Assignment for Day 8 – May 16, 2007

Read §4.4–4.6, pp. 204–232. You should also look at the Appendix on pp. 234-236, in order to get a sense of what is proved there and how it is proved.

A note on this week's assignment
The main focus of this week's assignment is the statement that if a function $f(x)$ is defined on an interval $[a, b]$ and is continuous there, then it has three very important properties: (1) It is bounded on $[a, b]$; (2) It *attains* its maximum and minimum values on that interval, and (3) It *has no gaps*. These are spelled out in §4.6 as Propositions 11, 12, and 13 on pages 228–230. These three propositions are crucial to the work of the last chapter where we prove that the area under the graph of a continuous function exists and that the Fundamental Theorem of Calculus is true.

∗∗∗ We will spend class time together next Wednesday working through the proof of Proposition 11 and, hopefully Proposition 12. So please make sure you work on these proofs as part of your preparation for class. Both proofs depend on Proposition 10.

In order to state and prove these propositions, however, we must define the concept of a *continuous function*. This depends on having a definition of the concept of a *limit of a function*. But haven't we already defined both of these concepts in §4.3? The answer to this question is that in §4.3, we only defined these concepts for the restricted class of monotonic functions. The work of §4.4 and §4.5 is to extend the notions of a limit of a function and continuous function from this restricted class of functions to the general case of *all* functions. The first step in this process in §4.4 where we extend the theory of continuous functions from the case of monotonic functions to the case of *piecewise monotonic functions*. The next step is in §4.5 where we make new definitions of the concepts of limit and continuity. In §4.6, we prove that functions that are continuous in this new and wider sense have the desired three properties.

Study Questions

1. As you read §4.4, you should write out your own (informal) definition of what it means for a function to be *piecewise monotonic*. Why do we need such a class of functions? How close is this class of functions to the class of *all* functions one would ever need in calculus?

2. What are the differences and connections between the definition of continuity in the case of a piecewise monotonic function and the definition we had in §4.3 of a continuous function?

3. In §4.3, we showed that a function that is monotonic and continuous on an interval $[a, b]$ satisfies the *Intermediate Value Property*—which means that its graph has no gaps. Explain how we are able to extend this proposition to the class of piecewise monotonic functions.

4. Describe informally what the differences and the connections are between the two properties of a function being *piecewise monotonic* and being *locally monotonic*?

5. In §4.3 we only dealt with monotonic functions and were able to define the concept of a limit in terms of the two concepts *least upper bound* and *greatest lower bound*. But in §4.5, we are dealing with functions that can oscillate wildly up and down. This requires a new concept of limit. This is very similar to something we already did in earlier sections when we discussed walks in which the walker possibly goes back and forth and when we discussed infinite sequences that are not monotonic. As you read the first six pages of §4.5, you should recall our earlier discussions and connect them to what is being said here. Then you should loosely paraphrase what this all means as we go from monotonic functions to functions that can oscillate wildly up and down.

6. As you read the material in §4.5, you should think about the two functions $W(x)$ and $D(x)$ defined on p.206. What do you think the limit is of each function is as x approaches 0? Using the definition of the limit of a function given on p.222, how would you support these two claims?

7. Using the two definitions of a continuous function given near the end of §4.5, you should discuss informally whether or not the function $f(x) = x^2$ is continuous at x=0. What about the function $g(x) = 1/x$ at x=0? What about the two functions $W(x)$ and $D(x)$ at x=0?

8. Consider the oscillating sequence that goes back and forth between –1 and 1: {1, –1, 1, –1, 1, –1….}. Is this sequence bounded? Does it have a limit? What, if anything, does the Convergent Subsequence Theorem say about this sequence?

9. You should study the proof of Proposition 10 in the same way we have studied earlier theorems that we have proved together in class. You should get as far as you can doing the same for Proposition 11, always considering the question of possible analogies between these two proofs in the same way we considered the question of analogies between the proofs for the area of a circle and the area under a graph.

E-Post Questions for Day 8

In your E-Post, you should give your responses to the following question:

> The concepts of limit and continuous function are "presented" (whatever that means) in every calculus textbook and in most introductory calculus courses. The definition of continuity found in at least one popular calculus book is paraphrased on pages 173-4. Is there anything that has been discussed in the present reading that you would find useful or valuable if you were about to teach these concepts in an introductory calculus course? *Please be honest and thoughtful.*

Weekly Writing Assignment for Day 8

Suppose you are writing a letter to someone who read and understood the material in §4.1–4.3, but this person will not be able to do any of the reading in §4.4–4.6. What happens in these later sections? What's likely to be important to know as he or she goes on to the next chapter?