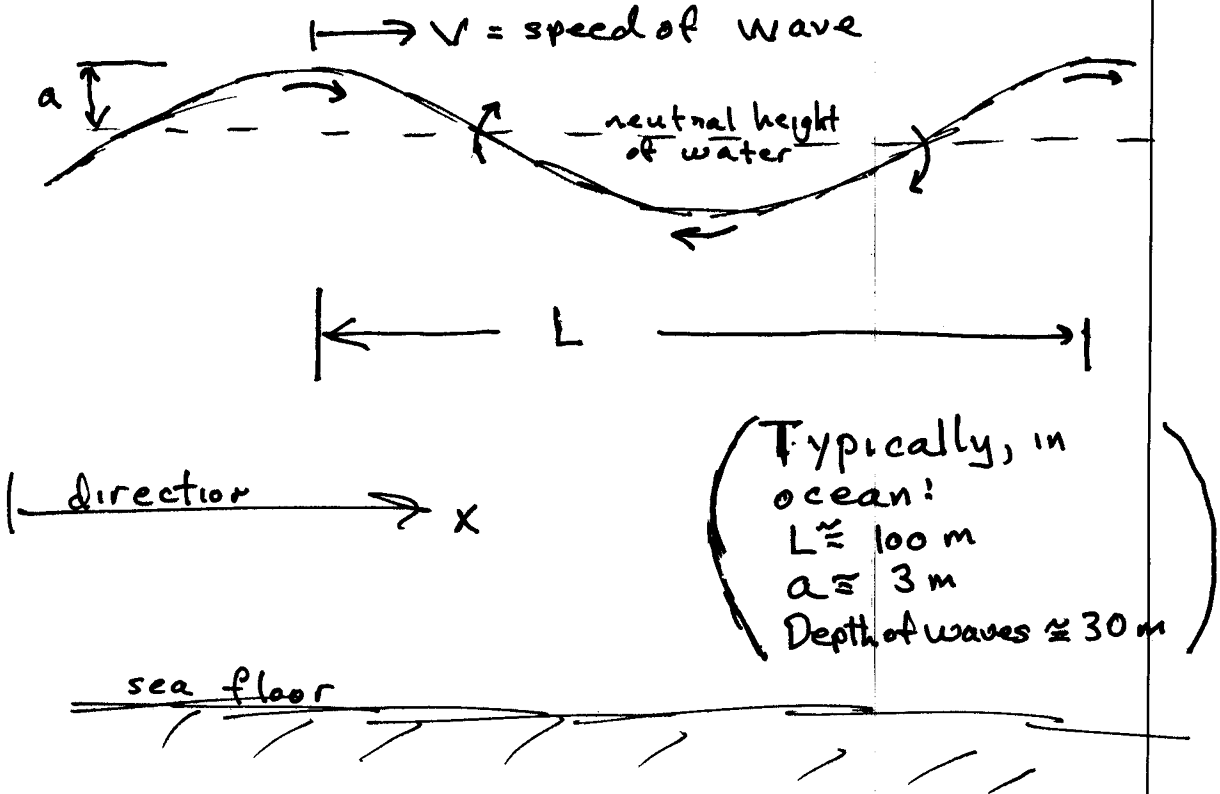
Lecture #22

Wave Energy

Pictured below is a wave at one time. The wave is moving from left to right.

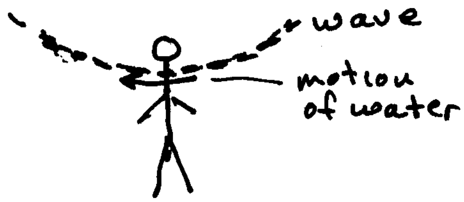


The arrows show how the water (i.e, "particles" of water) are moving.

The wavelength is the distance between crests.

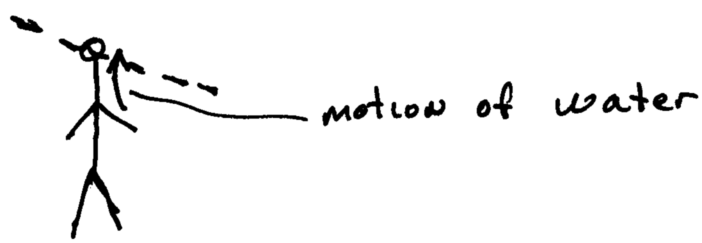
We can also picture the action of the waves at one position (x). Suppose we are standing the water at a fixed position.

Suppose we start noting the action of the waves when a wave valley is at our location. We assume the waves are moving from left to right.

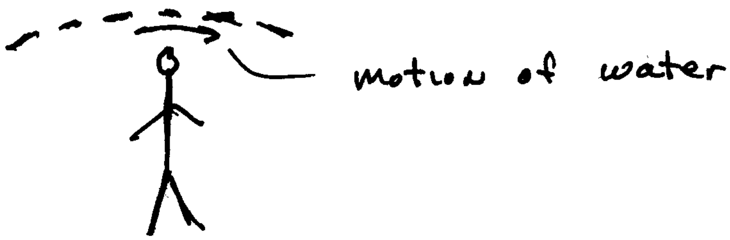


The direction of the water flow is to the left

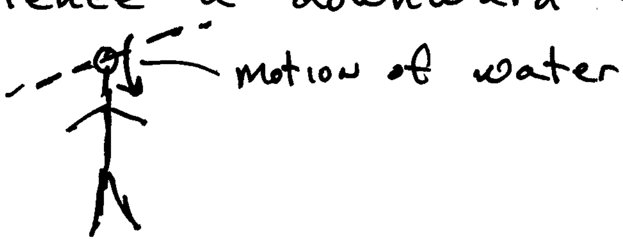
Now, the height of the wave builds. As the crest of the wave approaches we experience an upward flow of water:



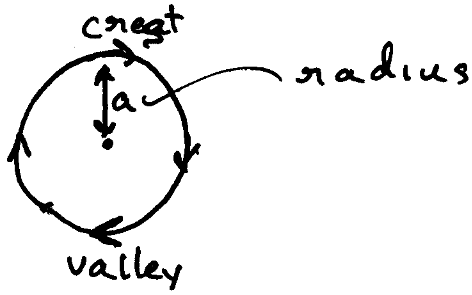
Now the crest hits our position. The flow of water is to the right



Next, the water level falls and we experience a downward flow of water



The the cycle repeats.
Note the motion of the water particles is circular:



The radius of this circle is a, where $a = \frac{1}{2} H$. H is the valley-to-crest height of the waves.

The time it takes one cycle to pass us is the wave period T.
The wave speed is $V = \frac{L}{T}$.
Note, the wave speed V is <u>NOT</u> the speed of the water particles.

The waves move in space (x) and time (t). The equation for a traveling wave of one frequency is

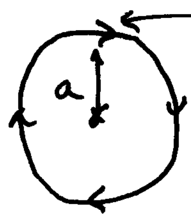
$$h = a \sin\left[2\pi\left(\frac{x}{L} - \frac{t}{T}\right)\right]$$

h: height at any x and t
a: $\frac{H}{2}$

The rotational frequency of the waves is related to T as

$$T = \frac{1}{N} = \frac{2\pi}{\omega}$$

where N = rev/sec and ω = radians/sec.
The circle of radius a also has these values, i.e.

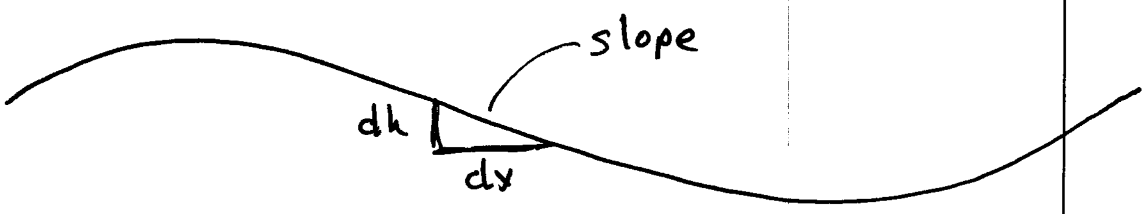


tangential speed

$$= u = 2\pi N a = \omega a$$
$$= \omega \frac{H}{2}$$

And since $V = \frac{L}{T} = \frac{\omega L}{2\pi}$

Let's look at our wave at one time. Let's take the differential of h with respect to x. This is the slope of the wave

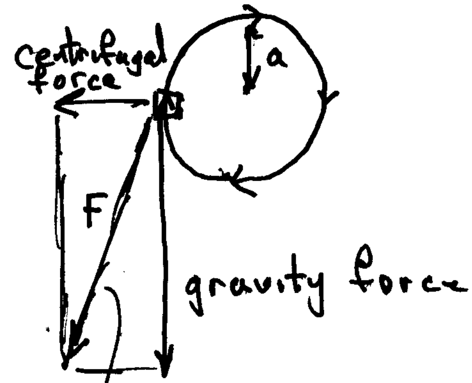


$$\frac{dh}{dx} = \frac{2\pi a}{L} \cos\left[2\pi\left(\frac{x}{L} - \frac{t}{T}\right)\right]$$

we now compare this to the forces acting on the water at the wave surface

There are two forces:

1) The force of gravity
2) The centrifugal force acting on the rotating water



The centrifugal force on particle of water of mass m is $ma\omega^2$

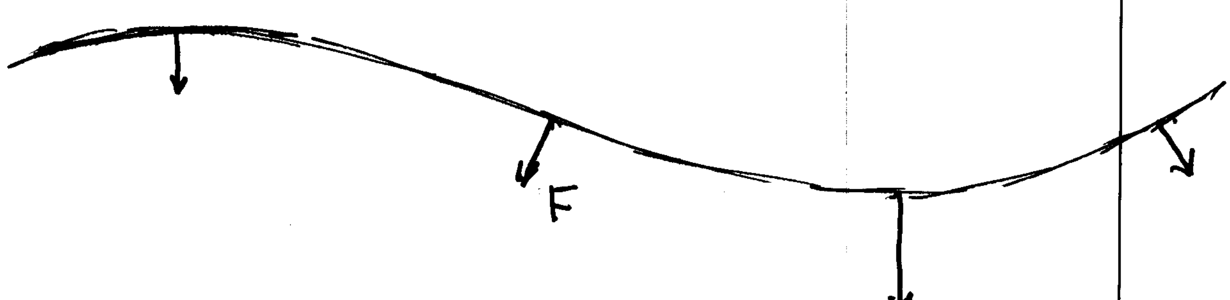
The force of gravity is mg where g is the acceleration of gravity

vector sum of forces = resultant force

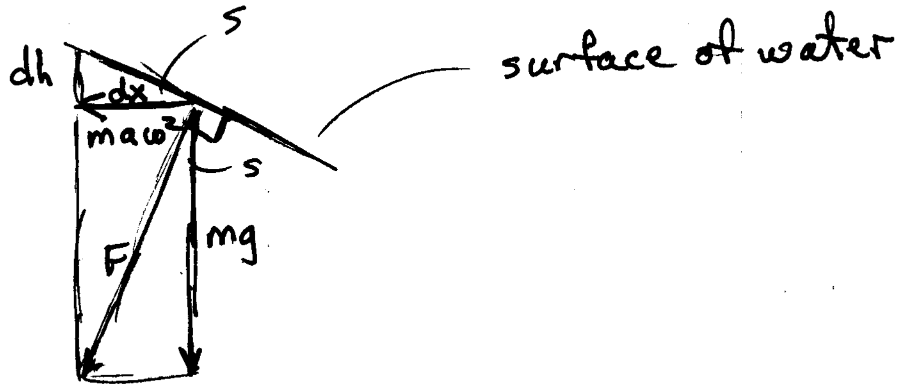
For the case depicted, with the water surface at the neutral height and with the centrifugal force to the left, dh/dx is max.

$$\frac{dh}{dx} = \frac{2\pi a}{L}$$

The resultant force is always normal to the surface of the water



Thus



$$\tan s = \underbrace{\frac{dh}{dx}}_{\frac{2\pi a}{L}} = \frac{ma\omega^2}{mg} = \frac{a\omega^2}{g}$$

Thus $$\frac{2\pi a}{L} = \frac{a\omega^2}{g}$$

and $$\boxed{L = 2\pi g/\omega^2}$$

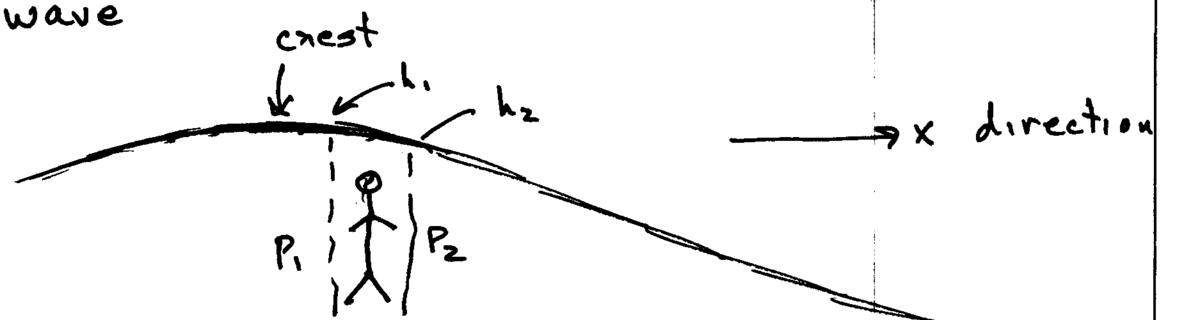
and $$V = \frac{\omega L}{2\pi} = \frac{\cancel{\omega}\,\cancel{2\pi} g}{\cancel{2\pi}\,\omega \cancel{^2}} = \frac{g}{\omega}$$

$$\boxed{V = \frac{g}{\omega}}$$

$$\boxed{V = \frac{gT}{2\pi}}$$

Waves with a long period travel faster!

Now we need to find the power of the waves. Power = force × velocity.
Suppose we are expepiencing a wave



$P_1 > P_2$ since depth of water is $h_1 > h_2$

Thus, the force (to the right) is

$$F = (P_1 - P_2) A = \rho g (h_1 - h_2) A$$

Since $h = a \sin\left[2\pi\left(\frac{x}{L} - \frac{t}{T}\right)\right]$, $h_1 - h_2 \sim a$

Thus

$$F_x \sim \rho g a A$$

The component of the velocity of the water in the x direction is in phase with height h.

Thus

$$u_x = \omega a \sin\left[2\pi\left(\frac{x}{L} - \frac{t}{L}\right)\right], \quad u_x \sim \omega a$$

Thus, power has the proportionality

$$P \sim \rho g a^2 \omega A$$

The face area = depth × $\underbrace{\text{length of wave into paper}}_{\text{"width" of wave}}$

In the deep ocean, the depth of the waves is proportional to L.

Thus

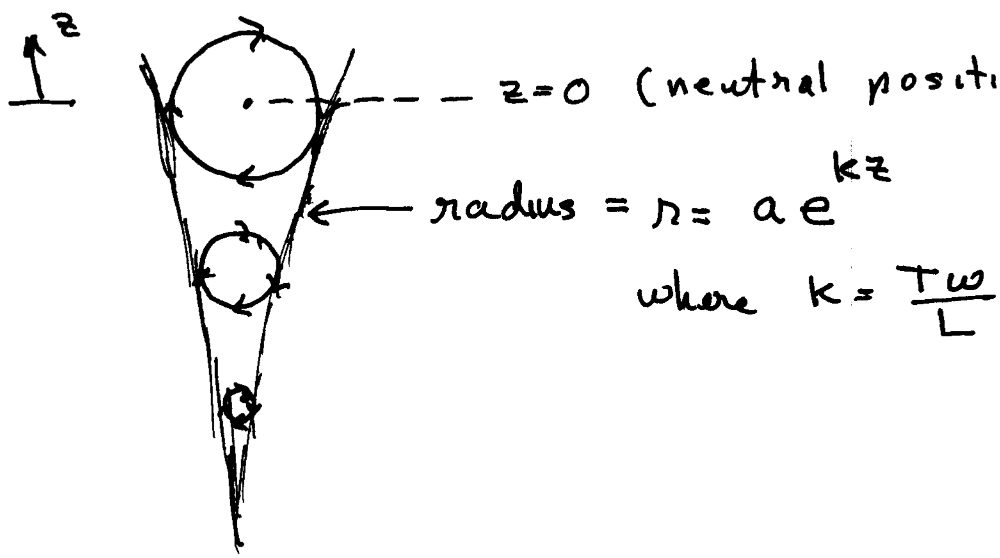
$$P' = \text{power/width} \sim \rho g a^2 \omega L$$

From above: $\omega L = 2\pi V = 2\pi \frac{gT}{2\pi} = gT$

Thus

$$P' \sim \rho g^2 a^2 T$$

Attention to detail, noting that the waves decay with depth as



gives the precise expression

$$\boxed{P' = \text{power/width} = \frac{\rho g^2 a^2 T}{8\pi} = \frac{\rho g^2 H^2 T}{32\pi}}$$