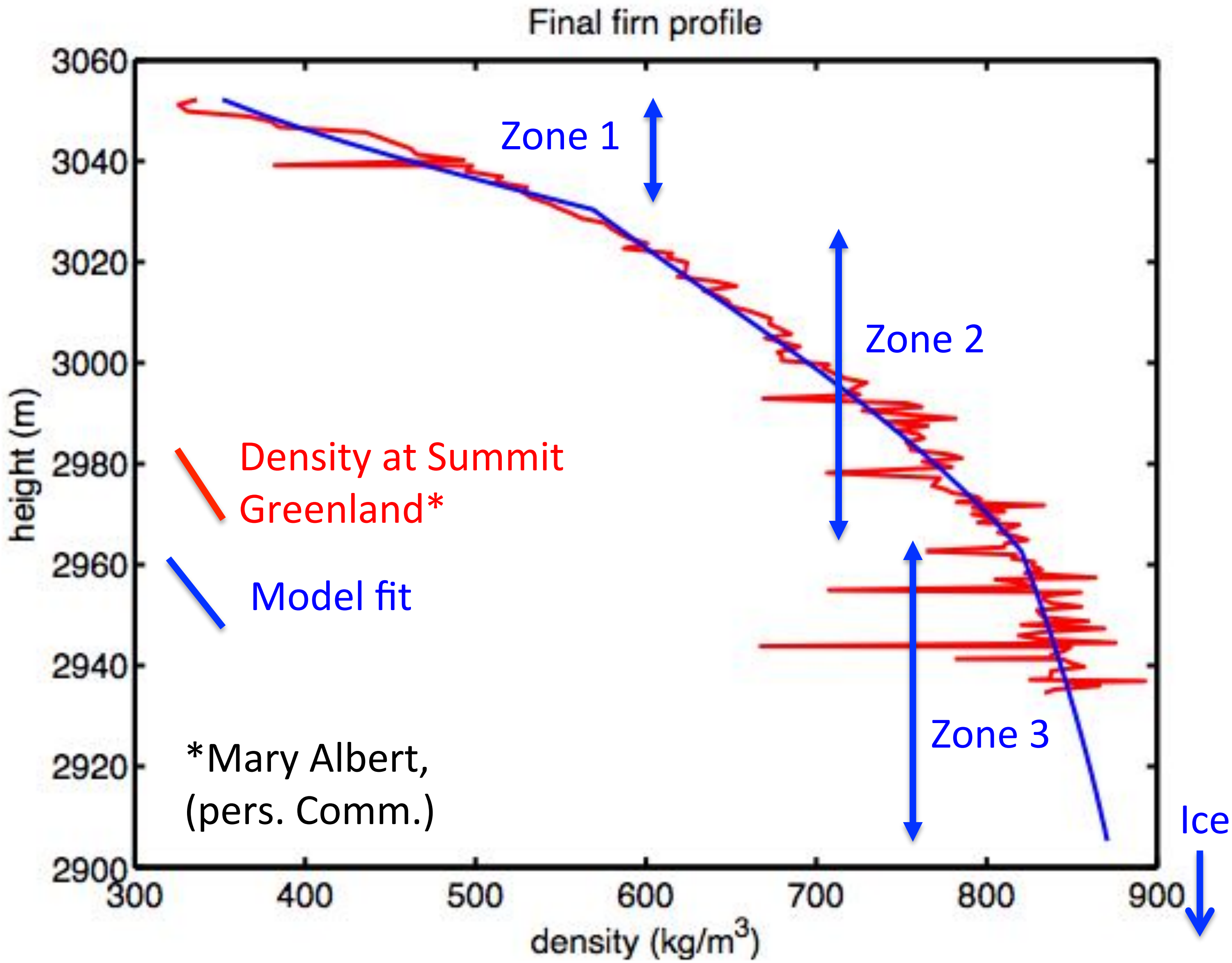
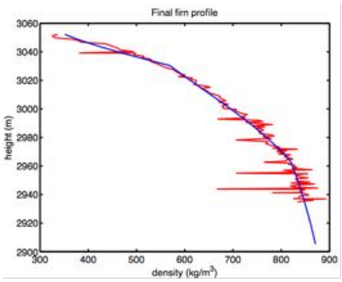

Firn physical properties model
Density
Fabric, Texture
Temperature
Permeability
Gas Transport
Shear Viscosity
Depth-Integrated Porosity
Age of Firn and Ice
Age of gas
Age Distribution
Flow of Thin “firny” Glaciers
Altimetry
Ice sheet mass changes
Δage
Fractionation
Concentration
Total gas

Firn-Density Structure





3 Zones of Dry-Firn Compaction

Zone	Physics	Boundaries
Zone 1 $\rho_0 < \rho < \rho_{550}$	Grain-boundary sliding	ρ_0 surface ρ_{550} 550 kg m^{-3}
Zone 2 $\rho_{550} < \rho < \rho_{pco}$	Strain and sintering	ρ_{550} 550 kg m^{-3} ρ_{BCO} bubble close-off
Zone 3* $\rho_{pco} < \rho < \rho_i$	Compression of bubble air	ρ_{pco} pore close-off ρ_i ice density

* Zone 3 is not so important for occluded gases

UW Community Firn Model

UW contribution to PIRE-ICEICS program
(Int'l Collaboration and Education in Ice-Core Science)

- Open access Python code (github)
- Modular – plug and play various published models
- Lagrangian firn and thermal evolution
- Gas transport
- Stable-isotope diffusion
- Horizontal strains
- FirnMICE (Firn Model InterComparison Experiment)
  Lundin et al., 2017. *J.Glaciol*. 63(239), 401-422.

Notation

- ρ = density
- z = depth
- t = time
- A = accumulation rate (kg m^{-2} a^{-1})
- θ = temperature (Kelvin)
- $\vec{S}$ = structural parameters

Units

- Most firn models (including H&L*) express accumulation rate A in units of

  Mass/Unit Area/Unit Time, e.g.

  - Mg m^{-2} yr^{-1} (ρ_i = 0.9)
  - kg m^{-2} yr^{-1} (ρ_i = 900)

Temperature θ will be expressed in degrees Kelvin

*although they switch between kg/m^2/yr and m/yr (water equiv.) without telling us.

Why less emphasis on melt-layer physics?

- Ice-core gas geochemists are not interested in samples from areas warm enough to melt.
- Maybe in an upcoming version … See Max.

Grain-boundary sliding

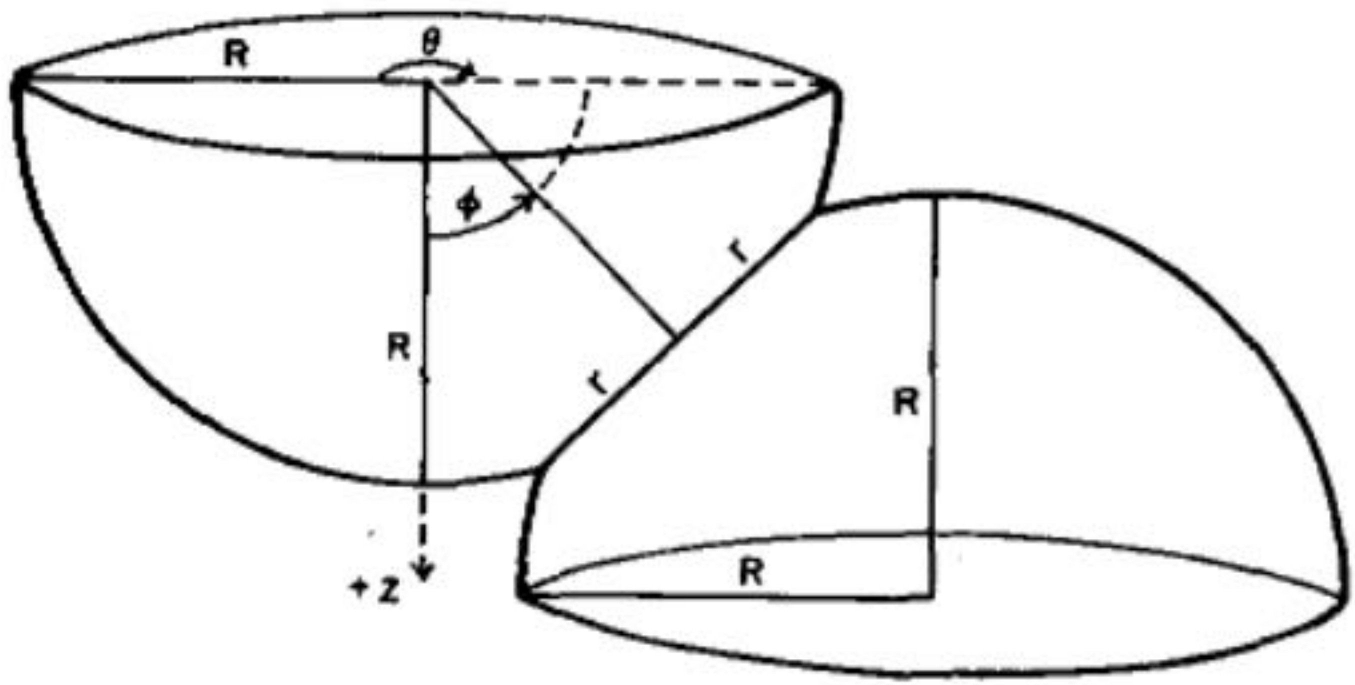


Fig. 2. Geometry of grains and bonds used in model. Variables are defined in text.

Alley, 1987. *J. Physique* 48(C1),249-256.

Coordination number

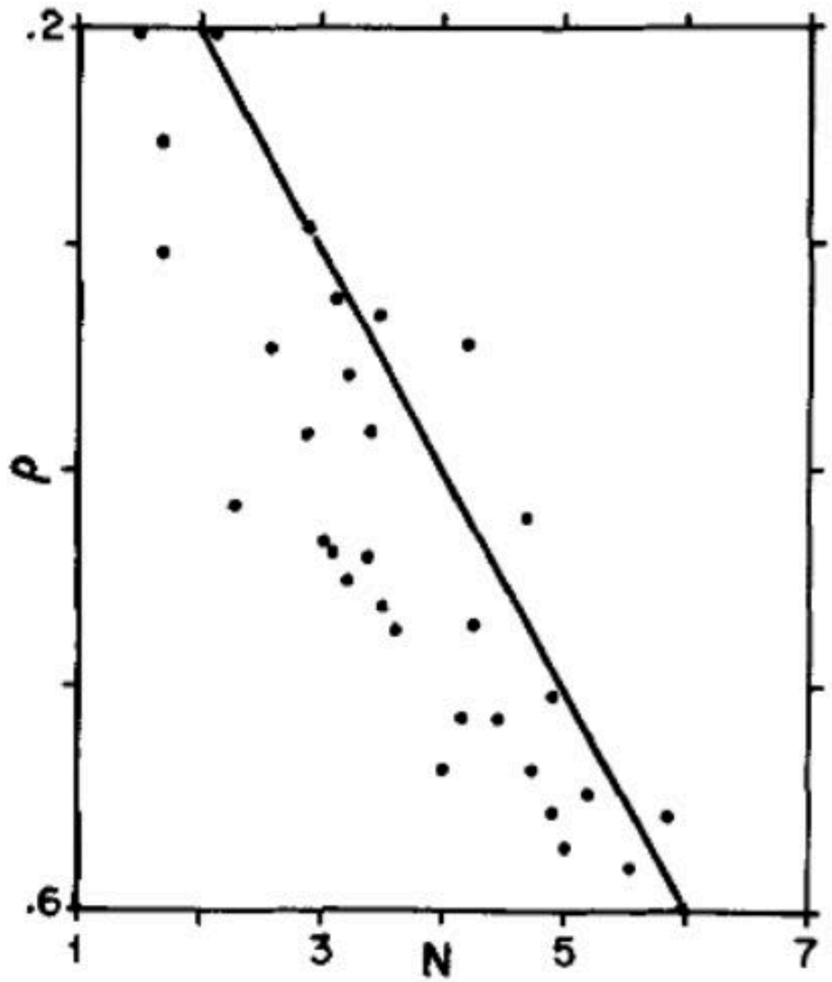


Fig. 1. Relative density (ρ) vs. coordination number (N) for firn from ridge BC and upstream B, West Antarctica, and from site A, Greenland. The approximation N = 10ρ also is shown. Use of an approximation that fits the data better would complicate the rate equations without changing them significantly.

Coordination number indicates the number of contacts where a grain touches its neighbors. Unless a grain is dangling free, N≥2.
For roughly equal-sized and rounded grains, N≤6 (limited by close-packing)

Alley, 1987. *J. Physique* 48(C1),249-256.

Levels of Empiricism

Fitting	**Assume**	**Examples**
Measured $\rho(z)$	Form of Dρ/Dt $f(A,\theta)$ Steady state	Herron & Langway others based on H&L
Measured $\rho(z)$ Measured Dρ/Dt	Form of Dρ/Dt structural dependence	Arthern et al. Morris & Wingham
Viscosity $\eta(\rho,\theta,\sigma)$ $\dot{\varepsilon}_{zz} = \sigma_{zz}/2\eta$	Viscous firn Pressure sintering	Lüthi & Funk Gagliardini et al.
Viscosity $\eta(\rho,\theta,\sigma)$ Measured Dρ/Dt Measured $\rho(z)$	Viscous firn Microstructural dependence e.g. pressure sintering, …	

$$\frac{\mathrm{D}\vec{s}}{\mathrm{D}t} = f\left(\rho, A, \theta, \vec{s}\right)$$

Issues

Steady State is an underlying assumption in some models

But …

- The interesting times are when climate changes
- Is a steady-state formulation appropriate?
- S.S. formulation can be generalized to transient in many ways (nonuniqueness).
- Which one (if any) is right?

- We can trust the time derivatives in conservation laws

Issues

Most models have a “physics-like basis” for their chosen form

- But, is it the correct form?
- Does it include all the important physical processes? (Global to microstructural)
- What assumptions were made?
- How to generalize steady-state models to solve transient problems?
- Role of conservation laws?

Issues

Most models are empirical – tuned to data

- At what level (global to microstructural) are the data to which is tuning done?

Data

- Global, long-term – $\rho(z)$, the most assumptions
- Instantaneous, Conservation laws - Dρ/Dt, Dθ/Dt
- Constitutive Laws - Viscosity $\eta(\rho,\theta,\sigma)$
- Structural parameters $\vec{s}$ that control viscosity (grain size, coordination numbers, their variance, chemical impurities, …
- Evolution of structural parameters - $\frac{\mathrm{D}\vec{s}}{\mathrm{D}t} = f\left(\rho, A, \theta, \vec{s}\right)$

Firn Workhorse: Herron and Langway (1980)*

Journal of Glaciology, Vol. 25, No. 93, 1980

FIRN DENSIFICATION: AN EMPIRICAL MODEL

By MICHAEL M. HERRON *and* CHESTER C. LANGWAY, JR

(Ice Core Laboratory, Department of Geological Sciences, State University of New York at Buffalo, 4240 Ridge Lea Road, Amherst, New York 14226, U.S.A.)

ABSTRACT. An empirical model of firn densification from the surface to the zone of pore close-off has been constructed. Fundamental rate equations have been derived for the first two stages of densification. In the first stage, for densities less than 0.55 Mg m^{-3}, the densification rate is proportional to the mean annual accumulation times the term $(\rho_i - \rho)$, where ρ is the density of the snow and ρ_i is the density of pure ice. The densification rate in the second stage, where 0.55 Mg m$^{-3} < \rho <$ 0.8 Mg m^{-3}, is proportional to the square root of the accumulation rate and to $(\rho_i - \rho)$. Depth–density and depth–age calculations from this model are compared with observation. Model accumulation rates are within about 20% of values obtained by other techniques. It is suggested that depth intervals of constant density in some Antarctic cores may represent a synchronous event in the 1880's when ten times the normal accumulation fell within a year or two.

* *J. Glaciol.* **25**(93), 373-385.

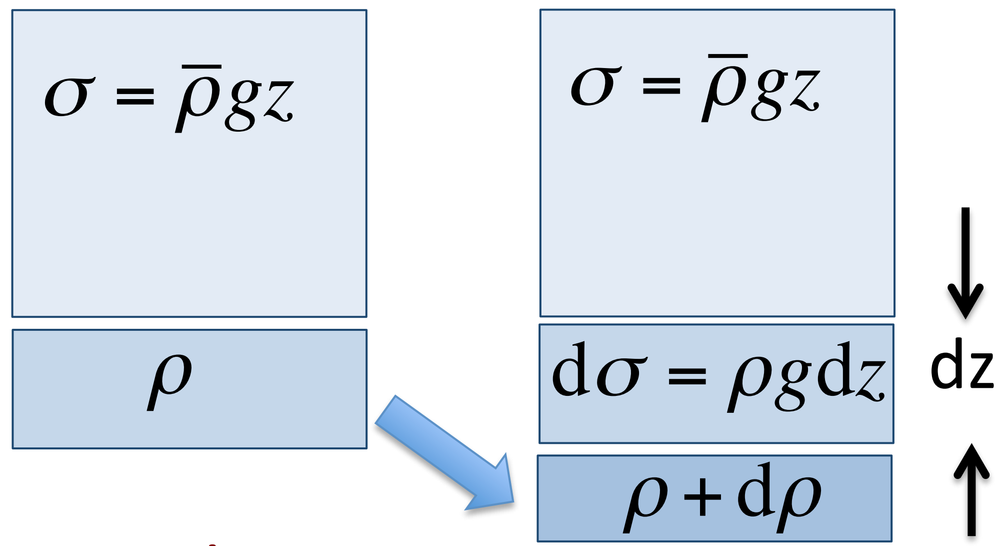
The Robin Assumption - Physics in Herron and Langway (1980)*

- Pioneering model for density profile $\rho(h)$ (h = depth).

Porosity $\boxed{\phi = 1 - \rho/\rho_i}$ (ρ_i = ice density)

- Fractional change of porosity ϕ is proportional to extra overburden load

$$d\sigma = \rho\, g\, dz$$



- This is a ***steady-state*** assumption.

$$\boxed{\frac{d\phi}{\phi} = \frac{d\rho}{\rho_i - \rho} \propto d\sigma = \rho g dz \quad (1)}$$

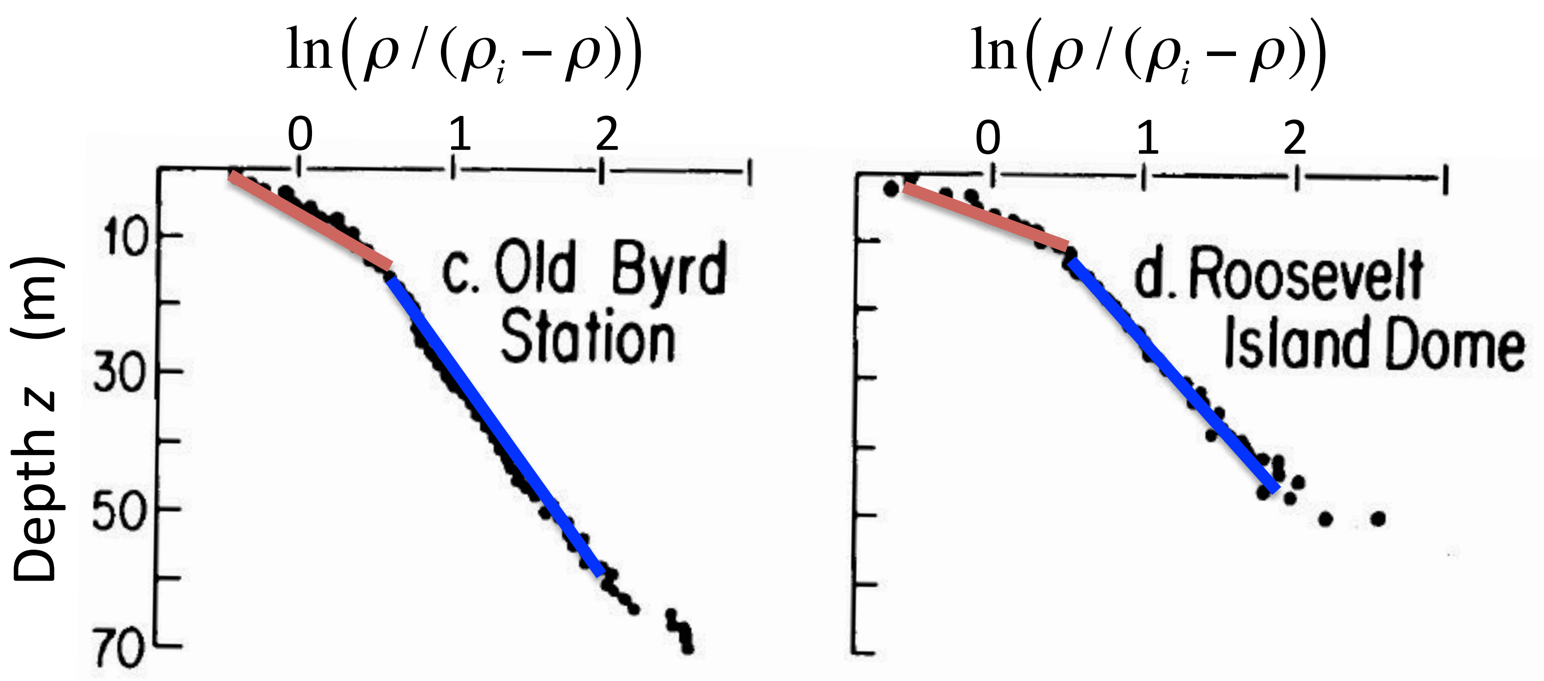
* *J. Glaciol.* **25**(93), 373-385.

$$\frac{\mathrm{d}\rho}{\rho_i - \rho} \propto \rho g \mathrm{d}z \quad (1)$$

(1) Re-arranged to give straight lines with slope C_1 in zone 1 and C_2 in zone 2.

$$C_j = \frac{\mathrm{d}\ln[\rho/(\rho_i - \rho)]}{\mathrm{d}z} \quad (2)$$

Slope C_j can be found from plots of measured $\rho(h)$ profiles. (j=1 or 2)



"Algebraic" Solution – depth-density profile $\rho(z)$

- Parameters $C_1(A,T)$ and $C_2(A,T)$ are derived from density data $\rho(z)$ in each zone at sites with various A and T.

$$C_j = \frac{\mathrm{d}\ln[\rho/(\rho_i - \rho)]}{\mathrm{d}z} \quad (2)$$

Equations (2) can be solved for the steady depth-density profile $\rho(z)$.

Zone 1 $\rho_0 < \rho < \rho_{55}$

$$\rho(z) = \rho_i \frac{\frac{\rho_0}{(\rho_i - \rho_0)}\exp(\rho C_1 z)}{1 + \frac{\rho_0}{(\rho_i - \rho_0)}\exp(C_1 z)} \quad (3a)$$

Zone 2 $\rho_{550} < \rho < \rho_{pco}$

$$\rho(z) = \rho_i \frac{\frac{\rho_{550}}{(\rho_i - \rho_{550})}\exp\left(\frac{\rho_i k_2}{A^{0.5}}(z - z_{550})\right)}{1 + \frac{\rho_{550}}{(\rho_i - \rho_{550})}\exp\left(\frac{\rho_i k_2}{A^{0.5}}(z - z_{550})\right)} \quad (3b)$$

"Algebraic" solution – Age of firn *Age*(ρ)

downward velocity
in steady state:

$$\frac{\mathrm{D}z}{\mathrm{D}t} = \frac{A}{\rho} \qquad (4)$$

A = accum. rate
(Mg m^{-2} yr^{-1})

$$C_j = \frac{\mathrm{d}\ln[\rho/(\rho_i - \rho)]}{\mathrm{d}z} \qquad (2)$$

(2) and (4) can be combined into (5)

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{C_j\, A}{\rho_i}\left(\rho_i - \rho\right) \qquad (5)$$

- C_j inferred from site $\rho(h)$.
- A is site accumulation.
- Steady state has now been assumed ***twice***.

But now we have introduced time into the equations …

“Algebraic” solution – *Age*(ρ)

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{C_j A}{\rho_i}\left(\rho_i - \rho\right) \qquad (5)}$$

Equations (5) can be solved (with an integrating factor) for the steady age-density profile *Age*(ρ).

$$Age(\rho) = \frac{\rho_i}{C_1 A}\ln\left(\frac{\rho_i - \rho_0}{\rho_i - \rho}\right) \qquad (6a)$$

Zone 1

$\rho_0 < \rho < \rho_{550}$

$$Age(\rho) = \frac{\rho_i}{C_2 A}\ln\left(\frac{\rho_i - \rho_{550}}{\rho_i - \rho}\right) + Age_{55} \qquad (6b)$$

Zone 2

$\rho_{550} < \rho < \rho_{\mathrm{pco}}$

General dependence on climate

$$\frac{D\rho}{Dt} = \frac{C_j\, A}{\rho_i}(\rho_i - \rho) \quad (5)$$

The empirical constants C_j in (5) depend on both accumulation rate A, and temperature T, and are different for each site and for each zone j.

It would be nice to relate all these C_j to a few climate parameters. H-L assumed:

- Power-law dependence on accumulation rate A^b.
- Arrhenius dependence on temperature T.

$$k_j(T) = K_j \exp\left({-Q_j}/{RT} \right)$$ in zone j.

$$\boxed{\frac{D\rho}{Dt} = k_j(T)\, A^{b_j} (\rho_i - \rho) \quad (8)}$$

Explicit climate-based parameters

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{C_j\,A}{\rho_i}\left(\rho_i - \rho\right) \quad (5)}$$

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_j(T)\,A^{b_j}(\rho_i - \rho) \quad (6)}$$

Comparing (5) and (6):

$$\boxed{C_j = \frac{\rho_i\,k_j(T)}{A^{1-b_j}} \quad (7)}$$

in Zone *j*

$$k_j(T) = K_j \exp\left({-Q_j}/{RT} \right)$$

- Parameters [b_1, K_1, Q_1; b_2, K_2, Q_2] determined from least-squares fit in (7), using C_1 and C_2 inferred from known density profiles, and known *A* and *T* at those sites.

“Algebraic” ***steady-state*** solution

Now density and age profiles can be calculated ***anywhere***

- just need *A, T, and* ρ_0 at the site.

Zone 1: $\rho_0 < \rho < \rho_{55}$

$$\rho(z) = \rho_i \frac{\frac{\rho_0}{(\rho_i - \rho_0)} \exp(\rho_i k_1 z)}{1 + \frac{\rho_0}{(\rho_i - \rho_0)} \exp(\rho_i k_1 z)}$$

$$Age(\rho) = \frac{1}{k_1(T)A} \ln\left(\frac{\rho_i - \rho_0}{\rho_i - \rho}\right)$$

(8a)

Zone 2: $\rho_{55} < \rho < \rho_{pco}$

$$\rho(z) = \rho_i \frac{\frac{\rho_{55}}{(\rho_i - \rho_{55})} \exp\left(\frac{\rho_i k_2}{A^{0.5}}(z - z_{55})\right)}{1 + \frac{\rho_{55}}{(\rho_i - \rho_{55})} \exp\left(\frac{\rho_i k_2}{A^{0.5}}(z - z_{55})\right)}$$

$$Age(\rho) = \frac{1}{k_2(T)A^{0.5}} \ln\left(\frac{\rho_i - \rho_{55}}{\rho_i - \rho}\right) + Age_{55}$$

(8b)

- This describes ***steady-state*** snapshots of the firn column.
- But, what about ***transients***?

Transient firn-compaction models

- At least 3 ways to calculate transient responses to step climate change from Herron and Langway.
  - Algebraic solution – immediate new "snapshot" $\rho(h)$
  - Diff. Eq. $D\rho/Dt$ – immediate new coefficients
  - Diff. Eq. $D\rho/Dt$ – evolving coefficients

- Firn Model Intercomparison Experiment (FirnMICE)
- Why do published models not give the same results when forced with the same climate?

* *J. Glaciol.* **25**(93), 373-385.

But what about transients?

We know that climate (A, θ) changes.

- If the changes are very slow, perhaps we can get away with continually updating a steady-state profile.
- Depends on what we mean by "slow" …
- Slow means large changes in climate occur only over time scales longer than the residence time of firn in the firn column.

- Today we explore only fast changes in accumulation rate A.

Another way: H-L Differential Equation formulation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_1(T)\,A\,(\rho_i - \rho) \quad \text{(6a) Zone 1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_2(T)\,A^{0.5}(\rho_i - \rho) \quad \text{(6b) Zone 2}$$

These equations are calibrated to follow firn parcels downward in a steady-state firn column.

Can we adapt them to follow parcels in a transient world?

Nonuniqueness:

- There are many ways to generalize from S.S. to transient behavior.
- Some ways are probably better than others.

Adapting the Diff. Eq.

How can we adapt (6a) and (6b) to a transient world?

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_1(T)A\left(\rho_i - \rho\right) \quad \text{(6a) Zone 1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_2(T)A^{0.5}\left(\rho_i - \rho\right) \quad \text{(6b) Zone 2}$$

Option 1:

Replace accumulation rate *A* and temperature *T* by their transient values *A*(*t*) and *T*(*t*) at the surface.

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_1\left(T\left[t\right]\right)A(t)\left(\rho_i - \rho\right) \quad \text{(10a) Zone 1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_2\left(T\left[t\right]\right)\left[A(t)\right]^{0.5}\left(\rho_i - \rho\right) \quad \text{(10b) Zone 2}$$

Advantage?

- Firn evolution responds to the changing climate.

Adapting H-L to a transient world

Option 2:

Replace accumulation rate *A* by its average value $\bar{A}(\rho,t)$ since the firn parcel fell as snow.

$$\bar{A}(\rho,t) = \int_{t-Age(\rho)}^{t} A(t')\mathrm{d}t'$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_1(T)\bar{A}(\rho,t)\left(\rho_i - \rho\right) \qquad \text{(11a)} \qquad \text{Zone 1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = k_2(T)\left[\bar{A}(\rho,t)\right]^{0.5}\left(\rho_i - \rho\right) \qquad \text{(11b)} \qquad \text{Zone 2}$$

Advantages?

- Firn evolution responds to the changing climate.
- The averaged accumulation rate $\bar{A}(\rho,t)$ is related to overburden load.

Option 3. Introduce load σ into H-L equations

" … S. J. Johnsen has kindly pointed out that Equation (4b) may be rewritten in terms of load σ as:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{k_1^2}{g} \frac{(\sigma - \sigma_{55})(\rho_i - \rho)}{\ln\left[(\rho_i - \rho_{55})(\rho_i - \rho)\right]} \qquad (4c)$$

for which the apparent activation energy … However, the utility of Equation (4b) lies in the ease of calculation of density profiles or accumulation rates …" *

But σ is not mentioned again in a Herron and Langway context … until adapted by Christo Buizert.

- Note that the Sigfus derivation is also based on steady-state assumptions.

*Herron and Langway (1980). *J. Glaciol.* **25**(93), 373-385.

Test: step change in climate

Why?

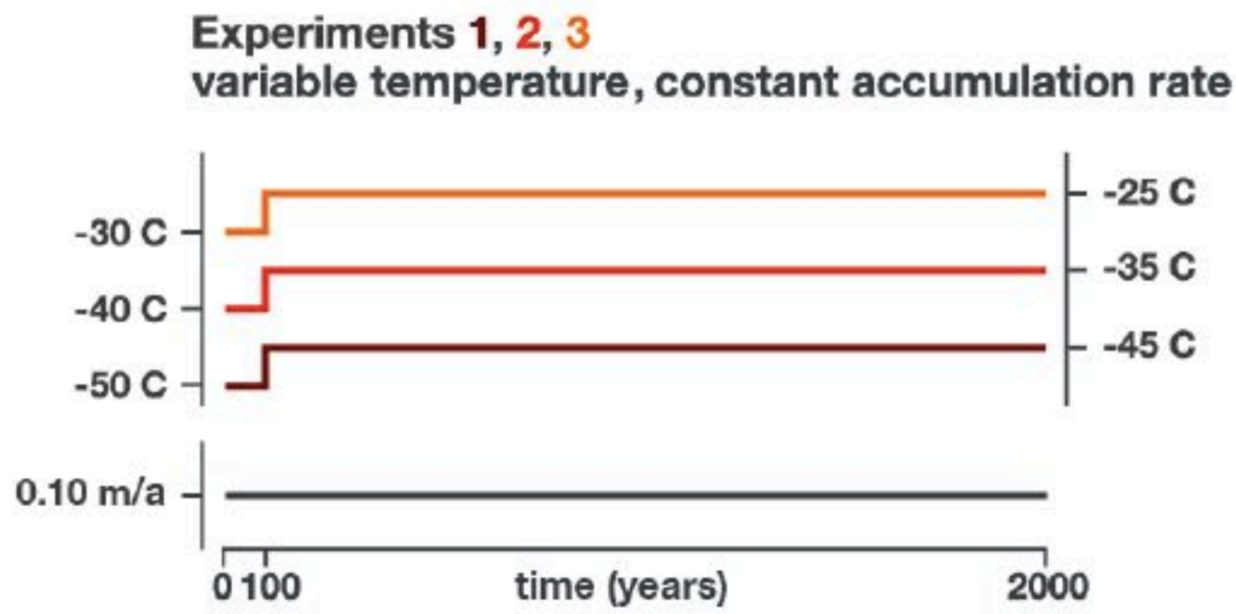
- Fast climate shifts happened in ice age.
- Gas-isotope fractionation in firn column was most interesting when temperature changed fast, or when firn-column height changed fast.
- Gas-trapping zone depends strongly on density, so pore-closeoff zone can move into ice of different age when temperature and load vary.

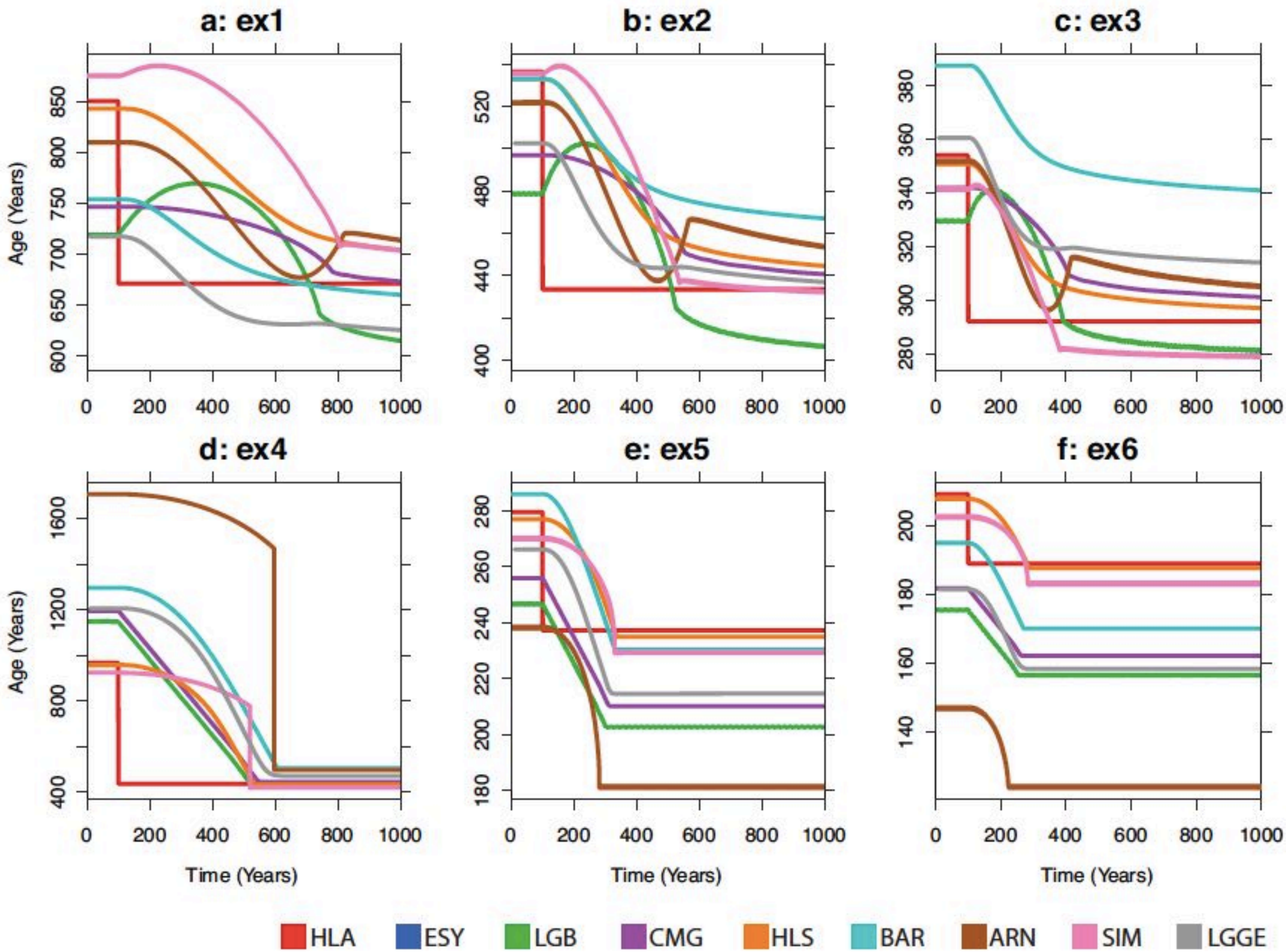
- Today we explore only abrupt changes in accumulation rate A.

NSF PIRE: Community Firn Model at UW

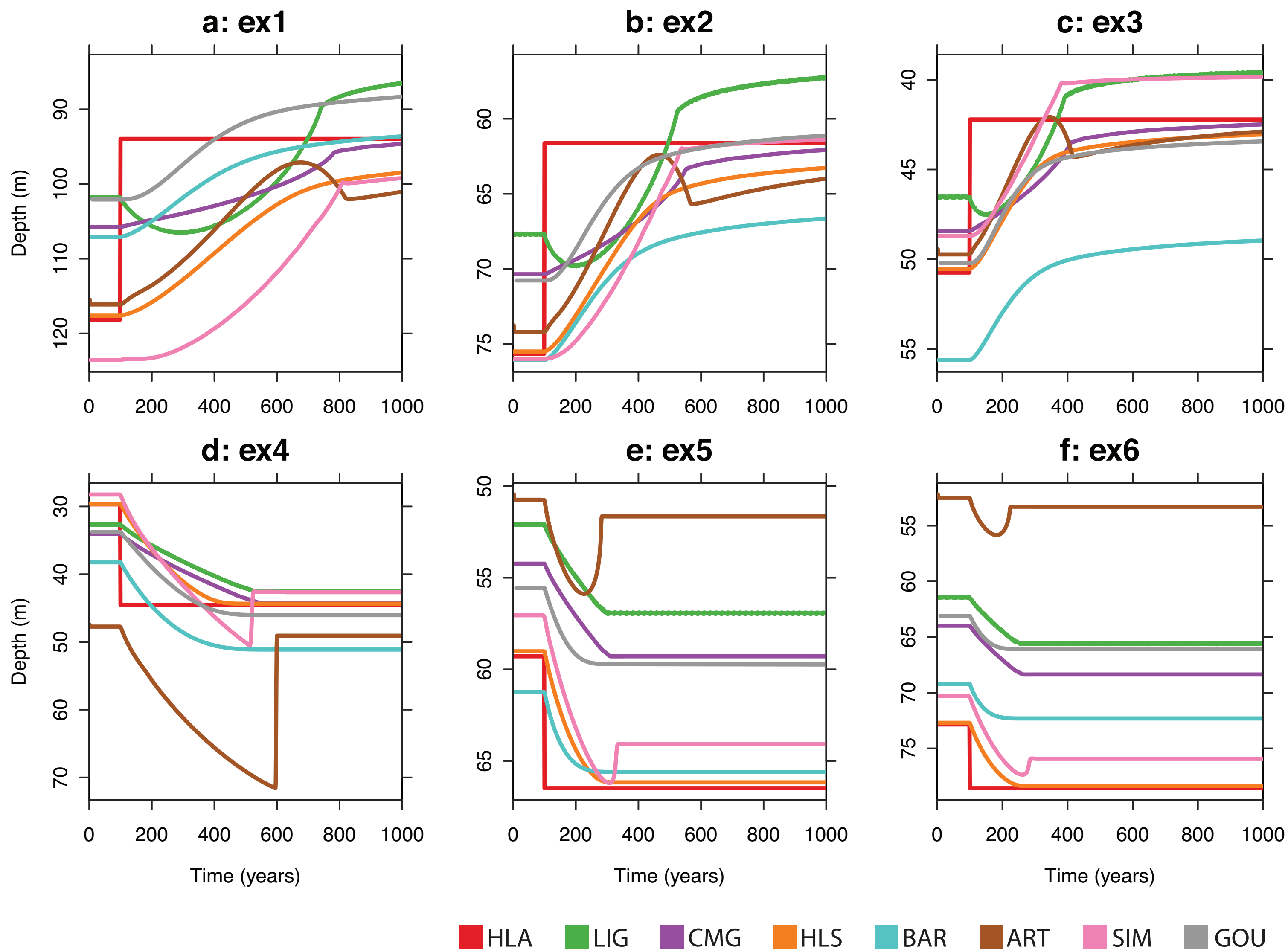
- Open-source, community model under development
- Incorporates physics:
  - Firn densification
  - Heat transfer
  - Gas transport
- Goals:
  - Constrain gas-age/ice-age offset Δage, especially at times of rapidly changing climate.
  - Improve understanding of firn dynamics.
  - Provide upper boundary condition for community ice-sheet models (?)
  - Infer mass changes from altimetry data.

FirnMICE Experiments

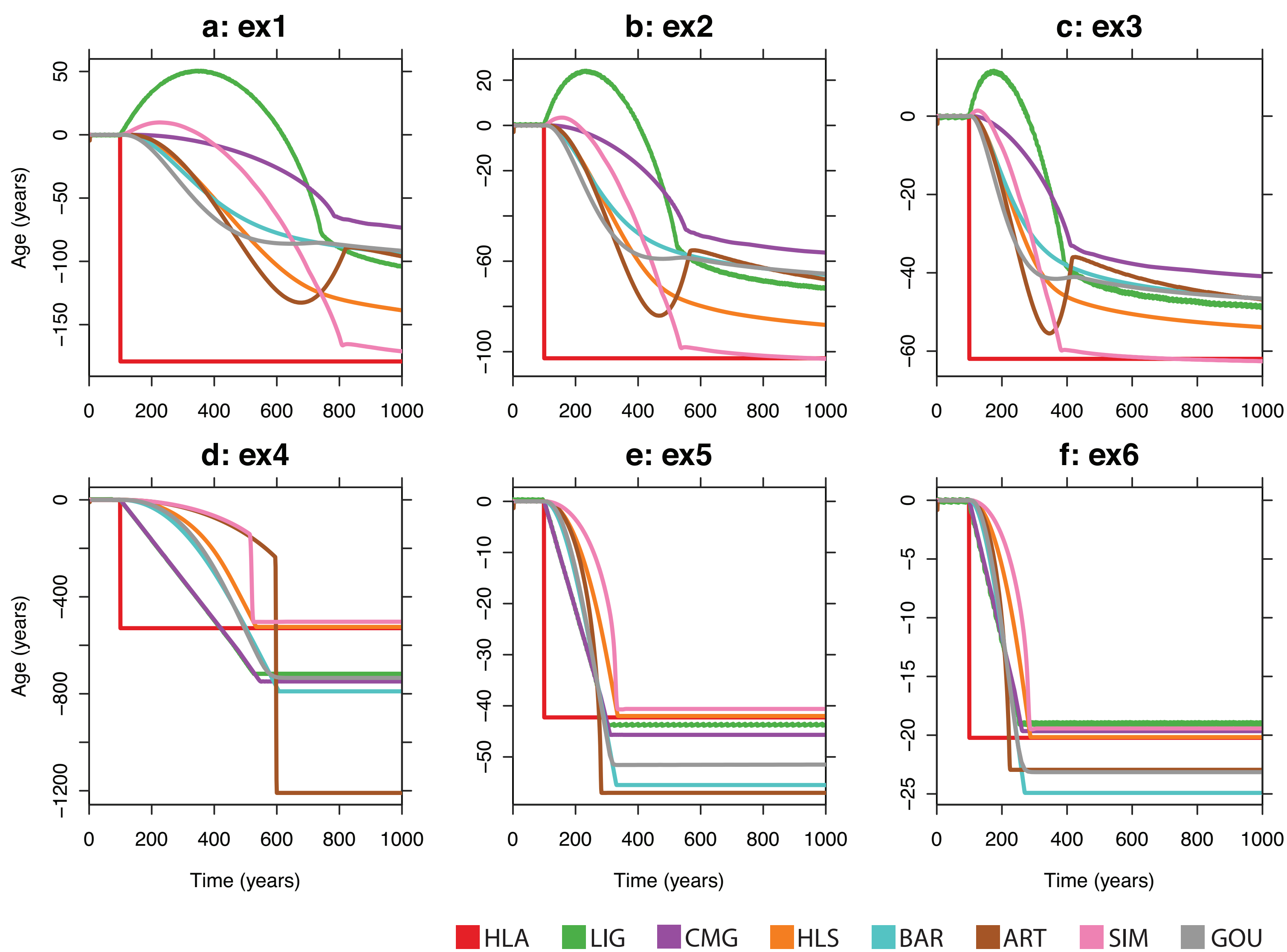



a: ex1
b: ex2
c: ex3
d: ex4
e: ex5
f: ex6
Age (Years)
Age (Years)
Time (Years)
Time (Years)
Time (Years)
HLA
ESY
LGB
CMG
HLS
BAR
ARN
SIM
LGGE

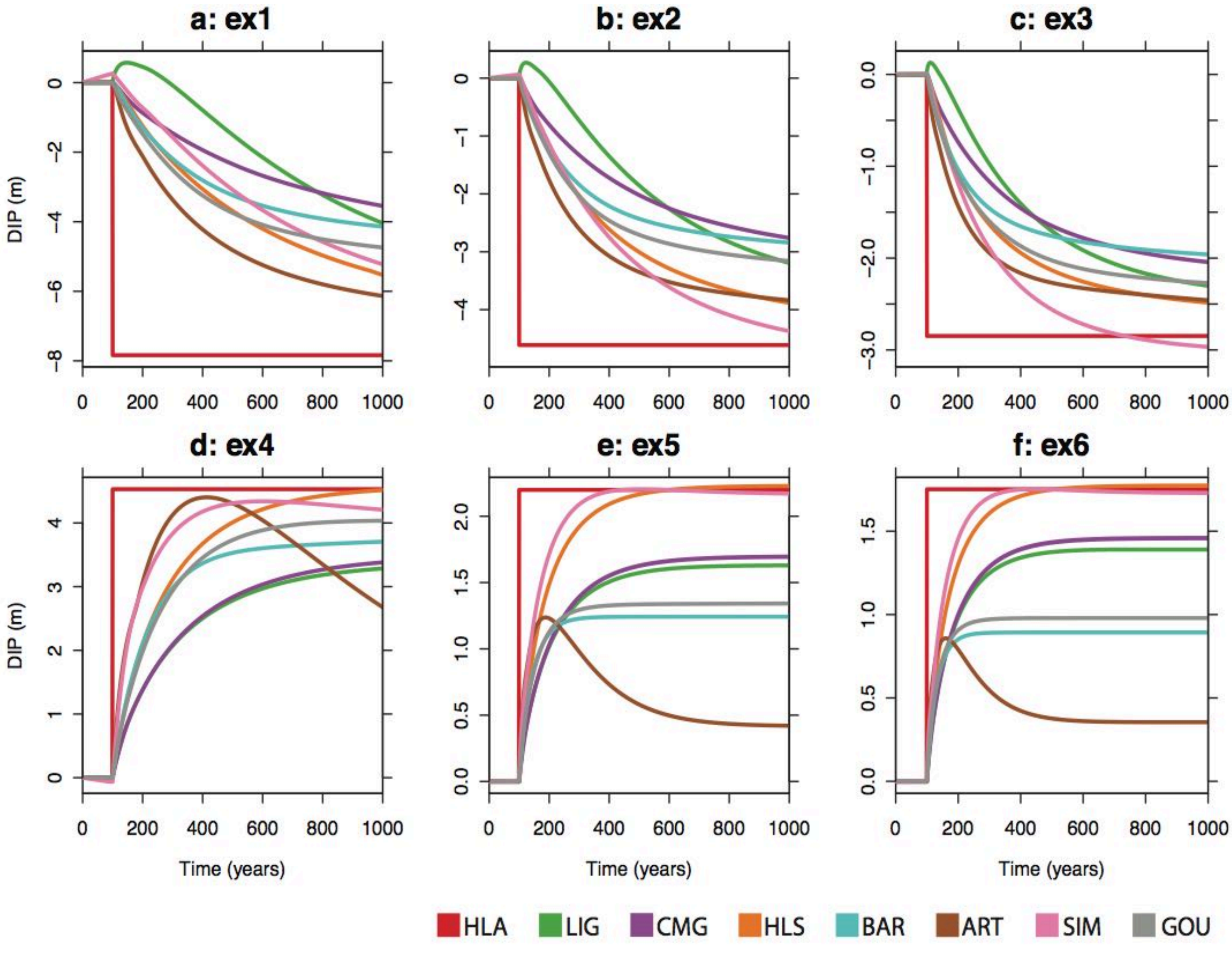
FirnMICE: Bubble close-off depth



FirnMICE: Bubble close-off age



Depth-integrated porosity



Continuum Mechanics for Firn

Definition:

Bulk strain rate $\dot{\varepsilon}(z,t)$ in terms of downward velocity *w*(*z*,*t*)

$$\dot{\varepsilon}(z,t) = \frac{\partial w(z,t)}{\partial z} \qquad (1)$$

Conservation of Mass:

Bulk strain rate in terms of density change

$$-\frac{1}{\rho(z,t)}\frac{D\rho}{Dt} = \dot{\varepsilon}(z,t) \qquad (2)$$

The *D* derivative indicates the Lagrangian (material-following) description of the motion

- Use time derivatives in conservation equations to express transients

Continuum Mechanics for firn

Overburden load

$\sigma(t,\tau)$ at time t on firn of age τ at depth z.

$$\sigma(t,\tau) = -g\int_{t-\tau}^{t} A(t')\,\mathrm{d}t' = -g\,\bar{A}\,\tau \quad (3)$$

Accumulation rate units:

$[A]$ = kg m^{-2} yr^{-1}

Constitutive relation:

for a material with viscosity $\eta(z,t)$

$$\dot{\varepsilon}(z,t) = \frac{1}{2\eta(z,t)}\sigma(z,t) \quad (4)$$

Now we can combine (2) and (4);

Mass conservation

$$\dot{\varepsilon}(z,t) = -\frac{1}{\rho(z,t)}\frac{D\rho}{Dt} \qquad (2)$$

Constitutive relation

$$\dot{\varepsilon}(z,t,\tau) = \frac{1}{2\eta(z,t,\tau)}\sigma(z,t,\tau) \qquad (4)$$

$$-\frac{1}{\rho(z,t)}\frac{D\rho}{Dt} = \frac{1}{2\eta(t,\tau)}\sigma(z,t) \qquad (5)$$

And $\sigma(t,\tau)$ is given by (3):

Overburden load:

$$\sigma(t,\tau) = -g\int_{t-\tau}^{t} A(t')\mathrm{d}t' = -g\bar{A}\tau \qquad (3)$$

Herron & Langway (1980) *J. Glaciol.*

Strain rate $$\frac{1}{\rho}\frac{D\rho}{Dt} = \dot{\varepsilon}$$

Viscous constitutive relation $$\dot{\varepsilon}(t,\tau) = \frac{1}{2\eta(t,\tau)}\sigma(t,\tau)$$

t=time, τ = firn parcel age

Lagrangian compaction rate In Zone 1:

$$\frac{D\rho}{Dt} = k_c \exp\left(\frac{-E_1}{RT}\right) A\left(\rho_i - \rho\right)$$

Overburden stress in terms of accumulation history:

$$\sigma(t,\tau) = g\int_{t-\tau}^{t} A(t')\,\mathrm{d}t' = -g\,\bar{A}\,\tau$$

Compaction rate in terms of σ in Zone 1:

$$\frac{1}{\rho(t,\tau)}\frac{D\rho}{Dt} = \frac{k_1}{g\tau}\exp\left(\frac{-E_c}{RT}\right)\left(\frac{\rho_i - \rho(t,\tau)}{\rho(t,\tau)}\right)\sigma(t,\tau)$$

Herron & Langway (1980) *J. Glaciol.*

$$-\frac{1}{\rho(t)}\frac{D\rho}{Dt} = \dot{\varepsilon}(t) \quad \dot{\varepsilon}(t) = \frac{1}{2\eta(t,\tau)}\sigma(t,\tau)$$

$$\frac{1}{\rho(t,\tau)}\frac{D\rho}{Dt} = \frac{k_1}{g\tau}\exp\left(\frac{-E_1}{RT}\right)\left(\frac{\rho_i - \rho(t,\tau)}{\rho(t,\tau)}\right)\sigma(t,\tau)$$

Combine to give viscosity implied in H&L in Zone 1

$$\boxed{\eta_1(t,\tau) = \frac{g}{2k_c}\exp\left(\frac{E_1}{RT}\right)\frac{\rho}{(\rho_i - \rho)}\tau}$$

What causes high viscosity in H&L Zone 1?

Cold temperature

gravity

High density

$$\eta_1(t,\tau) = \frac{g}{2k_c}\exp\left(\frac{E_1}{R\theta}\right)\frac{\rho}{(\rho_i - \rho)}\tau$$

near ice density

Firn age

Kind of sounds sensible, but it’s not exactly clear why gravity and age should increase instantaneous viscosity …

Herron & Langway (1980) *J. Glaciol.*

Similarly, in Zone 2 -

$$\frac{1}{\rho(t,\tau)}\frac{D\rho}{Dt} = \frac{k_2}{\sqrt{g\tau}}\exp\left(\frac{-E_2}{RT}\right)\left(\frac{\rho_i - \rho(t,\tau)}{\rho(t,\tau)}\right)\sqrt{\sigma(t,\tau)}$$

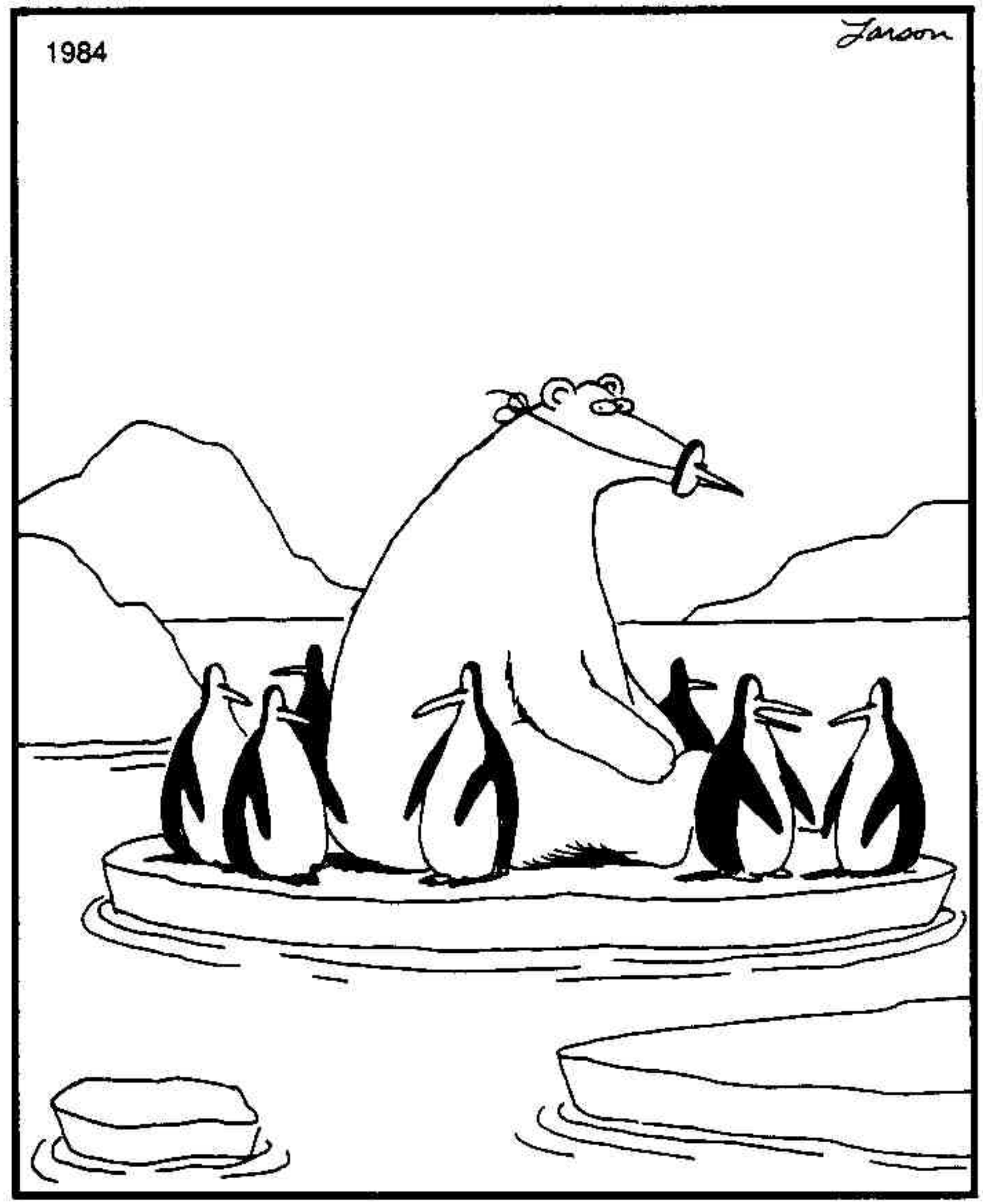
Nonlinear –
Depends on σ

Leads to H&L viscosity implied in Zone 2

$$\boxed{\eta_2(t,\tau) = \frac{\sqrt{g}}{2k_c}\exp\left(\frac{E_2}{RT}\right)\frac{\rho}{(\rho_i - \rho)}\sqrt{\tau}\sqrt{\sigma}}$$

Different dependencies on gravity and age ...

“And now Edgar’s gone …
Something’s going on around here.”



Incomplete physics

Physical processes that are not captured by the compaction equation are being “tuned” into τ and ρ

- The tuning captures their steady-state influences, not transient influences
- Correlated influences can be confounded

Arthern et al. (2010) *JGR*

$$\frac{D\rho}{Dt} = k_c(\rho_i - \rho)\exp\left(\frac{-E_c}{R\theta}\right)\left(\frac{\sigma}{r^2}\right)$$

$$\frac{Dr^2}{Dt} = k_g(\rho_i - \rho)\exp\left(\frac{-E_g}{R\theta}\right)$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = k_c\frac{(\rho_i - \rho)}{\rho}\left(\frac{1}{r^2}\right)\exp\left(\frac{-E_c}{R\theta}\right)\sigma$$

$$\boxed{\eta = \frac{1}{2k_c}\frac{\rho}{(\rho_i - \rho)}\exp\left(\frac{E_c}{R\theta}\right)\ r^2}$$

Arthern et al. (2010) *JGR*

If temperature is uniform and constant,

$$\frac{Dr^2}{Dt} = k_g(\rho_i - \rho)\exp\left(\frac{-E_g}{R\theta}\right)$$

Can be integrated to give

$$r^2 = k_g(\rho_i - \rho)\exp\left(\frac{-E_g}{R\theta}\right)\tau$$

Implying viscosity in Zone 1 of

$$\boxed{\eta_1(t,\tau) = \frac{k\ g}{2k_c}\frac{\rho}{(\rho_i - \rho)}\exp\left(\frac{E_c - E_g}{R\theta}\right)\tau}$$

Morris and Wingham (2014) *JGR*

$$\dot{\epsilon}_{zz} = -\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{k_0^*}{\rho_w\, g}\frac{(\rho_i - \rho)}{\rho}\,\frac{\left(1 - \tilde{M}_0 m\right)}{H(t,\tau)}\,\exp\left(\frac{-E_a}{R\,\theta(t,\tau)}\right)\,\sigma_{zz}$$

$$m = \frac{\rho - \rho_0}{\rho_i - \rho_0}$$

Deviations from a smooth profile $\rho_0(z)$ to capture summer-winter layers

$$\tilde{M}_0$$

Weighting given to seasonality

Morris and Wingham (2014) *JGR*

$$\dot{\epsilon}_{zz} = -\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{k_0^*}{\rho_w\, g}\,\frac{(\rho_i - \rho)}{\rho}\,\frac{\left(1 - \tilde{M}_0 m\right)}{H(t,\tau)}\,\exp\left(\frac{-E_a}{R\,\theta(t,\tau)}\right)\,\sigma_{zz}$$

$$H(t,\tau) = \int_{t-\tau}^{t} \exp\left(\frac{-E_a}{R\,\theta(s,\tau - t + s)}\right)\mathrm{d}s$$

Thermal-history factor – exposure to higher temperatures accelerates stiffening

$$H(t,\tau) = \exp\left(\frac{-E_H}{R\,\theta(t,\tau)}\right)\,\tau$$

E_H~E_a but slightly different due to seasonality and climate changes

Morris and Wingham (2014) *JGR*

$$\dot{\epsilon}_{zz} = -\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{k_0^*}{\rho_w\, g}\frac{(\rho_i - \rho)}{\rho}\,\frac{\left(1 - \tilde{M}_0 m\right)}{H(t,\tau)}\,\exp\left(\frac{-E_a}{R\,\theta(t,\tau)}\right)\,\sigma_{zz}$$

$$\boxed{\eta(\rho,\theta,m) = \frac{\rho_w\, g}{2k_0^*}\left(\frac{\rho}{\rho_i - \rho}\right)\frac{1}{\left(1 - \tilde{M}_0 m\right)}\,\exp\left(\frac{E_c - E_g}{R\,\theta}\right)\,\tau}$$

Cambridge debate- Arthern vs Morris

Both models invoke an additional thermal process

- Arthern – $r^2(t)$
  - larger grains contribute to greater viscosity over time
  - presumably due to lower curvature and greater contact areas

- Morris – $H(\tau,t)$
  - exposure to warmer temperatures increases viscosity over time
  - presumably due to stronger necks between grains

Freitag et al. (2013) *J. Glaciology*

AWI group has noted different compaction rates between summer and winter snow

- Correlation with impurities
- Ca^+ used as proxy for all impurities
- Herron and Langway model, with activation energy dependent on impurity concentration $[Ca^+]$

Zone 1 $$\eta = \frac{g}{2\kappa_1}\frac{\rho}{(\rho_i - \rho)}\exp\left(\frac{-E_1([Ca^+])}{R\,\theta}\right)\ \tau$$

Zone 2 $$\eta = \frac{g^{0.5}}{2\kappa_2}\left(\frac{\rho}{(\rho_i - \rho)}\right)\exp\left(\frac{E_2([Ca^+])}{R\theta}\right)\sigma^{0.5}\ \tau^{0.5}$$

Start from a constitutive relation

Firn represented as a linear isotropic compressible fluid
e.g. Lüthi and Funk (2000) Ann. Glaciol.

$$\dot{\epsilon}_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \left(1 - \frac{2\mu}{3\lambda}\right)\frac{\sigma_{kk}}{3}\delta_{ij}\right]$$

λ Bulk viscosity
μ Shear viscosity

Relation between (E, ν) and (λ, μ) systems

$$
\begin{aligned}
E &= \left(\frac{9\lambda\mu}{3\lambda+\mu}\right) \\
\nu &= \frac{1}{2}\left(\frac{3\lambda-2\mu}{3\lambda+\mu}\right)
\end{aligned}
$$

Start from a constitutive relation

Firn represented as a linear isotropic compressible fluid
e.g. Lüthi and Funk (2000) *Ann. Glaciol.*

Alternatively …

$$\dot{\epsilon}_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\frac{\sigma_{kk}}{3}\delta_{ij}\right]$$

E Viscous analog of Young's modulus
ν Viscous analog of Poisson's ratio

Transition to ice

Expressed in terms of deviatoric stress τ

$$\tau_{ij} = \sigma_{ij} + p\,\delta_{ij}$$

Where pressure is

$$p = -\frac{\sigma_{kk}}{3}$$

$$\dot{\epsilon}_{ij} = \left(\frac{1+\nu}{E}\right)\tau_{ij} + \left(\frac{1-2\nu}{E}\right)\frac{\sigma_{kk}}{3}\delta_{ij}$$

Using true constitutive relation

$$\dot{\epsilon}_{ij} = \left(\frac{1+\nu}{E}\right)\tau_{ij} + \left(\frac{1-2\nu}{E}\right)\frac{\sigma_{kk}}{3}\delta_{ij}$$

Can transition smoothly into incompressible form
When ν -> 1/2

e.g. Glen law

$$\dot{\epsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$$

Forms for compaction in one dimension

$$\dot{\epsilon}_{zz} = \left(\frac{1+\nu}{E}\right)\tau_{ij} + \left(\frac{1-2\nu}{E}\right)\left(\frac{1+\nu}{1-\nu}\right)\frac{\sigma_{zz}}{3}$$

$$\dot{\epsilon}_{zz} = \left(\frac{1}{E}\right)\left[\frac{(1-4\nu/3)(1+\nu)}{(1-\nu)}\right]\sigma_{zz}$$

E and ν, or λ *and* μ can be measured on samples or calculated from a micro model (metallurgy, ceramic powders, …)

Viscosity formulation

$$\dot{\epsilon}_{zz} = \left(\frac{1}{E}\right)\left[\frac{(1-4\nu/3)(1+\nu)}{(1-\nu)}\right]\sigma_{zz} \quad (1)$$

$$\dot{\varepsilon}(z,t) = \frac{1}{2\eta(z,t)}\sigma(z,t) \quad (2)$$

Compare (1) and (2) to find form of viscosity $\eta(E,\nu)$

- (E,ν) depend on density ρ, temperature θ, and firn structural parameters $\vec{s}(\rho,\theta,r^2,\ldots)$

Outlook

The area that needs work is evolution of the structural properties vector

$$\frac{\mathrm{D}\mathbf{s}}{\mathrm{D}t} = F_s(\rho, \theta, \mathbf{s}, \ldots)$$

viscosity depends on the structure

- Density evolution depends on viscosity

Thermal parameters depend on firn structure

- Temperature evolution depends on viscosity

Outlook

New micro-tomography measurements can illuminate structure and its evolution

- AWI, Dartmouth, other labs too

$$\frac{\mathrm{D}\mathbf{s}}{\mathrm{D}t} = F_s(\rho, \theta, \mathbf{s}, \ldots)$$

If a steady state assumption still must be made,

- Probably better to make it at the microstructure
- level, rather than at the density-depth level
- Applying physics deeper into the problem is probably better.

So, that’s all there is