## 2. STRESS, STRAIN, AND CONSTITUTIVE RELATIONS

Mechanics of materials is a branch of mechanics that develops relationships between the external loads applied to a deformable body and the intensity of internal forces acting within the body as well as the deformations of the body. External forces can be classified as two types: 1) surface forces produced by a) direct contact between two bodies such as concentrated forces or distributed forces and/or b) body forces which occur when no physical contact exists between two bodies (e.g., magnetic forces, gravitational forces, etc.). Support reactions are external surface forces that develop at the support or points of support between two bodies. Support reactions may include normal forces and couple moments. Equations of equilibrium (i.e., statics) are mathematical expressions of vector relations showing that for a body not to translate or move along a path then $\sum \overline{F} = 0$ . For a body not to rotate, $\sum \overline{M} = 0$. Alternatively, scalar equations in three-dimensional space (i.e., x, y, z) are:

$$\begin{aligned} &\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0 \\ &\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0 \end{aligned} \tag{2.1}$$

Internal forces are non external forces acting in a body to resist external loadings. The distribution of these internal forces acting over a sectioned area of the body (i.e., force divided by area, that is, stress) is a major focus of mechanics of materials. The response of the body to stress in the form of deformation or normalized deformation, that is, strain is also a focus of mechanics of materials. Equations that relate stress and strain are known as constitutive relations and are essential, for example, for describing stress for a measured strain.

### Stress

If an internal sectioned area is subdivided into smaller and smaller areas, $\Delta A$, two important assumptions must be made regarding the material: it is continuous and it is cohesive. Thus, as the subdivided area is reduced to infinitesimal size, the distribution of internal forces acting over the entire sectioned area will consist of an infinite number of forces each acting on an element, $\Delta A$, as a very small force $\Delta F$. The ratio of incremental force to incremental area on which the force acts such that: $\lim\limits_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$ is the stress which can be further defined as the intensity of the internal force on a specific plane (area) passing through a point.

Stress has two components, one acting perpendicular to the plane of the area and the other acting parallel to the area. Mathematically, the former component is expressed as a normal stress which is the intensity of the internal force acting normal to an incremental area such that:

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} \qquad (2.2)$$

where $+\sigma$ = tensile stress = "pulling" stress and $-\sigma$ = compressive stress = "pushing" stress. The latter component is expressed as a shear stress which is the intensity of the internal force acting tangent to an incremental area such that:

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A} \qquad (2.3)$$

The general state of stress is one which includes all the internal stresses acting on an incremental element as shown in Figure 2.1. In particular, the most general state of stress must include normal stresses in each of the three Cartesian axes, and six corresponding shear stresses.

Note for the general state of stress that $+\sigma$ acts normal to a positive face in the positive coordinate direction and a $+\tau$ acts tangent to a positive face in a positive coordinate direction. For example, $\sigma_{xx}$ (or just $\sigma_x$) acts normal to the positive x face in the positive x direction and $\tau_{xy}$ acts tangent to the positive x face in the positive y direction.

Although in the general stress state, there are three normal stress component and six shear stress components, by summing forces and summing moments it can be shown that $\tau_{xy} = \tau_{yx}; \tau_{xz} = \tau_{zx}; \tau_{yz} = \tau_{zy}$.

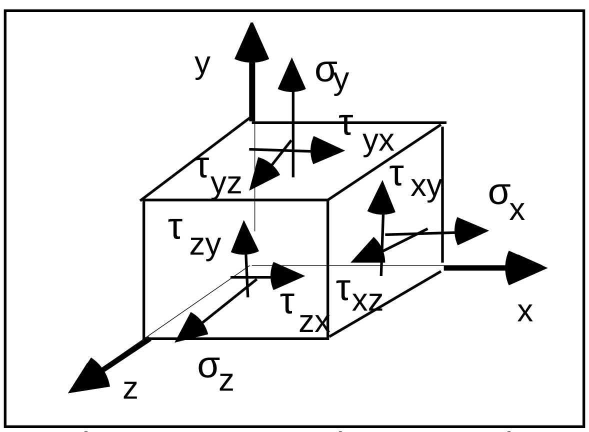



Figure 2.1 General and complete stress state shown on a three-dimensional incremental element.

Therefore the complete state of stress contains six independent stress components (three normal stresses, $\sigma_x;\sigma_y;\sigma_z$ and three shear stresses, $\tau_{xy};\tau_{yz};\tau_{xz}$) which uniquely describe the stress state for each particular orientation. This complete state of stress can be written either in vector form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \tag{2.4}$$

or in matrix form

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \tag{2.5}$$

The units of stress are in general: $\frac{Force}{Area} = \frac{F}{L^2}$. In SI units, stress is $Pa = \frac{N}{m^2}$ or MPa $= 10^6 \frac{N}{m^2} = \frac{N}{mm^2}$ and in US Customary units, stress is $psi = \frac{lb_f}{in^2}$ or ksi $= 10^3 \frac{lb_f}{in^2} = \frac{kip}{in^2}$.

Often it is necessary to find the stresses in a particular direction rather than just calculating them from the geometry of simple parts. For the one-dimensional case shown in Fig. 2.2, the applied force, P, can be written in terms of its normal, $P_N$, and tangential, $P_T$, components which are functions of the angle, θ, such that:

$$\begin{aligned} P_N &= P\cos\theta \\ P_T &= P\sin\theta \end{aligned} \tag{2.6}$$

The area, $A_\theta$, on which $P_N$ and $P_T$ act can also be written in terms of the area, A, normal to the applied load, P, and the angle, θ, such that

$$A_\theta = A/\cos\theta \tag{2.7}$$

The normal and shear stress relation acting on any area oriented at angle, θ, relative to the original applied force, P are:

$$\begin{aligned} \sigma_\theta &= \frac{P_N}{A_\theta} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta = \sigma\cos^2\theta \\ \tau_\theta &= \frac{P_T}{A_\theta} = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\cos\theta\sin\theta = \sigma\cos\theta\sin\theta \end{aligned} \tag{2.8}$$

where $\sigma$ is the applied unidirectional normal stress.

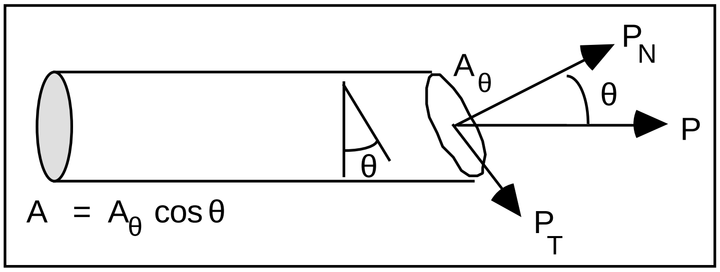



Figure 2.2  Unidirectional stress with force and area as functions of angle, θ

For the two dimensional case (i.e., plane stress case such as the stress state at a surface where no force is supported on the surface), stresses exist only in the plane of the surface (e.g., $\sigma_x ; \sigma_y ; \tau_{xy}$). The plane stress state at a point is uniquely represented by three components acting on a element that has a specific orientation (e.g., x, y) at the point. The stress transformation relation for any other orientation (e.g., x', y') is found by applying equilibrium equations ($\sum F = 0$ and $\sum M = 0$) keeping in mind that $F_n = \sigma A$ and $F_t = \tau A$. The rotated axes and functions for incremental area are shown in Fig. 2.3. The forces in the x and y directions due to $F_n = \sigma A$ and $F_t = \tau A$ and acting on the areas normal to the x and y directions are shown in Fig. 2.4

By applying simple statics such that in the x'-direction, $\sum F_{x'} = 0$ and

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \cos\theta \sin\theta$$

or (2.9)

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

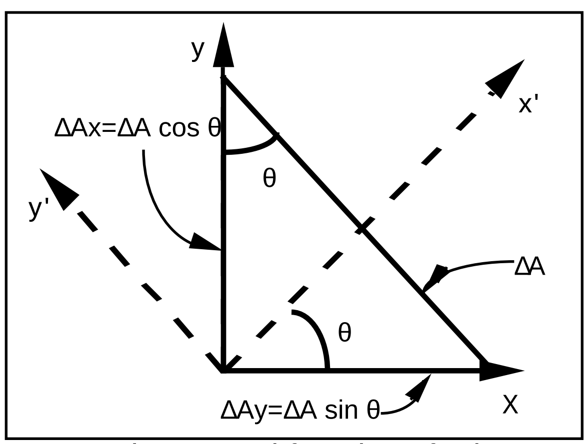



Figure 2.3 Rotated axes and functions for incremental area.

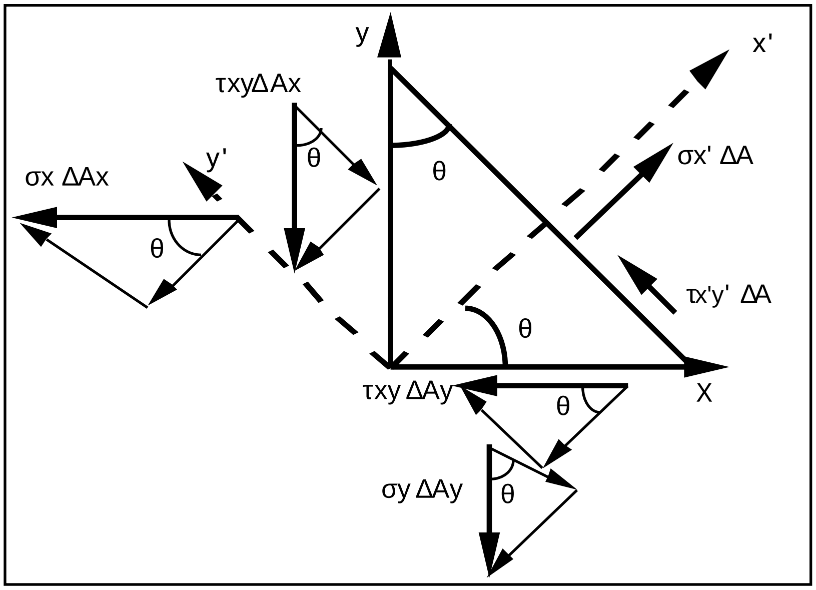


Figure 2.4 Rotated coordinate axes and components of stresses/forces.

Similarly, for the x'y'-direction, $\sum F_{y'} = 0$ and

$$\tau_{x'y'} = (\sigma_x - \sigma_y)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta + \sin^2\theta)$$

or (2.10)

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Finally, for the y' direction, $\sum F_{y'} = 0$ and

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta$$

or (2.11)

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

If the stress in a body is a function of the angle of rotation relative to a given direction, it is natural to look for the angle of rotation in which the normal stress is either maximum or nonexistent. A principal normal stress is a maximum or minimum normal stress acting in principal directions on principal planes on which no shear stresses act. Because there are three orthogonal directions in a three-dimensional stress state there are always three principal normal stresses which are ordered such that $\sigma_1 > \sigma_2 > \sigma_3$.

Mathematically, the principal normal stresses are found by determining the angular direction, θ, in which the function, $\sigma = f(\theta)$, is a maximum or minimum by differentiating $\sigma = f(\theta)$ with respect to θ and setting the resulting equation equal to zero such that $\frac{d\sigma}{d\theta} = 0$ before solving for the θ at which the principal stresses occur. Applying this idea to Eq. 2.9, gives

$$\frac{d\sigma}{d\theta} = 0 = \left(\sigma_x - \sigma_y\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta \qquad \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{2.12}$$

There are two solutions for the principal stress angle (i.e., for maximum and minimum) so that.

$$\begin{aligned} \theta_{N1} &= \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) \\ \theta_{N2} &= \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} + \pi\right) = \theta_{N1} + \frac{\pi}{2} \end{aligned} \tag{2.12}$$

Using trigonometry on the geometry shown in Fig. 2.5 results in

$$\begin{aligned} \tan 2\theta &= \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2} \\ \sin 2\theta &= \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \\ \cos 2\theta &= \frac{\left(\sigma_x - \sigma_y\right)/2}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \end{aligned} \tag{2.14}$$

Substituting the trigonometric relations of Eq. 2.14 back into Eq. 2.9 gives for the plane stress case:

$$\begin{aligned} \sigma_\theta &= \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \tan 2\theta_p &= \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2} \end{aligned} \tag{2.15}$$

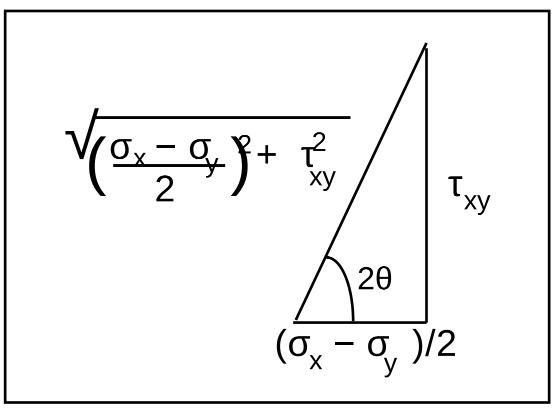



Figure 2.5  Geometric representation of the principal direction relations

Note that for the plane stress case in the x-y plane, $\sigma_z = 0$. Thus the 1 and 2 subscripts in Eq. 2.15 are only for the x-y plane and are not necessarily $\sigma_1$ and $\sigma_2$ for the three-dimensional general state of stress. Therefore, ordering of $\sigma_1$ and $\sigma_2$ of Eq. 2.15 is only preliminary, until they are compared to $\sigma_z$ and ordered according to convention $\sigma_1 > \sigma_2 > \sigma_3$. For example, for a particular plane stress state $\sigma_1$ and $\sigma_2$ found from Eq. 2.15 are 100 and 20 MPa, then the principal stresses are $\sigma_1 = 100$, $\sigma_2 = 20$, $\sigma_3 = 0$ MPa. However, if $\sigma_1$ and $\sigma_2$ found from Eq. 2.15 are 125 and -5 MPa, then the principal stresses are $\sigma_1 = 125$, $\sigma_2 = 0$, $\sigma_3 = -5$ MPa. Finally, if $\sigma_1$ and $\sigma_2$ found from Eq. 2.14 are -25 and -85 MPa, then the principal stresses are $\sigma_1 = 0$, $\sigma_2 = -25$, $\sigma_3 = -85$ MPa.

Performing a similar substitution of the trigonometric relations of Eq. 2.14 back into Eq. 2.10 gives for the plane stress case:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \quad \text{and} \quad \tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} \tag{2.16}$$

Note that the $\tau_{max}$ of Eq. 2.15 is only for the x-y plane. The maximum shear stress for the three-dimensional stress state can be found after the principal stresses are ordered $\sigma_1 > \sigma_2 > \sigma_3$ such that:

$$\tau_{1,3} = \frac{\sigma_1 - \sigma_3}{2} \tag{2.17}$$

Some general observations can be made about principal stresses.

a) In a principal direction, when $\tau = 0$, then $\sigma$ 's are maximum or minimum
b) $\sigma_{max}$ and $\sigma_{min}$ ($\tau_{max}$ and $\tau_{min}$) occur in directions 90° apart.
c) $\tau_{max}$ occurs in a direction midway between the directions of $\sigma_{max}$ and $\sigma_{min}$

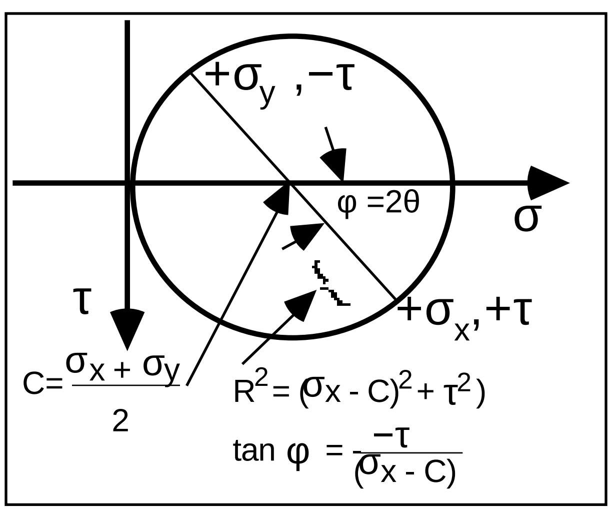



Figure 2.6 Mohr's circle representation of plane stress state

An interesting graphical relation occurs if the second equation in each of Eqs. 2.9 and 2.10 are squared and added together:

$$\left(\sigma_{x'} = \sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right)^2$$

$$+$$

$$\left(\tau_{x'y'} = \tau_\theta = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right)^2$$

$$=$$

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \qquad (2.18)$$

$$\equiv$$

$$(x-h)^2 \qquad + y^2 = \qquad r^2$$

The result shown in Eq. 2.18 is the equation for a circle (i.e., Mohr's circle) with radius, $r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ and displaced $h = \frac{\sigma_x + \sigma_y}{2}$ on the $x = \sigma_\theta$ axis as illustrated in Fig. 2.6. Examples of Mohr's circles are shown in Fig. 2.7. A procedure for developing Mohr's circle for plane stress is shown in the following section.

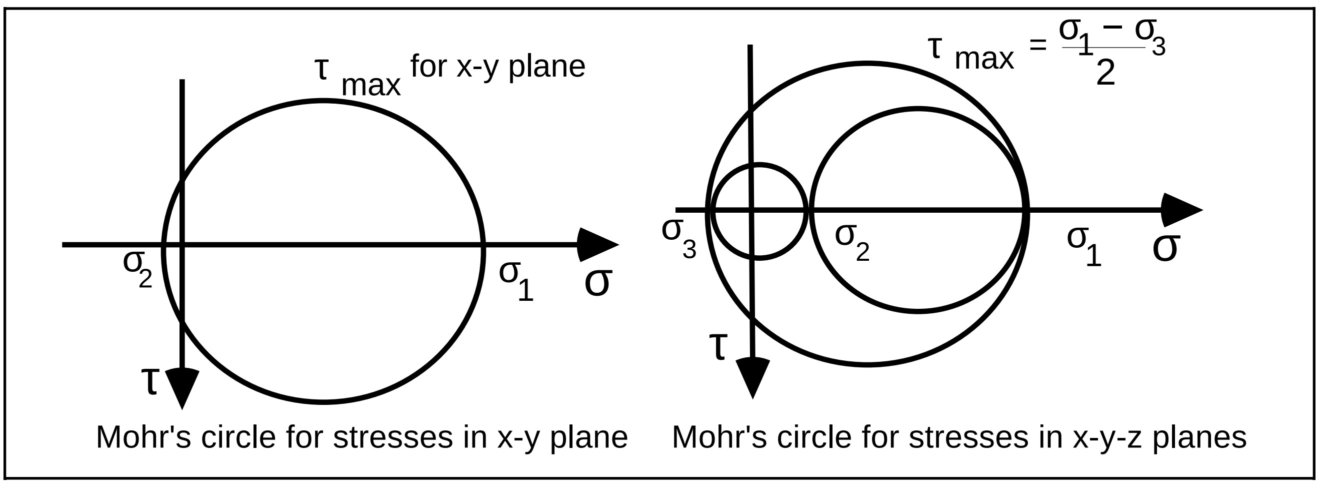



Figure 2.7 Examples of Mohr's circle for plane and three dimensional stress states.

**Graphical Description of State of Stress**

2-D Mohr's Circle

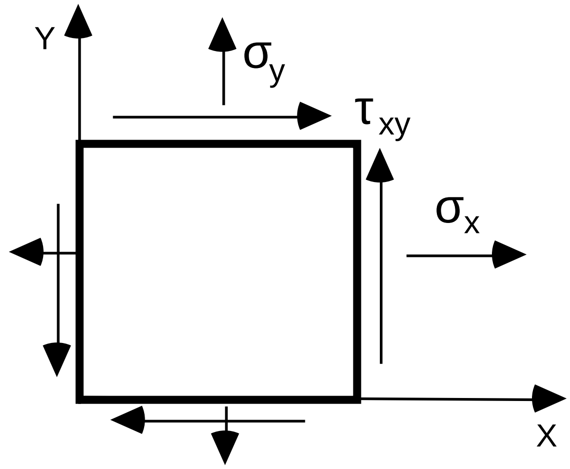



**Fig. M1-** Positive stresses acting on a physical element.

In this example all stresses acting in axial directions are positive as shown in Fig. M1.

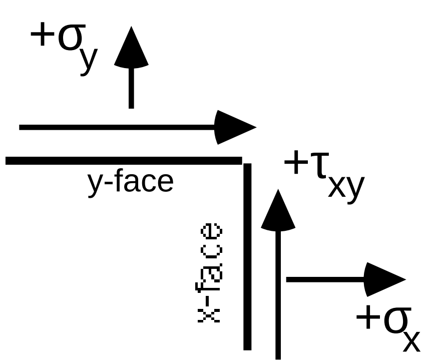



**Fig. M2** - Directionality of shear acting on x and y faces.

As shown in Figs. M2 and M3, plotting actual sign of the shear stress with x normal stress requires plotting of the opposite sign of the shear stress with the y normal stress on the Mohr's circle.

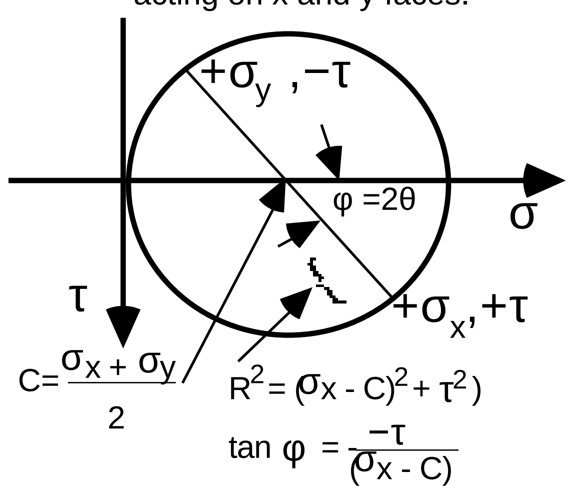



**Fig. M3** - Plotting stress values on Mohr's circle.

In this example $\sigma_x > \sigma_y$ and $\tau_{xy}$ is positive. By the convention of Figs. M2 and M3, $\varphi = 2\theta$ on the Mohr's circle is negative from the $+\sigma$ axis. (Mathematical convention is that positive angle is counterclockwise).

Note that by the simple geometry of Fig. M3, $\varphi = 2\theta$ appears to be negative while by the formula, $\tan 2\theta = 2\tau_{xy}/(\sigma_x - \sigma_y)$, the physical angle, $\theta$, is actually positive.

In-plane principal stresses are: $\sigma_1 = C+R$

$\sigma_2 = C - R$

Maximum in-plane shear stress is:

$\tau_{max} = R = (\sigma_1 - \sigma_2)/2$

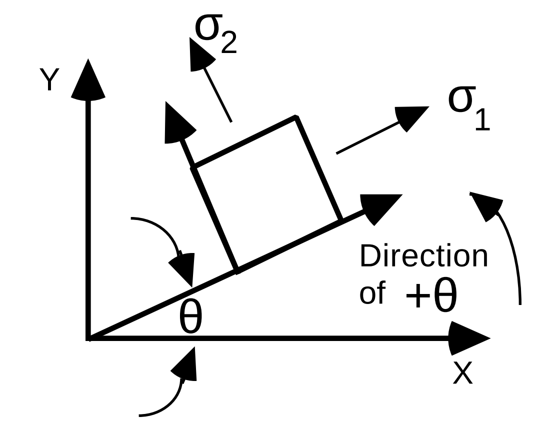



**Fig. M4 -** Orientation of physical element with only principal stresses acting on it.

The direction of physical angle, $\theta$, is from the x-y axes to the principal axes.

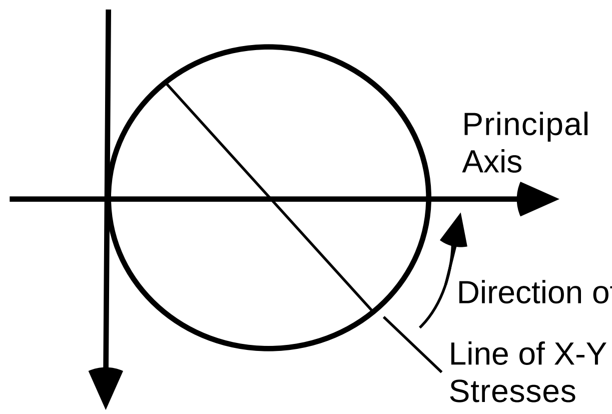



**Fig. M5** - Direction of q from the line of x-y stresses to the principal stress axis.

Note that the sense (direction) of the physical angle, $\theta$, is the same as on the Mohr's circle from the line of the x-y stresses to the axes of the principal stresses.

Same relations apply for Mohr's circle for strain except interchange variables as

$\sigma \Leftrightarrow \varepsilon$ and $\tau \Leftrightarrow \frac{\gamma}{2}$

Recall that all stress states are three-dimensional. Therefore, a more general method is required to solve for the principal stresses. One such method is to solve for the "eigenvalues" of the stress matrix where, $\sigma$ is the principal stress:

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} \quad (2.19)$$

Finding the determinant for this matrix and grouping terms gives:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \quad (2.20)$$

where the stress invariants, $(I_1, I_2, I_3)$, do not vary with stress direction:

$$\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2 \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 \end{aligned} \quad (2.21)$$

Note that if principal stresses are used in Eq. 2.21 for the $\sigma$ terms then all terms containing $\tau$ will be zero since by definition, principal stresses act in principal directions on principal planes on which $\tau = 0$.

Eq. 2.20 can then be solved for the three roots which when ordered $\sigma_1 > \sigma_2 > \sigma_3$ are the principal stresses. Eq. 2.20 can be plotted as $f(\sigma)$ vs $\sigma$ shown in Fig. 2.9 where the principal stresses are the values of $\sigma$ which occur when $f(\sigma) = 0$.

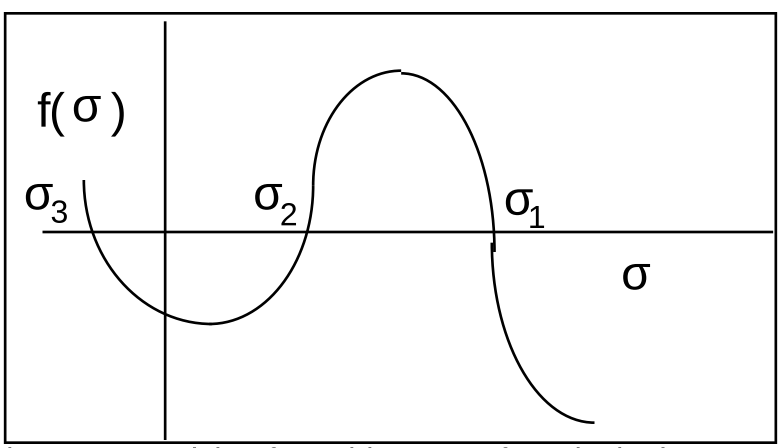


Figure 2.9 Solving for cubic routes for principal stresses

## Strain

Whenever a force is applied to a body, it will tend to change the body's shape and size. There changes are referred to as deformation. Size changes are known as dilatation (volumetric changes) and are due to normal stresses. Shape changes are known as distortion and are due to shear stresses.

In order to describe the deformation in length of line segments and the changes in angles between them, the concept of strain is used. Therefore, strain is defined as normalized deformations within a body exclusive of rigid body displacements

There are two type of strain, one producing size changes by elongation or contraction and the other producing shape changes by angular distortion.

Normal strain is the elongation or contraction of a line segment per unit length resulting in a volume change such that

$$\varepsilon = \lim_{B \to \text{A along n}} \frac{A'B' - AB}{AB} \equiv \frac{L_f - L_o}{L_o} \qquad (2.22)$$

where $+\varepsilon$ = tensile strain = elongation and $-\varepsilon$ = compressive strain = contraction

Shear strain is the angle change between two line segments resulting in a shape change such that:

$$\gamma = (\theta = \frac{\pi}{2}) - \theta' \approx \frac{\Delta}{h} \text{(for small angles )} \qquad (2.23)$$

where $+\gamma$ occurs if $\frac{\pi}{2} > \theta'$ and $-\gamma$ occurs if $\frac{\pi}{2} < \theta'$.

The general state is one which includes all the internal strains acting on an incremental element as shown in Fig. 2.10. The complete state of strain has six independent strain components (three normal strains, $\varepsilon_x; \varepsilon_y; \varepsilon_z$ and three engineering shear strains, $\gamma_{xy}; \gamma_{yz}; \gamma_{xz}$) which uniquely describe the strain state for each particular orientation. This complete state of strain can be written in vector form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad (2.24)$$

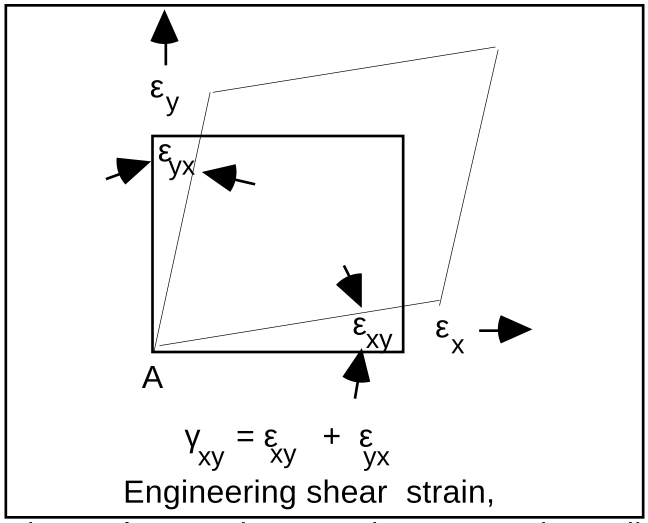



Figure 2.10 General and complete strain state shown on a three-dimensional incremental element.

Alternatively, the complete state of strain can be written in matrix form :

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{bmatrix} \qquad (2.25)$$

The units of strain In general: $\frac{Length}{Length} = \frac{L}{L}$, In SI units, strain is $\frac{\text{m}}{\text{m}}$ for $\varepsilon$ and $\frac{\text{m}}{\text{m}}$ or radian for $\gamma$ and in US Customary units, strain is $\frac{\text{in}}{\text{in}}$ for $\varepsilon$ and $\frac{\text{in}}{\text{in}}$ or radian for $\gamma$

Just as in the stress case it is often necessary to find the stresses in a particular direction rather than just calculating them from the geometry of simple parts. For the two-dimensional, plane strain condition (e.g., strain at a surface where no deformation occurs normal to the surface), strains exist only in the plane of the surface (e.g., $\varepsilon_x; \varepsilon_y; \gamma_{xy}$). The plane strain state at a point is uniquely represented by three components acting on a element that has a specific orientation (e.g., x, y) at the point. The strain transformation relation for any other orientation (e.g., x', y') is found by summing displacements in the appropriate directions keeping in mind that $\delta = \varepsilon L_o$ and $\Delta = \gamma h$ as shown in Fig. 2.11

Simply adding components of displacements in the x' direction, $\sum$ displacements in x 'direction for Q to Q * (see Fig. 2.12) gives

$$\varepsilon_{x'} = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\cos\theta\sin\theta$$

*or* (2.25)

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

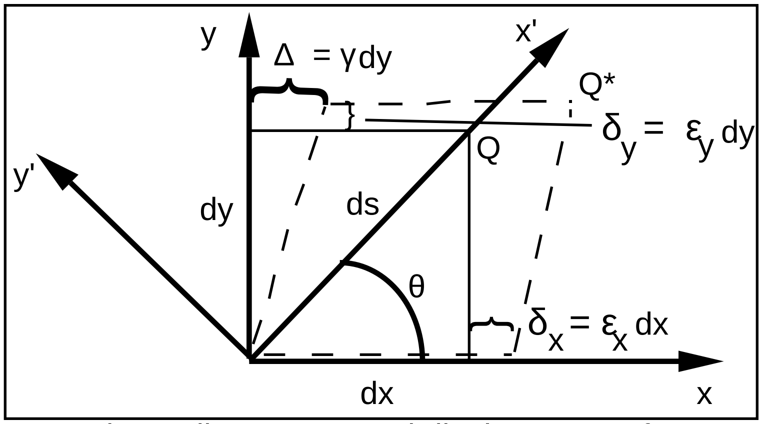


Figure 2.11 Rotated coordinate axes and displacements for x and y directions

Similarly, adding components of displacements in due to rotations of dx' and dy' $\sum$ rotation of dx ' and dy' (see Fig. 2.12) gives

$$\frac{\gamma_{x'y'}}{2} = (\varepsilon_x - \varepsilon_y)\cos\theta\sin\theta + \frac{\gamma_{xy}}{2}(\cos^2\theta + \sin^2\theta)$$

or (2.26)

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Finally, adding components of displacements in the y' direction, $\sum$ displacements in y 'direction for Q to Q * (see Fig. 2.12) gives

$$\varepsilon_{y'} = \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy}\cos\theta\sin\theta$$

*or* (2.27)

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

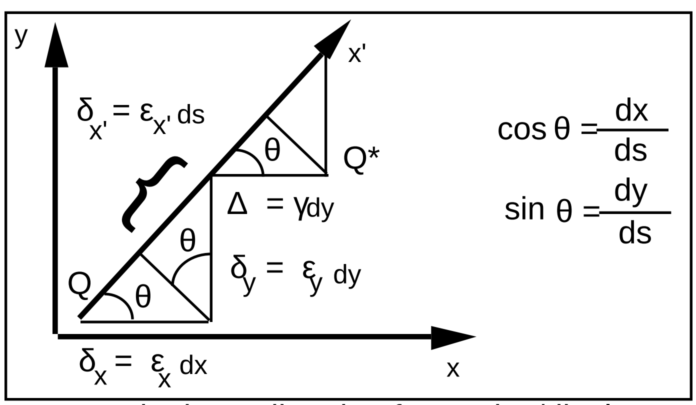


Figure 2.12 Displacements in the x' direction for strains/displacements in the x and y directions

Just as for the stress in a body which is a function of the angle of rotation relative to a given direction, it is natural to look for the angle of rotation in which the normal strain is either maximum or minimum. A principal normal strain is a maximum or minimum normal strain acting in principal directions on principal planes on which no shear strains act. Because there are three orthogonal directions in a three-dimensional strain state there are always three principal normal strains which are ordered such that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$.

Also as in the stress case, the principal strains for the plane strain case

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad (2.29)$$

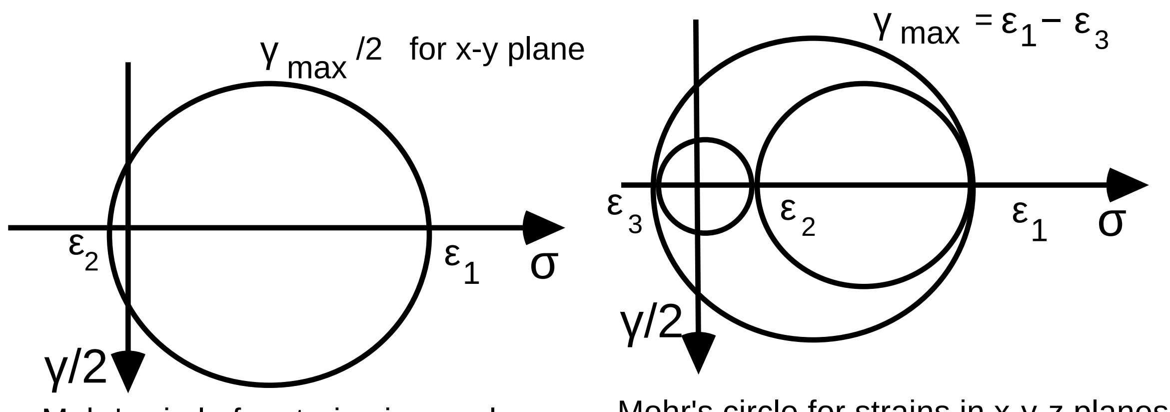
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

Note that for the plane strain case in the x-y plane, $\varepsilon_z = 0$. Thus, the 1 and 2 subscripts in Eq. 2.29 are only for the x-y plane and are not necessarily $\varepsilon_1$ and $\varepsilon_2$ for the three-dimensional general state of strain. Therefore, ordering of $\varepsilon_1$ and $\varepsilon_2$ of Eq. 2.29 is only preliminary, until they are compared to $\varepsilon_z$ and ordered according to convention $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$.

For the shear strain case

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)\tau_{xy}^2}\ , \qquad (2.30)$$

$$\varepsilon_{ave} = \frac{\varepsilon_x + \varepsilon_y}{2} \text{ and } \tan 2\theta_s = \frac{-\left(\varepsilon_x - \varepsilon_y\right)}{\gamma_{xy}}$$



Mohr's circle for strains in x-y plane
Mohr's circle for strains in x-y-z planes

Figure 2.13 Examples of Mohr's circle for strain.

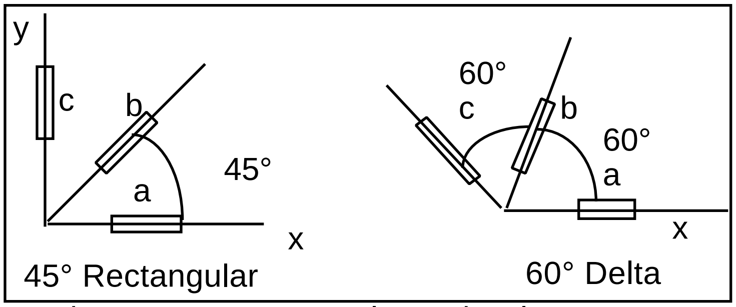


Figure. 2.14 Rectangular and Delta rosettes.

Note that the $\gamma_{max}$ of Eq. 2.30 is only the x-y plane. The maximum shear strain for the three-dimensional strain state can be found after the principal strains are ordered $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$ such that:

$$\gamma_{1,3} = \varepsilon_1 - \varepsilon_3 \tag{2.31}$$

As with stress, the complete strain state can be represented graphically as a Mohr's circle. Examples of Mohr's circles for strain are shown in Fig. 2.13. Note that the same procedure for developing Mohr's circle for the plain strain case can be used as with the plane stress case with by making the following substitutions: $\sigma \Leftrightarrow \varepsilon$ and $\tau \Leftrightarrow \frac{\gamma}{2}$.

An important application of the strain transformation relation of Eq. 2.26 is to experimentally determine the complete strain state since stress is an abstract engineering quantity and strain is measurable/observable engineering quantity. Equation 2.26 requires that three strain gages be applied at arbitrary orientations at the same point on the body to determine the principal strains and orientations. However, to simplify the data reduction, the three strain gages are applied at fixed angles usually in the form of 45° rectangular rosette or 60° Delta rosette as shown in Fig. 2.14. The resulting equations to determine the local (for the strain gage rosettes shown in Fig. 2.14) x-y strains in preparation for determining the principal strains:

| 45° Rectangular Rosette | 60° Delta Rosette | |
|---|---|---|
| $\varepsilon_x = \varepsilon_a$ | $\varepsilon_x = \varepsilon_a$ | |
| $\varepsilon_y = \varepsilon_c$ | $\varepsilon_y = \frac{1}{3}(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a)$ | (2.32) |
| $\gamma_{xy} = 2\varepsilon_b - (\varepsilon_a + \varepsilon_c)$ | $\gamma_{xy} = \frac{2}{\sqrt{3}}(\varepsilon_b - \varepsilon_c)$ | |

However, a limitation of measuring strains experimentally is that stresses are often required to determine the engineering performance of the component. Thus, equations which relate stress and strain are required.

Constitutive Relations

If the deformation and strain are the response of the body to an applied force or stress, then there must be some type of relations which allow the strain to be predicted from stress or vice versa. The uniaxial stress-strain case is a useful example to begin to understand this relation.

During uniaxial loading (see Fig. 2.15) by load, P, of a rod with uniform cross sectional area, A, and length, L, the applied stress, $\sigma$ ,is calculated simply as

$$\sigma = \frac{P}{A} \tag{2.33}$$

The resulting normal strain, $\varepsilon_L$, in the longitudinal direction can be calculated from the deformation response, $\Delta L$ , along the L direction:

$$\varepsilon_L = \frac{\Delta L}{L} \tag{2.34}$$

Another normal strain, $\varepsilon_T$, in the transverse direction can be calculated from the deformation response, $\Delta D$ , along the D direction:

$$\varepsilon_T = \frac{\Delta D}{D} \tag{2.35}$$

A plot of $\sigma$ vs. $\varepsilon_L$ (see Fig. 2.16) shows a constant of proportionality between stress and strain in the elastic region such that

$$y = mx + b \qquad \sigma = E\varepsilon_L \tag{2.36}$$

where $E = \frac{\Delta\sigma}{\Delta\varepsilon_L}$ is known as the elastic modulus or Young's modulus and Eq. 2.36 is known as unidirectional Hooke's law.

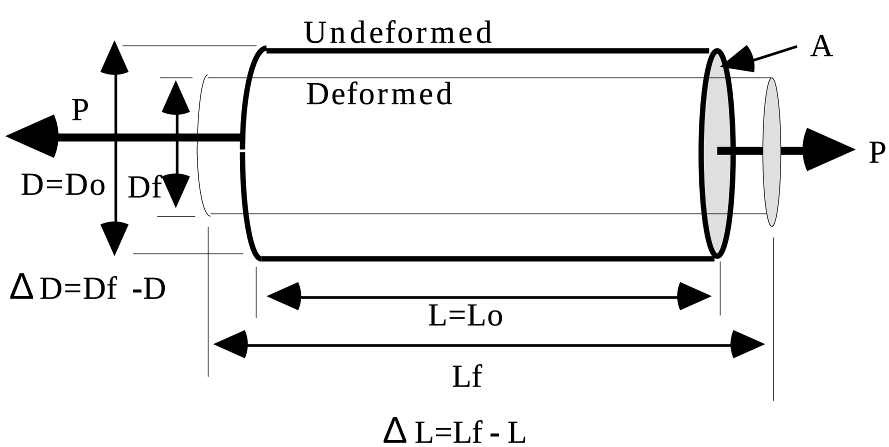



Figure 2.15 Uniaxially-loaded rod undergoing longitudinal and transverse deformation

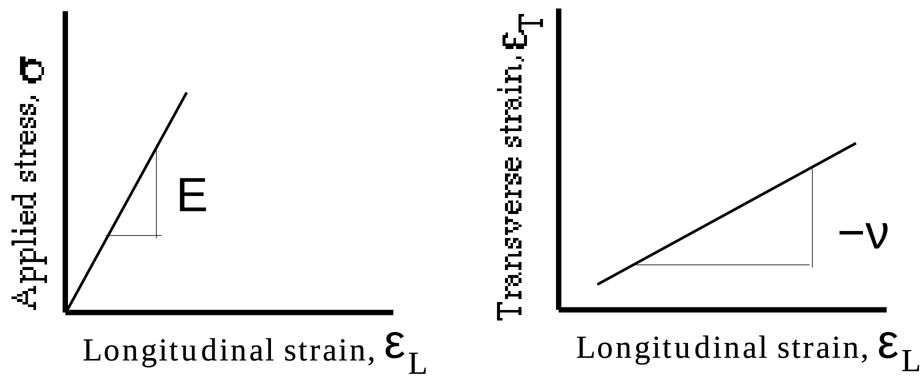



Figure 2.16 Plots of applied stress vs. longitudinal strain and transverse strain versus longitudinal strain

A plot of $\varepsilon_T$ vs. $\varepsilon_L$ (see Fig. 2.16) shows a constant of proportionality between transverse and longitudinal strains in the elastic region such that

$$y = mx + b \qquad \varepsilon_T = -\nu\varepsilon_L \tag{2.37}$$

where $\nu = -\dfrac{\varepsilon_T}{\varepsilon_L}$ is known as Poisson's ratio.

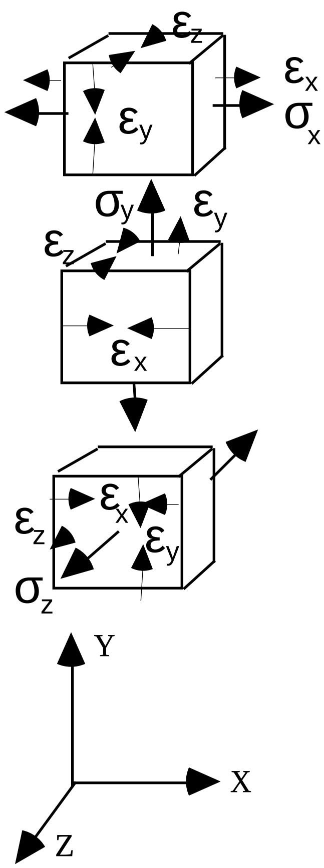
For the results for three different uniaxial stresses applied separately (see Fig. 2.17) in each of the orthogonal direction, x, y and z can give the general relations between normal stress and normal strain know as generalized Hooke's law if the strain components for each of the stress conditions are superposed:

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) \\ \varepsilon_y &= \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right) \\ \varepsilon_z &= \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) \end{aligned} \tag{2.38}$$

Since shear stresses are decoupled (i.e., unaffected) by stresses in other directions the relations for shear stress-shear strain are:

$$\begin{aligned} \gamma_{xy} &= \frac{1}{G}\tau_{xy} \\ \gamma_{xz} &= \frac{1}{G}\tau_{xz} \\ \gamma_{yz} &= \frac{1}{G}\tau_{yz} \end{aligned} \tag{2.39}$$

where the shear modulus is $G = \dfrac{\Delta\tau}{\Delta\gamma} = \dfrac{E}{2(1+\nu)}$.




| Stress | $\varepsilon_x$ | $\varepsilon_y$ | $\varepsilon_z$ |
|---|---|---|---|
| $\sigma_x$ | $E\sigma_x$ | $-\nu\varepsilon_x = -\nu E\sigma_x$ | $-\nu\varepsilon_x = -\nu E\sigma_x$ |
| $\sigma_y$ | $-\nu\varepsilon_y = -\nu E\sigma_y$ | $E\sigma_y$ | $-\nu\varepsilon_y = -\nu E\sigma_y$ |
| $\sigma_z$ | $-\nu\varepsilon_z = -\nu E\sigma_z$ | $-\nu\varepsilon_z = -\nu E\sigma_z$ | $E\sigma_z$ |

:

Figure 2.17 Development of generalized Hooke's law.

Generalized Hooke's law is usually written in matrix form such that:

$$\{\varepsilon\} = [S]\{\sigma\} \tag{2.40}$$

where $\{\varepsilon\}$ is the strain vector, $\{\sigma\}$ is the stress vector and $[S]$ is the compliance matrix. Expanded, Eq. 2.40 is:

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} =
\begin{bmatrix}
\frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\
\frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\
\frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \tag{2.41}
$$

Although Eq. 2.41 is a very convenient form, it is not that useful since stress is rarely known with strain being the unknown. Instead, strain is usually measured experimentally and the associated stress needs to be determined. Therefore, the inverse of the compliance matrix needs to be found such that

$$\{\sigma\} = [C]\{\varepsilon\} \tag{2.42}$$

where $[C]$ is the stiffness matrix such that $[C] = [S]^{-1}$. Expanding Eq. 2.42 gives

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} =
\begin{bmatrix}
\frac{E}{(1+\nu)}\left(1+\frac{\nu}{(1-2\nu)}\right) & \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{E}{(1+\nu)}\left(1+\frac{\nu}{(1-2\nu)}\right) & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{E}{(1+\nu)}\left(1+\frac{\nu}{(1-2\nu)}\right) & 0 & 0 & 0 \\
0 & 0 & 0 & G & 0 & 0 \\
0 & 0 & 0 & 0 & G & 0 \\
0 & 0 & 0 & 0 & 0 & G
\end{bmatrix}
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{2.43}
$$

Generalized Hooke's can be simplified somewhat for the special case of plane stress in the x-y plane since $\sigma_z = 0$. Being orthogonal to the x-y plane, Since $\sigma_z$ is a also a principal stress by definition, all the shear stresses associated with the z-direction are also zero. Thus, the stress-strain relations for plane stress in the x-y plane become

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} =
\begin{bmatrix}
\frac{E}{(1-\nu^2)} & \frac{\nu E}{(1-\nu^2)} & 0 \\
\frac{\nu E}{(1-\nu^2)} & \frac{E}{(1-\nu^2)} & 0 \\
0 & 0 & G
\end{bmatrix}
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.44}
$$

For the special case of plane stress, although $\sigma_z = 0$, the strain in the z-direction is not zero but instead can be determined such that

$$\text{Plane stress}: \ \sigma_z = 0, \ \varepsilon_z \neq 0 = \frac{-\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) \tag{2.45}$$

Similarly for the special case of plane strain, although, $\varepsilon_z$=0, the stress in the z-direction is not zero but instead can be determined such that

$$\text{Plane strain} \;:\; \varepsilon_z = 0,\ \sigma_z \neq 0 = \nu(\sigma_x + \sigma_y) \tag{2.46}$$

For the special case of hydrostatic pressure, no distortion takes place, only size changes. In this case the hydrostatic stress is

$$\boxed{\sigma_H = \frac{\left(\sigma_x + \sigma_y + \sigma_x\right)}{3}} \tag{2.47}$$

and the dilatation (i.e., volumetric change) is

$$\varepsilon_V = \frac{\Delta V}{V} = \left(\varepsilon_x + \varepsilon_y + \varepsilon_x\right) \tag{2.48}$$

The ratio between the hydrostatic stress and the dilation is a special combination of elastic constants called the bulk modulus.

$$k = \frac{\sigma_H}{\varepsilon_V} = \frac{\left(\sigma_x + \sigma_y + \sigma_x\right)}{3\left(\varepsilon_x + \varepsilon_y + \varepsilon_x\right)} = \frac{E}{3(1-2\nu)} \tag{2.49}$$

It is useful to know that elastic constants are related to the atomic structure of the material and thus are not affected by processing or component fabrication. For example, E is related to the repulsion/attraction between two atoms. The force-displacement curve for this interaction is shown in Fig. 2.18. Since this is the uniaxial case, recall that $\sigma = \frac{P}{A}$ and $\varepsilon = \frac{\Delta L}{L} = \frac{x}{x_e}$. Therefore, since E is defined as $E = \frac{\Delta\sigma}{\Delta\varepsilon} = \frac{d\sigma}{d\varepsilon}$ with $d\sigma = \frac{dP}{A}$ and $\varepsilon = \frac{dx}{x_e}$ then $E = \frac{d\sigma}{d\varepsilon} = \frac{x_e}{A}\frac{dP}{dx}$. Note from Fig. 2.18 that $\frac{dP}{dx}$ is the slope of the force-displacement curve, thus making the elastic modulus E fixed by atomic interaction. From a materials standpoint, covalent and ionic bonds such as those in ceramics are stiff leading to high elastic moduli in those materials. Metallic bonds are intermediate in stiffness leading to intermediate elastic moduli in metals. Secondary bonds such as those found in polymers are least stiff leading to low elastic moduli in those materials.

For Poisson's ratio, it is useful to consider the sphere model of atomic structure as shown in Fig. 2.19. Before deformation the triangle between centers has a transverse side length of $\sqrt{3}R$, a longitudinal side length of R and a hypotenuse of 2R. After longitudinal deformation, the transverse side length is $\sqrt{3}R$ -dx, the longitudinal side

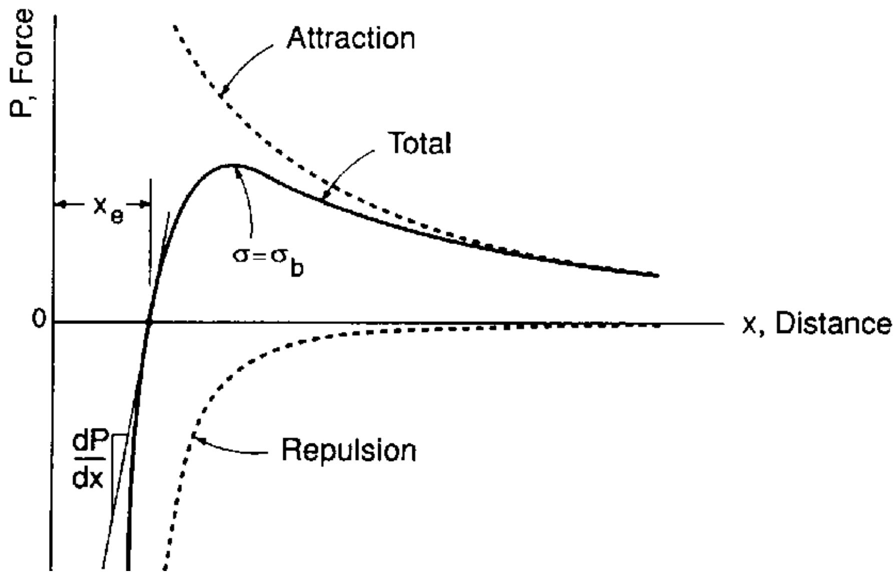



Figure 2.18 Repulsion-attraction force-displacement curve between two atoms.

length is R+dy and the hypotenuse is still 2R. Applying Pythagorean's theorem after deformation gives

$$(2R)^2 = \left(\sqrt{3}R - dx\right)^2 + (R + dy)^2 \tag{2.50}$$

Expanding Eq. 2.50 and eliminating higher order terms leaves

$$2Rdy = 2\sqrt{3}Rdx \qquad \frac{dy}{dx} = \sqrt{3}\,. \tag{2.51}$$

Recalling that the longitudinal and transverse strains can be written in terms of the deformations

$$\begin{aligned} \varepsilon_y &= \frac{(R + dy) - R}{R} = \frac{dy}{R} \qquad dy = R\varepsilon_y \\ \varepsilon_x &= \frac{\left(\sqrt{3}R - dx\right) - \sqrt{3}R}{\sqrt{3}R} = \frac{-dx}{\sqrt{3}R} \qquad dx = -\sqrt{3}R\varepsilon_x \end{aligned} \tag{2.52}$$

Combining the two terms for dy and dx in Eq. 2.52 and equating them to Eq. 2.51 gives

$$\frac{dy}{dx} = \frac{R\varepsilon_y}{-\sqrt{3}R\varepsilon_x} = \sqrt{3} \qquad -\frac{\varepsilon_y}{\varepsilon_x} \equiv -\frac{1}{\nu} = 3 \tag{2.53}$$

which shows that from a simple sphere model and expected deformations, Poisson's ratio is fundamentally linked to atomic interactions giving a value of 1/3 which is the range of many dense materials.

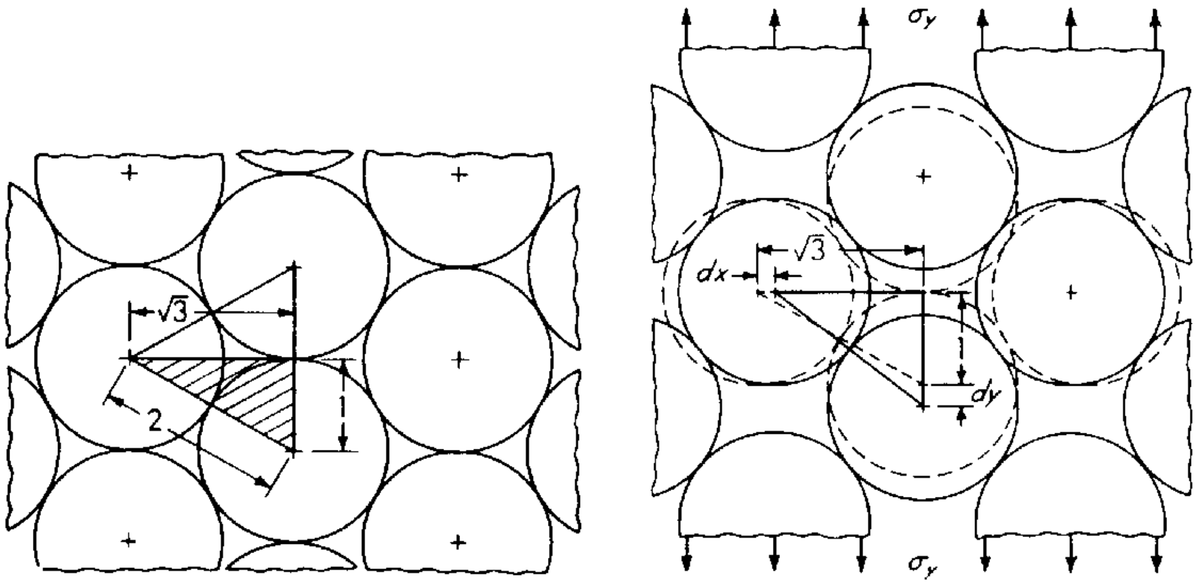


Figure 2.19 Undeformed and deformed sphere model for atomic structure and determination of Poisson's ratio