ME 354 LAB #4: DISCUSSION OF THE TORSION TEST

Each lab section performed a torsion test of a cylindrical 6061-T6 aluminum specimen. The specimen was mounted in a Technovate model 9041 Torsion Tester. A top view is shown in Figure 1. The cylindrical specimen was clamped in two 52.3 mm dia grips. The top grip was held (essentially) fixed via two wire ropes. The bottom grip (not shown in Figure 1) was rotated by means of a threaded loading rod and/or loading lever.

The angle through which the bottom grip (and hence the lower end of the specimen) was rotated was measured using a pointer and angular scale. The force induced in the wire ropes as torque was applied to the specimen was sensed indirectly by means of a lever system and force gage, as shown in Figure 1.

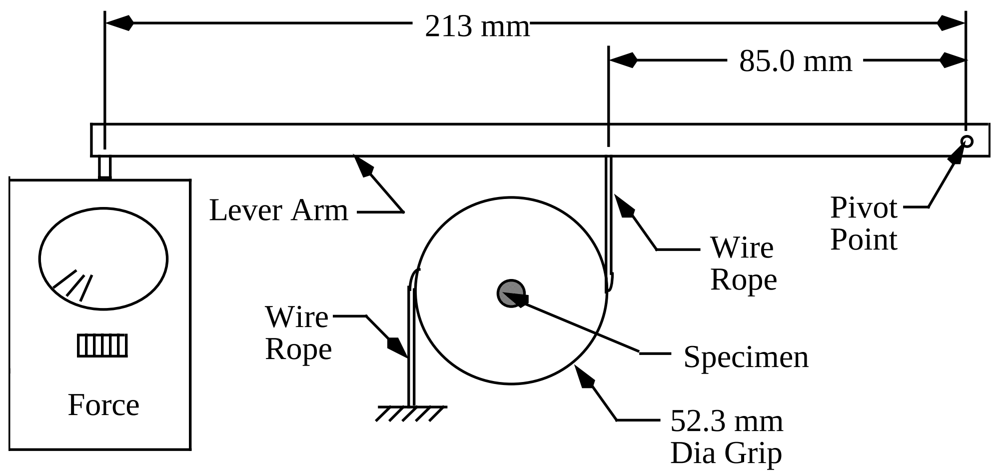

Figure 1: Cylindrical Specimen Mounted in Torsion Tester (Top View)

Tensile tests of 6061-T6 aluminum were conducted during ME354 Lab #3. The following properties can be inferred from this data*:

● Young's modulus for 6061-T6: $E = 68.5\ GPa$

● True stress-true curve modeled using the "Power-Hardening Relationship" (i.e., eq. 4.28 or 12.8 in the Dowling textbook):

strength coefficient: $H = 413\ MPa$

strain hardening exponent: $n = 0.0633$

(Note: in accordance with eq 12.10, these values imply a yield strength of *294 MPa*)

* These properties were inferred by Prof. M. Tuttle based on the data collected during the lab on Monday 27 January 2003. The properties you inferred from data you collected should be similar, but will probably not be numerically identical.

● True stress-true curve modeled using the "Ramberg-Osgood Model" (i.e., eq. 12.13 in the Dowling textbook):

strength coefficient: $H = 407\ MPa$

strain hardening exponent: $n = 0.0490$

Poisson's ratio was not measured; assume $\nu = 0.34$. One objective of this lab is to use these properties (i.e., properties measured during the tension test) to predict the T versus (θ/L) curve measured *during the torsion test*. A formal lab report describing your work is due two weeks after your lab session. The following two items must appear in your lab report:

Table 1: A table with 6 columns is shown on the following page. Complete the first 5 columns of this table using the data collected during the torsion test. In the last column enter the torque predicted at the angle of twist based on the power-hardening model. The steps that should be followed to obtain these predictions are summarized in a following section of this document, titled "Background Information".

Table 2: A table with 6 columns

Item 3: Demonstrate whether the response of the cylindrical specimen subjected to a torque was well predicted using properties measured in tension by plotting measured and predicted torque versus (θ/L) on the same graph.

Table 1: Experimental measurements and predicted torques based on the Power-Hardening model

Angle (deg)	Force (N)	Angle (rad)	θ/L	Measured Torque (N-m)	Predicted Torque (Power-Hardening) (N-m)
0					
2					
4					
6					
8					
10					
12					
14					
16					
18					
20					
22					
24					
26					
28					
30					
35					
40					
45					

Table 2: Experimental measurements and predicted torques based on the Ramberg-Osgood model

$\tilde{\tau}_c$ (MPa)	$\tilde{\gamma}_{pc}$	$\tilde{\gamma}_{ec}$	$\tilde{\gamma}_c$	θ / L	Predicted Torque (Ramberg-Osgood) (N-m)
0					
50					
100					
150					
200					
250					
270					
280					
290					
300					
305					
310					
315					
320					
325					
330					
335					
340					
345					
350					
355					

Background Information

Preliminary Discussion:

● In this lab we tested a cylindrical shaft of radius *c* and length *L*, subjected to a pure torque *T*. Calculation of the stresses and strains induced by this loading is based on the following experimental observation:

"a radial line which is straight *before* loading remains a straight radial line *after* loading"

This observation leads to the conclusion that the shear strain γ increases linearly with the radial distance from center of the shaft (*r*):

$$\gamma_{xy}(r) = \frac{r\theta}{L} \qquad \text{(1a)}$$

where θ is the angle of twist, measured in radians...that is, θ is the angle that the cross-section at one end of the shaft has rotated with respect to the cross-section at the other end (θ was measured during the test...). Equation (1a) indicates that

- γ_{xy} is zero along the shaft centerline (at $r = 0$), and

- γ_{xy} is a maximum at the outer surface of the shaft ($\gamma_{xy} = \gamma_{\max} = c\theta / L$ at $r = c$).

Therefore Eq (1a) can also be written:

$$\gamma_{xy}(r) = \frac{r}{c}\gamma_{\max} \qquad \text{(1b)}$$

● Refer to Figure 3.12 and section 13.4.2 of the Dowling textbook. The torque applied to a circular shaft is related to the shear stress induced at any radial position according to Eq 13.52 (repeated here as Eq 2):

$$T = 2\pi \int_0^c \tau_{xy} r^2 dr \qquad \text{(2)}$$

To evaluate this integral we must specify how stress is related to strain. We will here consider three possibilities: (a) linear elastic, (b) nonlinear, power-hardening model, and (c) nonlinear, Ramberg-Osgood model.

If the material is linear-elastic (which requires that stresses are relatively low such that yielding does not occur), then according to Hooke's Law ($\tau_{xy} = G\gamma_{xy}$) the shear stress also increases linearly with *r*:

$$\tau_{xy} = G\gamma_{xy} = \frac{Gr\theta}{L}$$

This result can be rearranged as follows:

$$r = \frac{\tau_{xy} L}{G\theta} \qquad (3)$$

In this case integration of Eq (2) leads to the well-known "torsion formula":

$$\tau_{xy} = \frac{Tr}{J} \quad \Rightarrow \quad T = \frac{\tau_{xy} J}{r}$$

Or, equivalently:

$$T = GJ\frac{\theta}{L} \qquad (4)$$

Now, the original experimental observation ("...straight radial lines remain straight radial lines...") holds true even *if the shaft is plastically deformed*. Hence, Eq (1) is valid even if the shaft is loaded beyond the yield point. However, Eqs (2-4) are based on the assumption of linear-elastic behavior, and therefore these equations are not valid once yielding has occurred. An idealized sketch showing the distribution of shear strains and stresses both before and after yielding is shown in Figure 2. If stresses are low and yielding does not occur, then both shear stress and shear strains increase linearly from the shaft center, and reach maximum values at the outer radius. However, after yielding only the shear strain increases linearly. After yielding an "elastic core" develops, and at radial positions outside this core the material has been plastically deformed and the shear stress distribution is nonlinear. The radial position at which yielding is initiated can be predicted using Eq (3):

$$r_y = \frac{\tau_o L}{G\theta} \qquad (5)$$

where τ_o = the shear yield strength.

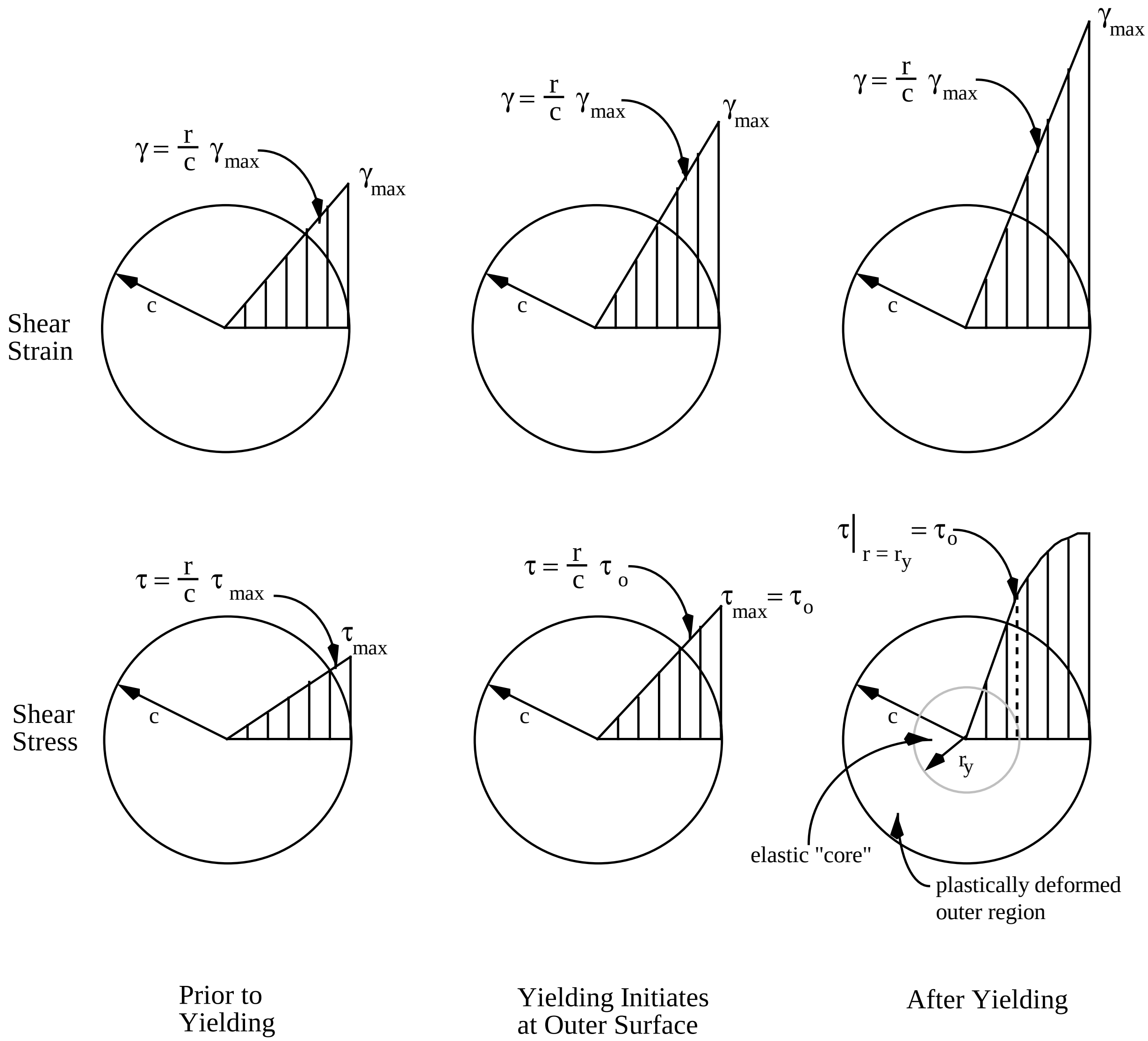

Figure 2: Distribution of shear stress and shear strain in a shaft with circular cross-section, subjected to a torque *T*

● We will use the concept of "effective stress" and "effective strain" in our analysis. The effective stress is listed in the Dowling textbook as Eq (7.37, 7.38):

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2} \qquad (6)$$

Since we are interested in stresses well beyond yielding, it is appropriate to use true stresses in Eq (6):

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left[(\tilde{\sigma}_x - \tilde{\sigma}_y)^2 + (\tilde{\sigma}_y - \tilde{\sigma}_z)^2 + (\tilde{\sigma}_z - \tilde{\sigma}_x)^2 + 6(\tilde{\tau}_{xy}^2 + \tilde{\tau}_{yz}^2 + \tilde{\tau}_{zx}^2)\right]^{1/2} \qquad (7)$$

“It can be shown” that the effective strain is related to true strains according to:

$$\bar{\varepsilon} = \frac{\sqrt{2}}{3}\left[(\tilde{\varepsilon}_x - \tilde{\varepsilon}_y)^2 + (\tilde{\varepsilon}_y - \tilde{\varepsilon}_z)^2 + (\tilde{\varepsilon}_z - \tilde{\varepsilon}_x)^2 + \frac{3}{2}(\tilde{\gamma}_{xy}^2 + \tilde{\gamma}_{yz}^2 + \tilde{\gamma}_{zx}^2)\right]^{1/2} \qquad (8)$$

Equation (8) does not appear in the Dowling textbook, but is equivalent to Eq 12.22.

*****HERE ARE SOME IMPORTANT OBSERVATIONS ABOUT EFFECTIVE STRESS AND EFFECTIVE STRAIN*****

● <u>What was the effective stress during the uniaxial tensile test</u>? During the tension test the only stress applied was the axial true stress, $\tilde{\sigma}_x$. No other stresses were present:

$$\tilde{\sigma}_y = \tilde{\sigma}_z = \tilde{\tau}_{xy} = \tilde{\tau}_{yz} = \tilde{\tau}_{zx} = 0$$

Substituting these conditions into Eq (7), we find:

$$\bar{\sigma} = \tilde{\sigma}_x$$

....during the tension test the effective stress $\bar{\sigma}$ was identical to the true stress $\tilde{\sigma}_x$

● <u>What was the effective strain during the uniaxial tensile test?</u> We measured the true axial strain, $\tilde{\varepsilon}_x$, during the tension test. After yielding, $\nu \to 1/2$. We can therefore assume that, after yielding:

$$\tilde{\varepsilon}_y = \tilde{\varepsilon}_z = \frac{-\tilde{\varepsilon}_x}{2}$$

$$\tilde{\gamma}_{xy} = \tilde{\gamma}_{yz} = \tilde{\gamma}_{zx} = 0$$

Substituting these conditions into Eq (8), we find:

$$\bar{\varepsilon} = \tilde{\varepsilon}_x$$

....after yielding during the tension test the effective strain $\bar{\varepsilon}$ was identical to the true axial strain $\tilde{\varepsilon}_x$.

● <u>What was the effective stress during the torsion test</u>? During the torsion test the only stress applied was the true shear stress, $\tilde{\tau}_{xy}$. No other stresses were present during the torsion test:

$$\tilde{\sigma}_x = \tilde{\sigma}_y = \tilde{\sigma}_z = \tilde{\tau}_{yz} = \tilde{\tau}_{zx} = 0$$

Substituting these conditions into Eq (7), we find:

$$\bar{\sigma} = \sqrt{3}\ \tilde{\tau}_{xy}$$

.... during the torsion test the effective stress ($\bar{\sigma}$) equaled the true shear stress ($\tilde{\tau}_{xy}$) multiplied by a constant factor ($\sqrt{3}$)

● <u>What was the effective strain during the torsion test</u>? The only strain induced during the torsion test was the true shear strain $\tilde{\gamma}_{xy}$. No other strains were present:

$$\tilde{\varepsilon}_x = \tilde{\varepsilon}_y = \tilde{\varepsilon}_z = \tilde{\gamma}_{yz} = \tilde{\gamma}_{zx} = 0$$

Substituting these conditions into Eq (8), we find:

$$\bar{\varepsilon} = \frac{1}{\sqrt{3}}\tilde{\gamma}_{xy}$$

.... during the torsion test the effective strain ($\bar{\varepsilon}$) equaled the true shear strain ($\tilde{\gamma}_{xy}$) multiplied by a constant factor ($1/\sqrt{3}$).

These observations are important during application of the “power-hardening” and “Ramberg-Osgood” models, as discussed in the following subsections.

<u>Power-Hardening Model</u>: As a part of the uniaxial tensile test data reduction, we fit the measured true axial stress-true axial strain response for 6061-T6 aluminum to the following expression:

$$\tilde{\sigma}_x = H\tilde{\varepsilon}_x^n$$

....that is, we have "measured" *H* and *n* for 6061-T6 aluminum. As pointed out above, during the *tension* test: $\bar{\sigma} = \tilde{\sigma}_x$ and $\bar{\varepsilon} = \tilde{\varepsilon}_x$. Furthermore, during the *torsion* test: $\bar{\sigma} = \sqrt{3}\ \tilde{\tau}_{xy}$ and $\bar{\varepsilon} = \frac{1}{\sqrt{3}}\tilde{\gamma}_{xy}$. We can therefore write the following:

$$\tilde{\sigma}_x = H\tilde{\varepsilon}_x^n$$

$$\Downarrow$$

$$\bar{\sigma} = H\bar{\varepsilon}^n$$

$$\Downarrow$$

$$\sqrt{3}\ \tilde{\tau}_{xy} = H\left(\frac{\tilde{\gamma}_{xy}}{\sqrt{3}}\right)^n$$

$$\Downarrow$$

$$\tilde{\tau}_{xy} = \frac{H}{\sqrt{3}}\left(\frac{\tilde{\gamma}_{xy}}{\sqrt{3}}\right)^n \qquad (10)$$

**** Eq (10) relates the true shear stress ($\tilde{\tau}_{xy}$) and true shear strain ($\tilde{\gamma}_{xy}$) present during the <u>*torsion test*</u> using material properties (*H*, *n*) measured <u>*during the tension test*</u>.****

We are now ready to predict the measured torque versus (*θ*/*L*) response. The total applied torque equals the sum of the torque acting over the elastic core plus the torque acting over the plastically-deformed outer region:

$$T_{total} = T_{elastic} + T_{plastic} \qquad (11)$$

The elastic core extends over $0 < r < r_y$, and over this region $\tilde{\tau}_{xy} = \frac{r}{r_y}\tilde{\tau}_o$. Therefore:

$$T_{elastic} = \int_0^{r_y}(\tilde{\tau}_{xy})(r)(2\pi r)dr = \int_0^{r_y}\left(\frac{r}{r_y}\tilde{\tau}_o\right)(r)(2\pi r)dr = \frac{2\pi\tilde{\tau}_o}{r_y}\int_0^{r_y} r^3 dr$$

$$T_{elastic} = \frac{\pi\tilde{\tau}_o r_y^3}{2} \qquad (12)$$

The plastic region extends over $r_y < r < c$, and over this region

$$\tilde{\tau}_{xy} = \frac{H}{\sqrt{3}}\left(\frac{\tilde{\gamma}_{xy}}{\sqrt{3}}\right)^n = \frac{H}{\sqrt{3}}\left(\frac{r\theta}{\sqrt{3}L}\right)^n.$$

Therefore:

$$T_{plastic} = \int_{r_y}^{c} (\tilde{\tau}_{xy})(r)(2\pi r)dr = \int_{r_y}^{c}\left[\frac{H}{\sqrt{3}}\left(\frac{r\theta}{\sqrt{3}L}\right)^n\right](r)(2\pi r)dr$$

$$T_{plastic} = \frac{2\pi H}{\sqrt{3}}\left(\frac{\theta}{\sqrt{3}L}\right)^n \int_{r_y}^{c} r^{n+2}dr = \frac{2\pi H}{\sqrt{3}}\left(\frac{\theta}{\sqrt{3}L}\right)^n \left[\frac{r^{n+3}}{n+3}\right]_{r_y}^{c}$$

$$T_{plastic} = \frac{2\pi H}{\sqrt{3}(n+3)}\left(\frac{\theta}{\sqrt{3}L}\right)^n \left[c^{n+3} - r_y^{n+3}\right] \qquad (13)$$

Equations (12) and (13) allow us to predict the torque versus (θ/L) response based on the Power-Hardening model. To summarize:

a) For a specified angle of twist per unit length, use Eq 5 to calculate the radial position at which yielding is predicted to occur:

Note:

(a) if $r_y^{pred} > c$, then yielding is not predicted...in other words, the stress-strain response is predicted to be linear across the entire cross-section.

(b) if $r_y^{pred} < c$, then yielding is predicted. In this case an elastic core (with radius r_y^{pred}) and outer plastically-deformed region (with inner and outer radii r_y^{pred} and c, respectively) is predicted.

b) Calculate the predicted elastic torque, using Eq (4) or Eq (12) as appropriate, which corresponds to the angle of twist per unit length.

c). If yielding is predicted, calculate the predicted plastic torque using Eq (13) which corresponds to the angle of twist per unit length

d) Use Eq (11) to calculate the predicted total predicted torque that corresponds to the angle of twist per unit length.

Ramberg-Osgood Model: An alternate approach is to fit the tensile test data for 6061-T6 aluminum to the Ramberg-Osgood model, using the process described in Section 12.2.4 of the Dowling textbook:

$$\tilde{\varepsilon} = \frac{\tilde{\sigma}}{E} + \left(\frac{\tilde{\sigma}}{H}\right)^{1/n}$$

As discussed in Section 13.4.1, the concepts of effective stress and effective strain implies that for the torsion test:

$$\tilde{\gamma}_{xy} = \frac{\tilde{\tau}_{xy}}{G} + \sqrt{3}\left(\frac{\sqrt{3}\tilde{\tau}_{xy}}{H}\right)^{1/n} \tag{14}$$

where: $G = \dfrac{E}{2(1+\nu)}$

The mathematical manipulation to follow is greatly simplified if we define a “shear” strength coefficient:

$$H_\tau = \frac{H}{3^{(n+1)/2}}$$

This allows us to write Eq (14) as:

$$\tilde{\gamma}_{xy} = \frac{\tilde{\tau}_{xy}}{G} + \left(\frac{\tilde{\tau}_{xy}}{H_\tau}\right)^{1/n} \tag{15}$$

We now wish to integrate Eq (2), using the stress-strain relationship defined by Eq (15). First, using Eq (1b), note:

$$r = \frac{c}{\tilde{\gamma}_{max}}\tilde{\gamma}_{xy} = \frac{c}{\tilde{\gamma}_{max}}\left[\frac{\tilde{\tau}_{xy}}{G} + \left(\frac{\tilde{\tau}_{xy}}{H_\tau}\right)^{1/n}\right] \tag{16}$$

Since γ_{max} is a constant (for a specified torque, *T*), and the radius *c* is obviously a constant, we have:

$$dr = \frac{c}{\gamma_{max}} d\gamma_{xy}$$

From Eq (15):

$$\frac{d\tilde{\gamma}_{xy}}{d\tilde{\tau}_{xy}} = \frac{1}{G} + \frac{1}{n\tilde{\tau}_{xy}}\left(\frac{\tilde{\tau}_{xy}}{H_\tau}\right)^{1/n}$$

Or:

$$d\tilde{\gamma}_{xy} = \left[\frac{1}{G} + \frac{1}{n\tilde{\tau}_{xy}}\left(\frac{\tilde{\tau}_{xy}}{H_\tau}\right)^{1/n}\right] d\tilde{\tau}_{xy} \qquad (17)$$

Combining Eq (2), (16) and (17):

$$T = 2\pi \int_0^c \tau_{xy} \left\{ \frac{c}{\tilde{\gamma}_{\max}} \left[\frac{\tilde{\tau}_{xy}}{G} + \left(\frac{\tilde{\tau}_{xy}}{H_\tau}\right)^{1/n} \right] \right\}^2 \left\{ \frac{c}{\gamma_{\max}} \left(\left[\frac{1}{G} + \frac{1}{n\tilde{\tau}_{xy}} \left(\frac{\tilde{\tau}_{xy}}{H_\tau}\right)^{1/n} \right] d\tilde{\tau}_{xy} \right) \right\}$$

Completing this integral it can (eventually!) be shown:

$$T = 2\pi c^3 \tilde{\tau}_{\max} \left[\frac{\frac{1}{4} + \frac{2n+1}{3n+1}\beta_\tau + \frac{n+2}{2n+2}\beta_\tau^2 + \frac{1}{n+3}\beta_\tau^3}{(1+\beta_\tau)^3} \right] \qquad (18)$$

where:

$$\beta_\tau = \frac{\tilde{\gamma}_{pc}}{\tilde{\gamma}_{ec}} \qquad (19a)$$

$$\tilde{\gamma}_{pc} = \left(\frac{\tau_c}{H_\tau}\right)^{1/n} \qquad (19b)$$

$$\tilde{\gamma}_{ec} = \frac{\tau_c}{G} \qquad (19c)$$

$$\tilde{\gamma}_c = \tilde{\gamma}_{ec} + \tilde{\gamma}_{pc} \qquad (19d)$$

Equations (18) and (19) allow us to predict the torque versus (θ/L) response based on the Ramberg-Osgood model. The process is summarized in Table 2:

a) For a specified value of the shear stress induced at the outer radius ($\tilde{\tau}_c$; column 1, Table 2):

- calculate the corresponding plastic true strain using Eq 19b ($\tilde{\gamma}_{pc}$; column 2, Table 2)
- calculate the corresponding elastic true strain using Eq 19c ($\tilde{\gamma}_{ec}$; column 3, Table 2)
- calculate the corresponding total true strain induced at the outer radius using Eq 19d ($\tilde{\gamma}_c$; column 4, Table 2)
- calculate the corresponding (θ/L) using Eq 1 (column 5, Table 2)

b) Calculate the corresponding predicted torque using Eq 18 (T, column 6, Table 2)