Tackling the Sonar Equation

LO: Apply characteristics of sound in water to calculate sound levels.

John K. Horne
Sonar Equation: Single Target

\[ V_o = SL + G_1 + TS + 2D_i(\phi, \theta) - 40\log(r) - 2\alpha r + G_{tvg} + G_{rec} \]

where:
- \( V_o \) = voltage out (also EL echo level)
- \( SL \) = transducer source level (at a specific transmit level)
- \( G_1 \) = through system gain, at 1m
- \( TS \) = target strength (acoustic size)
- \( D_i(\phi, \theta) \) = directivity index (i.e. 0 dB for on-axis targets)
- \( 40 \log(r) \) = two-way transmission (spreading) loss at range \( r \)
- \( \alpha \) = absorption coefficient
- \( G_{tvg} \) = time-varied-gain (20 or 40 \( \log(r) \))
- \( G_{rec} \) = receiver gain
Source Level Cal Measurement

\[ SL = 20\log\left(\frac{i_{p-p}}{8}\right) + S_i \]

where:
- \( i_{p-p} \) = peak to peak current to transducer
- \( S_i \) = transducer transmitting response
  (pressure on axis at 1 m produced by 1 unit electrical power (units amps))

Source Level in sonar equation is a pressure from a source \((p_o)\)

\[ SL = 20\log(p_o) \]

Example:

\[ i_{p-p} = 40 \text{ A} \quad S_i = 209 \text{ dB} \parallel 1 \text{ \mu Pa} \]

\[ SL = 20\log(40/8) + 209 = 223 \text{ dB} \parallel 1 \text{ \mu Pa} \]
Source Level Measurement

The oscilloscope $v_{pp}$ (volts) is converted to $V_{so}$ (dBV):

$$V_{so} = 20 \cdot \log\left(\frac{v_{pp}}{2/1.414}\right)$$

The Sonar equation for the one-way transmission to the standard:

$$V_{so} = SL - TL + S_s$$

$$TL_{cal} = 20 \cdot \log(R_{cal}) + \alpha R_{cal}$$

$SS$ is a calibration value provided with the standard, therefore:

$$SL = V_{so} + TL_{cal} - S_s$$
Sonar Equation (log form)

\[ V_o = SL + G_1 + TS + 2D_i(\phi, \theta) - 40 \log(r) - 2\alpha r + G_{tvg} + G_{rec} \]

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Through System Gain: $G_1$

- receive sensitivity of echosounder
- dependent on range compensation (i.e. 20 or 40 log TVG)

$$G_1 = V_{det} - L - 40\log(r_{cal}) + 2\alpha r_{cal} - G_{rec}$$

where:

$V_{det} = \text{voltage detected}$
$L = \text{transducer diameter}$
$r_{cal} = \text{calibration range}$
$\alpha = \text{absorption loss}$
$G_{rec} = \text{receiver gain}$
$G_1$ Measurement

$V_s$ is dB$_v$ equiv of $v_{pp}$; $V_{det}$ is dB$_v$ equiv of $v_{det}$.

$v_{det} = G + V_s + TS - TL$

$TL_{cal} = 20 \log(R_{cal}) + \alpha R$

TS is a calibration value supplied with the standard.

Remembering that $G = G_1 + G_{tvg} + RG$,

with a 40 log TVG characteristic:

$G_{tvg} = 40 \log(R_{cal}) + 2\alpha R_{cal}$

$G_1 = v_{det} - G_{tvg} - RG - V_s + TS - TL_{cal}$
Calibration Sheet: SL and $G_1$

**Sum Channel Detected 12 kHz Output**

Calibration Readings

\[
V_{12kHz} = \frac{0.408}{volts \ (rms)}
\]

\[
V_{det} = \frac{4.81}{dB \ (det)}
\]

TVG Gain

\[
G(40) = (40 \ log \ R_{cal} + 2a \ R_{cal})
\]

\[
G(40) = 42.00 \ dB
\]

Sensitivity at 1 m

\[
G_1 = G_x - G(40) - R_g
\]

\[
G_1 = -171.87 \ dB/uPa \ @ \ 1m
\]

**20 Log R Channel Detected Output**

Calibration Readings

\[
v_{det} = \frac{0.575}{volts \ (peak)}
\]

\[
V_{det} = \frac{-4.81}{dBV \ (det)}
\]

TVG Gain

\[
G(20) = (20 \ log \ R_{cal} + 2a \ R_{cal})
\]

\[
G(20) = 21.00 \ dB
\]

Sensitivity at 1 m

\[
G_1 = G_x - G(20) - R_g
\]

\[
G_1 = -150.90 \ dB/uPa \ @ \ 1m
\]

**Transmission Loss**

\[
TL = 20 \ log \ Rs + aR
\]

\[
TL = 15.71 \ dB
\]

**Source Level**

\[
SL = V_{so} - S_s + TL
\]

\[
SL = V_{so} - S_s + TL
\]

$$
\text{Source Level (SL)}
$$

<table>
<thead>
<tr>
<th>Transmit Power (dB)</th>
<th>Standard Transducer</th>
<th>Source Level (dBuPa @ 1 m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.0</td>
<td>8.13</td>
<td>216.79</td>
</tr>
<tr>
<td>14.0</td>
<td>-14.06</td>
<td>210.86</td>
</tr>
<tr>
<td>8.0</td>
<td>0.01</td>
<td>204.93</td>
</tr>
<tr>
<td>2.0</td>
<td>-6.56</td>
<td>198.36</td>
</tr>
</tbody>
</table>
Sonar Equation (log form)

\[ V_o = SL + G_1 + TS + 2D_i(\phi, \theta) - 40\log(r) - 2\alpha r + G_{tvg} + G_{rec} \]

where:

\( V_o \) = voltage out (also EL echo level)

\( SL \) = transducer source level (at a specific transmit level)

\( G_1 \) = through system gain, at 1m

\( TS \) = target strength (acoustic size)

\( D_i(\phi, \theta) \) = directivity index (i.e. 0 dB for on-axis targets)

\( 40 \log(r) \) = two-way transmission (spreading) loss at range \( r \)

\( \alpha \) = absorption coefficient

\( G_{tvg} \) = time-varied-gain (20 or 40 log(r))

\( G_{rec} \) = receiver gain
Target Strength TS

- acoustic size of target (e.g. fish or zooplankton)
- ability of an object to reflect sound to the source
- linear measure: backscattering cross section $\sigma_{bs}$ units m$^2$
- measured as a ratio of sound intensities or pressures ($I \propto p^2$)

$$\sigma_{bs} = \frac{I_r}{I_i} = \frac{p_r^2}{p_i^2}$$

$$TS = 10\log(I_r) - 10 \log(I_i) = 20\log(p_r) - 20\log(p_i)$$

$$TS = 10\log(\sigma_{bs})$$
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Transducer Energy Transmission

monostatic transducer or transceiver: transmits and receives from same source

$$D_i = 10\log(D) = 10\log(I_0/\|I\|)$$

fingernail traces (i.e. boomerangs): due to differences in range and intensities
Equivalent Ideal Beam Pattern

\[ \psi = \int_{4\pi} b^2 d\Omega \]

where \( b \) is the radius of the transducer.

10log(\( \Psi \)) = 10log(\( \beta_1 \beta_2 / 5800 \))

where \( \beta \) is the active length of the transducer.

If square or circular transducer:

10log(\( \Psi \)) = 10log(\( \beta^2 / 5800 \))

where \( \beta \) is the active length of the transducer.

\[ \psi = \left( \frac{4.853}{kD} \right)^2 \]

where \( k \) is the wavenumber and \( D \) is the diameter of the transducer.
Integrated Beam Pattern Factor

- one-way loss in signal intensity due to the angle of the target relative to the acoustic axis

\[ I = k \left( 10^{-2\alpha} \frac{R}{R^4} b^2(\theta, \phi) \sigma_{bs} \right) \]

\[ TS = 10 \log_{10} \sigma_{bs} \]
Effect of Beam Pattern

- transmit response (i.e. acoustic level) is highest along acoustic axis

- receive response (i.e. echo level) is highest along acoustic axis

- echo received from a target will decrease off axis due to transmit and receive losses

- echo amplitude of a target depends on acoustic size of fish and position in beam
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Transmission Loss

Geometric Spreading
- pressure decreases as the 1/distance from source
- spherical spreading from a point source (e.g. transducer)
- 2-way spreading increases as range^2

\[ \frac{I_o}{I} = \left(\frac{r}{r_o}\right)^2 \]

\[ TL = 10\log\left(\frac{I_o}{I}\right) = 20\log\left(\frac{r}{r_o}\right) \]

if \( r_o = 1 \text{ m} \)

then one way TL = 20 log(r)

and two way TL = 40log(r)
Sonar Equation (log form)

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Absorption

- attenuation of pressure due to friction ($\alpha$, units nepers/m or dB/m))
- proportional to range
- dependent on frequency: increases proportional to the square of frequency
- higher in salt water than fresh water
Absorption Loss

One way: $\alpha r$, units dBm$^{-1}$

Two way: $2\alpha r$, units dBm$^{-1}$
Total transmission loss (two way): $40 \log(r) + 2\alpha r$
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where:

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- \( G_{rec} \) = receiver gain
Range Compensation: TVG

Time Varied Gain
- amplification applied to received echo to compensate for transmission loss due to beam spreading
- constant TVG is main reason why ‘scientific’ echosounders cost lots

Single target: small relative to wavelength
individual targets can be resolved

one way spreading loss = $1/r^2$
two way spreading loss = $1/r^4$

Log form: $10\log(r^4) = 40\log(r)$
Range Compensation: TVG

Multiple targets: assumes constant density
individual targets can not be resolved
spreading is range dependent
collection is large relative to beam width

one way spreading loss = 1/r
two way spreading loss = 1/r^2

Log form: 10\log(r^2) = 20\log(r)
TVG Again

20 log(r)
- Echo level for fish at range $r \propto 1/r^2$
- $(\text{Echo level})^2 \propto 1/r^4$
- # fish @ $r$ increases with area of beam (i.e. $1/r^2$)
So, squared signal $\propto r^2(1/r^4) = 1/r^2$
Squared signal in dB $\propto 10\log(1/r^2) = -20\log(r)$
$G_{tvg}$: Time Varied Gain

Individual targets: $40\log(r)$  
Multiple targets: $20\log(r)$

where $v_1(t) = \text{uncompensated voltage}$, $a(t) = \text{receiver gain}$, $v(t) = \text{compensated voltage}$
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- \( G_{tvg} \) = time-varied-gain (20 or 40 \( \log(r) \))
- \( G_{rec} \) = receiver gain
G_{rec}: System Receiver Gain

- amplification applied to received echo to center dynamic range of echosounder
- some manufacturers user selected: range -12 dB to +12 dB
- other manufacturers user sets minimum detected and then adds range (typically 36 dB)
Sonar Equation Example

You are on the NOAA R/V Oscar Dyson in the Gulf of Alaska. You are interested in the length distribution of juvenile walleye pollock in Barnabus Trough. You have a 120 kHz echosounder and the Traynor et al. publication that tells you: \( TS = 20 \log(L_{cm}) - 66 \). You measure a target strength of \(-45\) dB re \(1\ \mu\text{Pa}\) from a fish at 100m range. The water is \(10^\circ\) C with a salinity of 35, resulting in an absorption coefficient of 38.7 dB/m. The system is set so that you have a source level of 216.78 dB re \(1\ \mu\text{Pa}\). From the transducer calibration parameter sheet you know that the directivity index is \(-5\) dB re \(1\ \mu\text{Pa}\), and the through system gain is \(171.87\) dB re \(1\ \mu\text{Pa}\).

What is the voltage recorded on your echosounder and what is the length of the fish?
Juvenile Walleye Pollock Length

\[ V_0 = SL + G_1 + TS + 2D_i(\phi, \theta) - 40\log(r) - 2\alpha r + G_{tvg} + G_{rec} \]

where:

- \( V_0 \) = voltage out (also EL echo level)
- \( SL = 216.78 \text{ dB re } 1 \mu\text{Pa} \) transducer source level
- \( G_1 = -171.87 \text{ dB re } 1 \mu\text{Pa} \) through system gain, at 1m
- \( TS = -45 \text{ dB re } 1 \mu\text{Pa} \) target strength
- \( D_i(\phi, \theta) = -5 \mu\text{Pa} \) directivity index
- \( 40\log(r) = 80 \text{ dB re } 1 \mu\text{Pa} \) two-way transmission loss at range \( r \)
- \( \alpha = 0.0387 \text{ dB/m} \) (120 kHz, 10°C, 35 ppt) absorption coefficient
- \( G_{tvg} = 80 \text{ dB re } 1 \mu\text{Pa} \) 40 log(r) time-varied-gain
- \( G_{rec} = 0 \text{ dB re } 1 \mu\text{Pa} \) receiver gain

**Conditions:**
- Frequency = 120 kHz
- Target Range = 100 m
- \( H_2O \) Temp = 10°C
- Salinity = 35 ppt
Sonar Equation Example

\[ V_o = SL + G_1 + TS + 2D(\phi, \theta) - 40\log(r) - 2\alpha r + G_{tvg} + G_{rec} \]

\[ V_o = 216.79 + (-171.87) + (-45) + 2(-5) - 80 - 7.74 + 80 + 0 \]

\[ V_o = -17.82 \text{ dB}_v \]

\[ 20\log(\text{volts}) = \text{dB}_v \quad 10^{\text{dBv}/20} = \text{volts} \]

\[ 10^{\text{dBv}/20} = 0.12853 \text{ volts} \]

\[ TS = 20\log(L)-66 \]

\[ L = 11.22 \text{ cm} \]