Energy methods:

These are methods based on linear elastic behavior and conservation of energy, i.e. the work done by external forces equals the energy stored in the structure under load.

Energy $U = Fx/2 = F^2/2k$ where F is the applied force, x is the distance moved in the direction of the force at its point of application and k is the elastic stiffness of the part, again in the direction of the force at its point of application.

In tension: $U = \frac{F^2 L}{2AE}$

In torsion: $U = \frac{T^2 L}{2GJ}$

In bending: $U = \int \frac{M^2 dx}{2EI}$

Castigliano's Theorem:

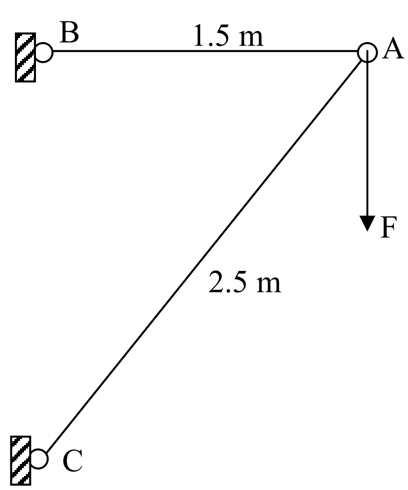
This is a powerful approach to solving a wide range of deflection analysis situations.

The displacement corresponding to any force applied to an elastic stricture and collinear with that force is equal to the partial derivative of the total strain energy with respect to that force.

i.e $\delta_i = \frac{\partial U}{\partial F_i}$ where δ_i is the displacement at the point of application of force F_i in the direction of F_i.

e.g.

The truss members are steel rods with a 50 mm diam.
The load F is 4 kN. Find the deflection at the point A



Statics solution gives F_{AB} = 0.75 F = 3 kN (tension)
F_{AC} = -1.25 F = 5 kN (compression)

Rod area = π x $.05^2/4 = 0.001963$ m^2

Total energy

$$U = \frac{(0.75F)^2 \times 1.5}{2 \times 0.001963 \times 205e9} + \frac{(1.25F)^2 \times 2.5}{2 \times 0.001963 \times 205e9}$$

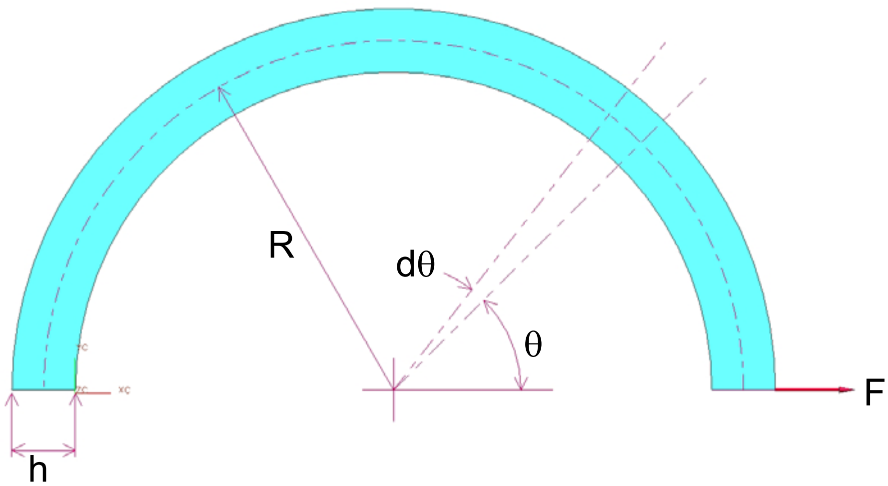
= 5.902e-9 x F^2 N.m.

$$\delta_A = \frac{\partial U}{\partial F_A} = 1.18 \times 10^{-8} F \quad N.m$$

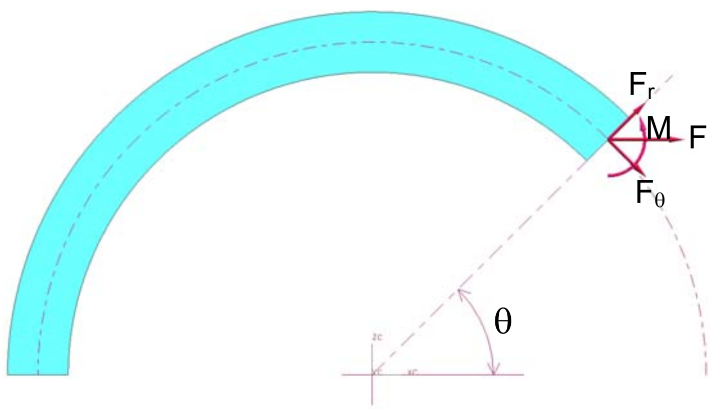
giving δ_A = 0.047 mm

Deflection of curved members:

Consider the curved frame shown in (a) below. We want to find the deflection of the frame due to force F, in the direction of F and at its point of application.



(a)



(b)

Consider the strain energy in the element defined by the angle dθ. The force F is resolved into components F_r and F_θ. There are three parts of the strain energy:

1. due to axial force F_θ we have $dU_1 = \dfrac{F_\theta^2 R d\theta}{2AE}$
2. due to transverse force F_r we have $dU_2 = \dfrac{CF_r^2 R d\theta}{2AG}$ where C = 1.5 is the correction factor for a rectangular cross section in shear.

3. due to bending moment M we have $dU_3 = \dfrac{M^2 R d\theta}{2EI}$ (for R/h > 10 only)

The total strain energy is thus:

$$U = \int \frac{F_\theta^2 R d\theta}{2AE} + \int \frac{CF_r^2 R d\theta}{2AG} + \int \frac{M^2 R d\theta}{2EI}$$

The required deflection is:

$$\delta = \frac{\partial U}{\partial F} = \int_0^\pi \frac{F_\theta R}{AE}\left(\frac{\partial F_\theta}{\partial F}\right) d\theta + \int_0^\pi \frac{CF_r R}{AG}\left(\frac{\partial F_r}{\partial F}\right) d\theta + \int_0^\pi \frac{MR}{EI}\left(\frac{\partial M}{\partial F}\right) d\theta$$

From the figures we find:

$$M = FR\sin\theta \qquad \frac{\partial M}{\partial F} = R\sin\theta$$

$$F_\theta = F\sin\theta \qquad \frac{\partial F_\theta}{\partial F} = \sin\theta$$

$$F_r = F\cos\theta \qquad \frac{\partial F_r}{\partial F} = \cos\theta$$

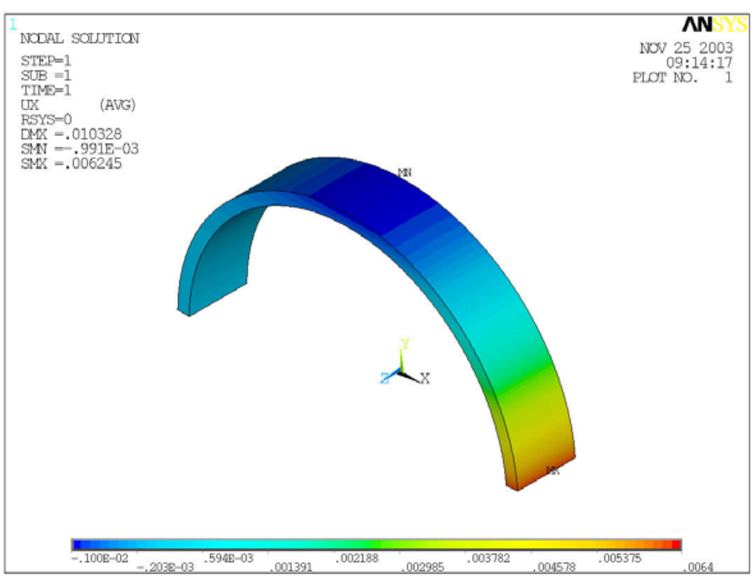
Substituting these gives:

$$\delta = \frac{FR}{AE}\int_0^\pi \sin^2\theta\, d\theta + \frac{CFR}{AG}\int_0^\pi \cos^2 d\theta + \frac{FR^3}{EI}\int_0^\pi \sin^2 d\theta$$

and, $\delta = \dfrac{\pi FR}{2AE} + \dfrac{\pi CFR}{2AG} + \dfrac{\pi FR^3}{2EI}$ and, if R/h is large, the first two terms will be small

and hence an approximate solution is $\delta \approx \dfrac{\pi FR^3}{2EI}$

FEA comparison: a 4" ID, semicircular aluminum object similar to Figure (a) above has a wall thickness of 0.15" and a width of 0.75". The force F is 1.0 lb. Using the equation above gives δ = 0.006218". FEA analysis of this case gives δ = 0.006245".



A Circular Ring Subjected to Diametral Loading

A circular ring subjected to a tensile load P along a diameter is shown in the following sketch. This geometry is often referred to as a "proving ring." Commercial load cells based on a proving ring may be (a) sensing the change in inner diameter, or (b) sensing the strain induced at various points in the member using strain gages (as in this lab).

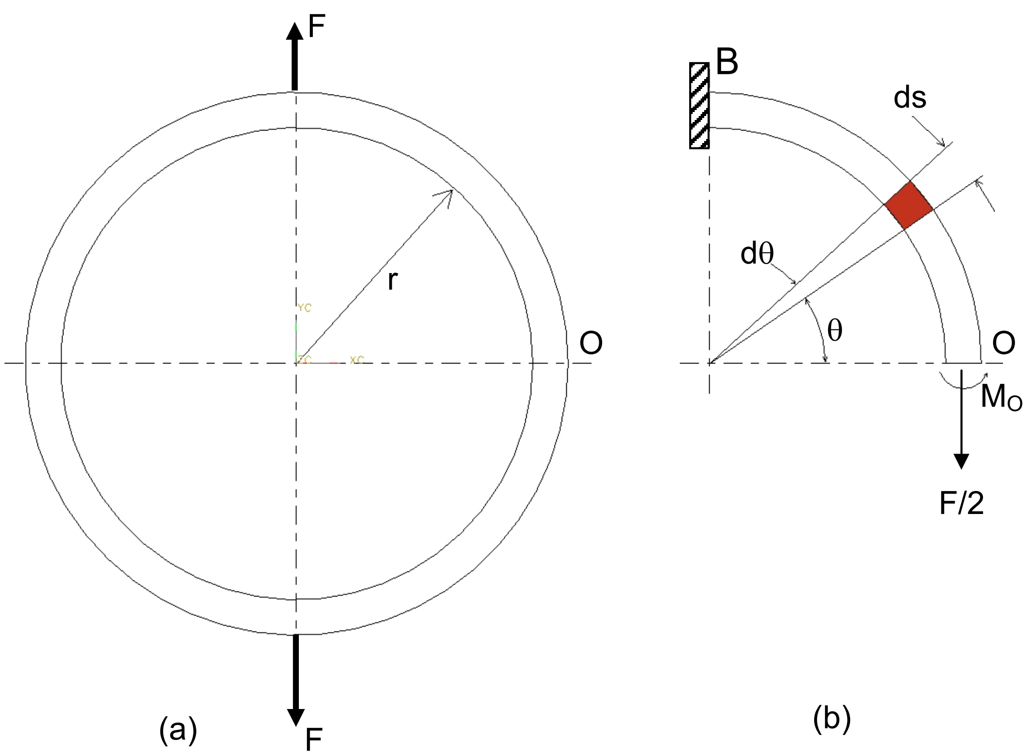
A free-body diagram obtained by making an imaginary cut along the horizontal axis is also shown in the sketch. Due to symmetry considerations:

(a) The shear load at the horizontal cut must be zero (note that horizontal shear loads have therefore not been included in the free-body diagram).

(b) An internal normal force with magnitude $(P/\ 2)$ is induced at the horizontal cross-section.

The bending moment M_o is indeterminate, and therefore cannot be determined directly from the equations of equilibrium. However, M_o can be determined using strain-energy methods as shown above:

Consider a thin ring loaded by two equal and opposite forces as shown:



(a) (b)

Since, by symmetry, the section at O does not rotate, we have by Castigliano:

$$\frac{\partial U}{\partial M_O} = 0$$ where U is the strain energy for a single quadrant.

Now consider the element *ds* shown in (b). At this section the bending moment is:

$$M = M_O - \frac{F}{2}(r - x) = M_O - \frac{Fr}{2}(1 - \cos\theta) \quad as\ x = r\cos\theta$$

The strain energy is:

$$U = \int \frac{M^2 ds}{2EI} = \int_0^{\pi/2} \frac{M^2 r\, d\theta}{2EI} \qquad as \quad ds = r d\theta$$

Hence, from above we have:

$$\frac{\partial U}{\partial M_O} = \frac{r}{EI}\int_0^{\pi/2} M \frac{\partial M}{\partial M_O} d\theta = 0$$

$and,\, from\ above, we\ get \quad \partial M / \partial M_O = 1,\, hence,$

$$\int_0^{\pi/2} M\, d\theta = \int_0^{\pi/2} \left[M_O - \frac{Fr}{2}(1 - \cos\theta) \right] d\theta = 0$$

$Solving\ gives:$

$$M_O = Fr\left(\frac{1}{2} - \frac{1}{\pi}\right) \qquad ....1$$

$hence,\ from\ above\ we\ get$

$$M = \frac{Fr}{2}\left(\cos\theta - \frac{2}{\pi}\right)$$

$The\ greatest\ value\ occurs\ at\ B, where\ \theta = \pi/2\ and\ is$

$$M_B = -\frac{Fr}{\pi}$$

Curved beam theory implies that a uniaxial state of stress is induced along the horizontal diameter. Since linear elastic behavior has been assumed, the total stress induced at the horizontal axis is the sum of the stress caused by the normal force and the stress caused by the bending moment. That is:

$$\sigma = \frac{F}{2A} + \frac{M_o c_i}{Aer_i}$$ at inner surface and $$\sigma = \frac{F}{2A} + \frac{M_o c_o}{Aer_o}$$ at the outer surface.

The strains and, hence the strain gauge outputs are readily found from these stresses.