NOTE: These notes represent selected highlights of ME354 and are not intended to replace conscientious study, attendance of lecture, reading of the textbook, completion of homework assignments, and performance of laboratory work. These notes are corrected, modified, and upgraded periodically with date and latest version number appearing in the header.

Mechanics of Materials - a branch of mechanics that develops relationships between the external loads applied to a deformable body and the intensity of internal forces acting within the body as well as the deformations of the body
External Forces - classified as two types: 1) surface forces produced by a) direct contact between two bodies such as concentrated forces or distributed forces and/or b) body forces which occur when no physical contact exists between two bodies (e.g., magnetic forces, gravitational forces, etc.).
Internal Forces - non external forces acting in a body to resist external loadings
Support Reactions - surface forces that develop at the support or points of support between two bodies. Support reactions may include normal forces and couple moments.
Equations of Equilibrium - mathematical expression of vector relations showing that for a body not to translate or move along a path then $\sum \overline{F} = 0$. $\sum \overline{M} = 0$ for a body not to rotate. Alternatively, scalar equations in 3-D space (i.e., x, y, z) are:

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$

$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

Some nomenclature used in these notes

Roman characters

a - crack length; **A**- area; $\mathbf{A_f}$ - final area; $\mathbf{A_o}$ - original area; **c** - distance from neutral axis to farthest point from neutral axis or Griffith flaw size; **C**- center of Mohr's circle; **E**- elastic modulus (a.k.a., Young's modulus); **F** - force or stress intensity factor coefficient; **FS** - factor of safety; **G** - shear modulus (a.k.a. modulus of rigidity); **I** - moment of inertia; **J** - polar moment of inertia; **K** - strength coefficient for strain hardening; **K** - stress intensity factor, **k** - bulk modulus; **L** - length; $\mathbf{L_f}$ - final length; $\mathbf{L_o}$ - original length; **M** or **M(x)** - bending moment; **m** - metre (SI unit of length) or Marin factor for fatigue; **N** - Newton (SI unit of force) or fatigue cycles; $\mathbf{N_f}$ - cycles to fatigue failure; **n** - strain hardening exponent or stress exponent; **P** - applied load; $\mathbf{P_{cr}}$ - critical buckling load; $\mathbf{P_{SD}}$ - Sherby-Dorn parameter; $\mathbf{P_{LM}}$ - Larson-Miller parameter; **p -** pressure; **Q** - first moment of a partial area about the neutral axis or activation energy; **R** - radius of Mohr's circle or radius of shaft/torsion specimen or stress ratio; $\mathbf{S_f}$ - fracture strength; $\mathbf{S_{uts}}$ or $\mathbf{S_u}$ - ultimate tensile strength; **r** - radius of a cylinder or sphere; $\mathbf{S_y}$ - offset yield strength; **T** - torque or temperature; $\mathbf{T_{mp}}$ - melting temperature; **t** - thickness of cross section or time; $\mathbf{t_f}$ - time to failure; **U** - stored energy; $\mathbf{U_r}$ - modulus of resilience; $\mathbf{U_t}$ - modulus of toughness; **V** or **V(x)** - shear force; **v** or **v(x)** - displacement in the "y" direction; **w(x)** - distributed load; **x** or **X** - coordinate direction or axis; **y** or **Y** - coordinate direction or axis; **z** or **Z** - coordinate direction or axis;

Greek characters

Δ - change or increment; ε - normal strain or tensoral strain component; ε_o - normal strain at σ_o; φ - angle or angle of twist; γ - engineering shear strain; ν - Poisson's ratio; ω - angular velocity; ρ - variable for radius or radius of curvature; σ - normal stress; σ_1 , σ_2 , σ_3 - greatest, intermediate, and least principal normal stresses; σ' - effective stress; σ_o - proportional limit, elastic limit, or yield stress; τ - shear stress; τ_{max} - maximum shear stress; τ_o - yield shear strength; θ - angle; θ_p - principal normal stress angle; θ_s - maximum shear stress angle

Stress

Stress: i) the ratio of incremental force to incremental area on which the force acts such that: $\lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$.

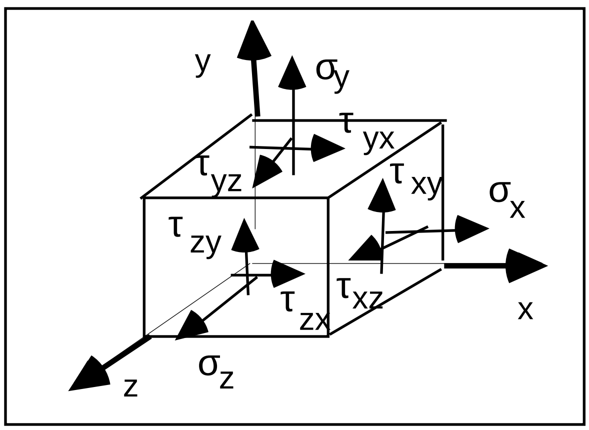
ii) the intensity of the internal force on a specific plane (area) passing through a point.

Normal Stress: the intensity of the internal force acting normal to an incremental area such that: $\sigma = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$

Note: $+\sigma =$ tensile stress = "pulling" stress
and $-\sigma =$ compressive stress = "pushing" stress

Shear Stress: the intensity of the internal force acting tangent to an incremental area such that: $\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A}$

General State of Stress: all the internal stresses acting on an incremental element



Note: A $+\sigma$ acts normal to a positive face in the positive coordinate direction and a $+\tau$ acts tangent to a positive face in a positive coordinate direction

Note: Moment equilibrium shows that $\tau_{xy} = \tau_{yx}; \tau_{xz} = \tau_{zx}; \tau_{yz} = \tau_{zy}$

Complete State of Stress: Six independent stress components (3 normal stresses, $\sigma_x; \sigma_y; \sigma_z$ and 3 shear stresses, $\tau_{xy}; \tau_{yz}; \tau_{xz}$) which uniquely describe the stress state for each particular orientation

Units of Stress: In general: $\frac{Force}{Area} = \frac{F}{L^2}$,

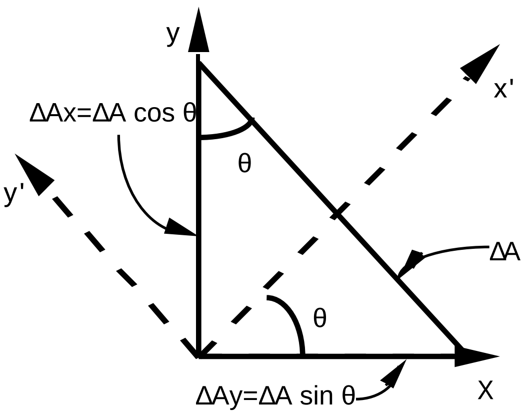
In SI units, $Pa = \frac{N}{m^2}$ or MPa $= 10^6 \frac{N}{m^2} = \frac{N}{mm^2}$

In US Customary units, $psi = \frac{lb_f}{in^2}$ or ksi $= 10^3 \frac{lb_f}{in^2} = \frac{kip}{in^2}$

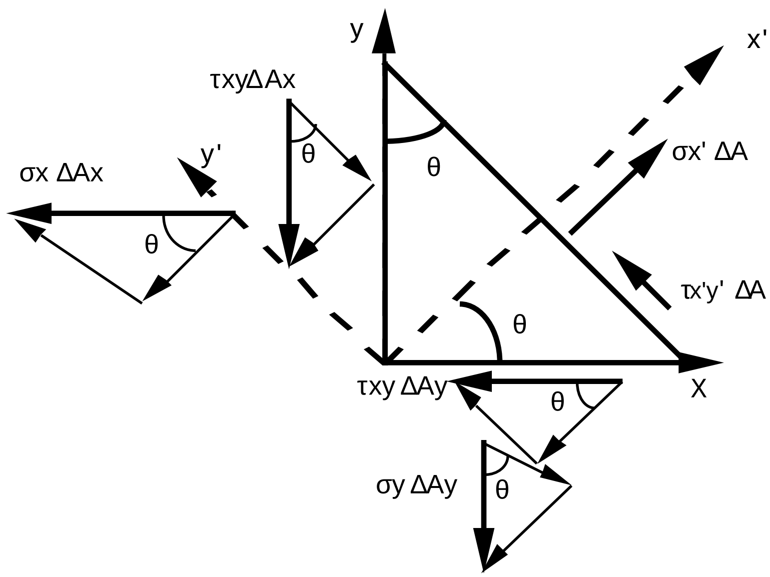
Stress Transformation

For the plane stress condition (e.g., stress state at a surface where no load is supported on the surface), stresses exist only in the plane of the surface (e.g., $\sigma_x ; \sigma_y ; \tau_{xy}$)

The plane stress state at a point is uniquely represented by three components acting on a element that has a specific orientation (e.g., x, y) at the point. The stress transformation relation for any other orientation (e.g., x', y') is found by applying equilibrium equations ($\sum F = 0$ and $\sum M = 0$) keeping in mind that $F_n = \sigma A$ and $F_t = \tau A$



Rotated coordinate axes and areas for x and y directions



Rotated coordinate axes and components of stress/forces for original coordinate axes

$\sum F_{x'} = 0$ gives

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta \;\; \textit{or} \;\; \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$\sum F_{y'} = 0$ gives

$$\tau_{x'y'} = (\sigma_x - \sigma_y)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta + \sin^2\theta) \;\text{or}\; \tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Similarly, for a cut in the y' direction,

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta \;\; \textit{or} \;\; \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

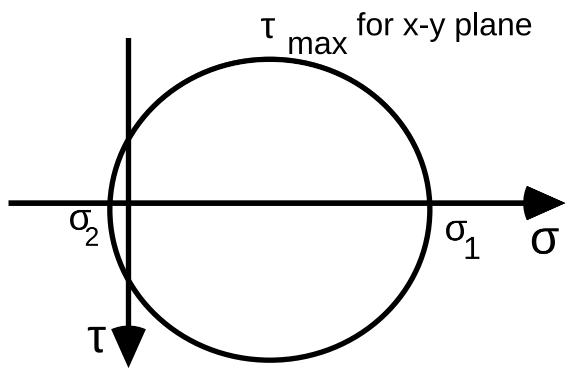
Principal Normal Stress - maximum or minimum normal stresses acting in principal directions on principal planes on which no shear stresses act.
Note that $\sigma_1 > \sigma_2 > \sigma_3$

For the plane stress case $\sigma_{1,2} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ and $\tan 2\theta_p = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$

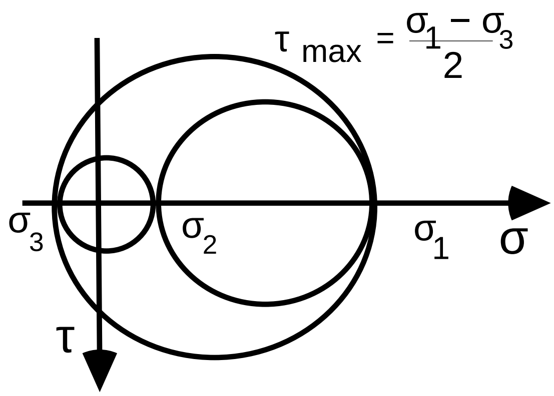
and $\tau_{max} = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$, $\sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2}$ and $\tan 2\theta_s = \dfrac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}$

Mohr's Circles for Stress States - graphical representation of stress

Examples of Mohr's circles



Mohr's circle for stresses in x-y plane



Mohr's circle for stresses in x-y-z planes

Graphical Description of State of Stress

<u>2-D Mohr's Circle</u>

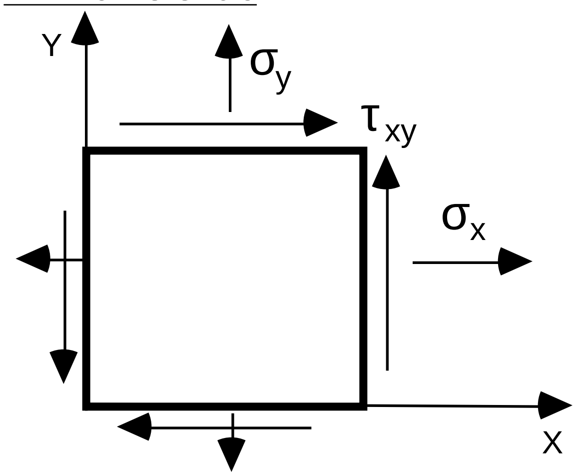


Fig. 1- Positive stresses acting on a physical element.

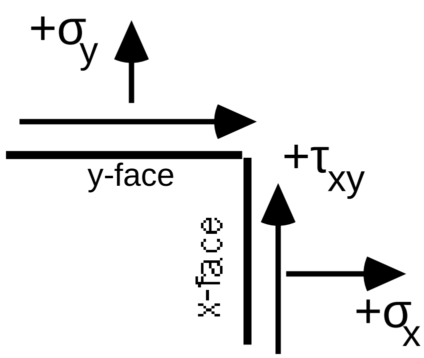
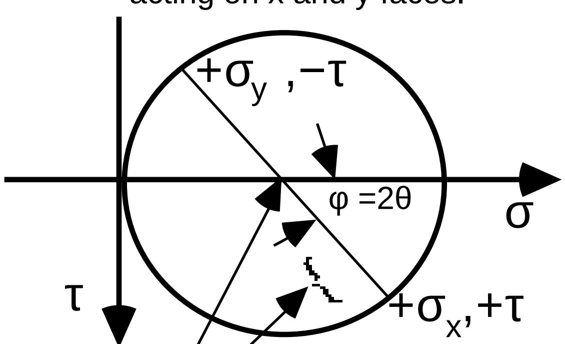


Fig. 2 - Directionality of shear acting on x and y faces.



$C = \frac{\sigma_x + \sigma_y}{2}$

$R^2 = (\sigma_x - C)^2 + \tau^2)$

$\tan \varphi = \frac{-\tau}{(\sigma_x - C)}$

Fig. 3 - Plotting stress values on Mohr's circle.

In this example all stresses acting in axial directions are positive as shown in Fig. 1.

As shown in Figs. 2 and 3, plotting actual sign of the shear stress with x normal stress requires plotting of the opposite sign of the shear stress with the y normal stress on the Mohr's circle.

In this example $\sigma_x > \sigma_y$ and τ_{xy} is positive. By the convention of Figs. 2 and 3, $\varphi = 2\theta$ on the Mohr's circle is negative from the $+\sigma$ axis. (Mathematical convention is that positive angle is counterclockwise).

Note that by the simple geometry of Fig. 3, $\varphi = 2\theta$ appears to be negative while by the formula, $\tan 2\theta = 2\tau_{xy}/(\sigma_x - \sigma_y)$, the physical angle, θ, is actually positive.

In-plane principal stresses are: $\sigma_1 = C+R$
$\sigma_2 = C - R$

Maximum in-plane shear stress is:

$$\tau_{max} = R = (\sigma_1 - \sigma_2)/2$$

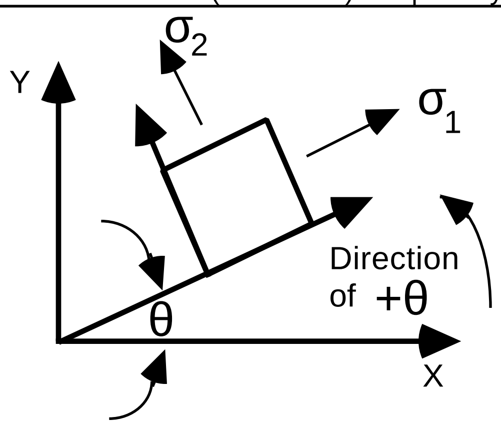


Fig. 4 - Orientation of physical element with only principal stresses acting on it.

The direction of physical angle, θ, is from the x-y axes to the principal axes.

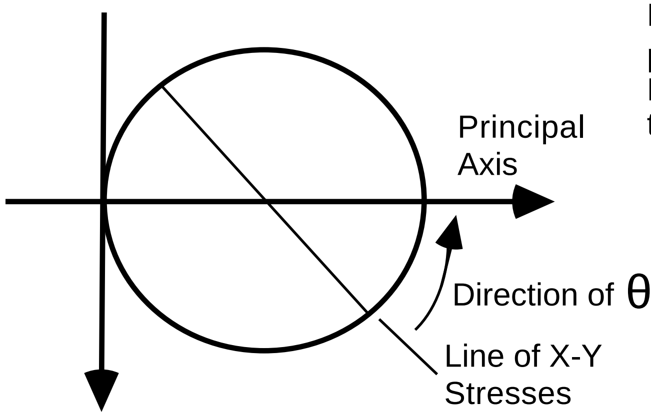


Fig. 5 - Direction of θ from the line of x-y stresses to the principal stress axis.

Note that the sense (direction) of the physical angle, θ, is the same as on the Mohr's circle from the line of the x-y stresses to the axes of the principal stresses.

Same relations apply for Mohr's circle for strain except interchange variables as

$\sigma \Leftrightarrow \varepsilon$ and $\tau \Leftrightarrow \frac{\gamma}{2}$

Strain

Strain: normalized deformations within a body exclusive of rigid body displacements

Normal Strain: elongation or contraction of a line segment per unit length such that

$$\varepsilon = \lim_{B \to \text{A along n}} \frac{A'B' - AB}{AB} \equiv \frac{L_f - L_o}{L_o}$$ and a volume change results.

Note: $+\varepsilon =$ tensile strain = elongation

and $-\varepsilon =$ compressive strain = contraction

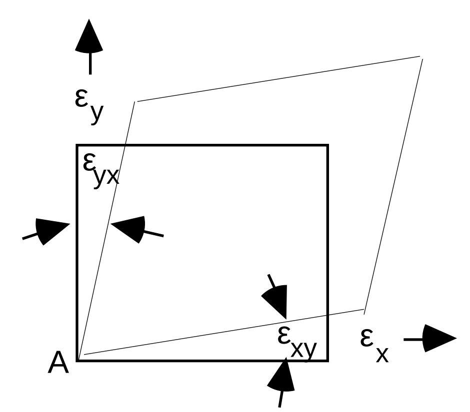
Shear Strain: the angle change between two line segments such that

$$\gamma = (\theta = \frac{\pi}{2}) - \theta' \approx \frac{\Delta}{h} \text{(for small angles)}$$ and a shape change results.

Note: $+\gamma$ occurs if $\frac{\pi}{2} > \theta'$

and $-\gamma$ occurs if $\frac{\pi}{2} < \theta'$

General State of Strain: all the internal strains acting on an incremental element



Engineering shear strain,

$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx}$

Complete State of Strain: Six independent strain components (3 normal strains, $\varepsilon_x; \varepsilon_y; \varepsilon_z$ and 3 engineering shear strains, $\gamma_{xy}; \gamma_{yz}; \gamma_{xz}$) which uniquely describe the strain state for each particular orientation

Units of Strain: In general: $\frac{Length}{Length} = \frac{L}{L}$,

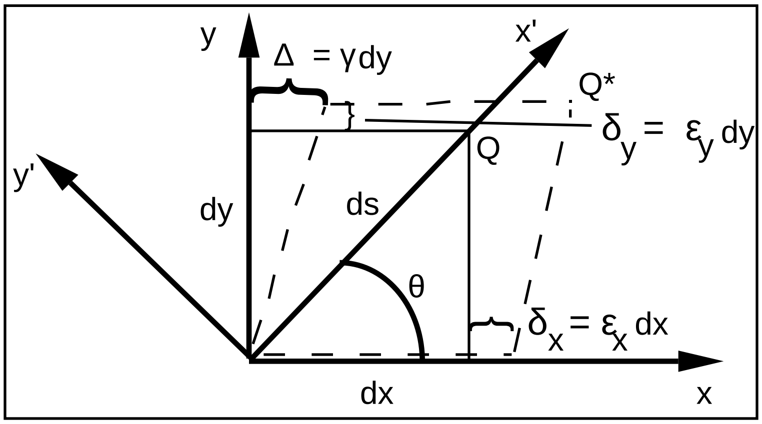
In SI units, $\frac{\text{m}}{\text{m}}$ for ε and $\frac{\text{m}}{\text{m}}$ or radian for γ

In US Customary units, $\frac{\text{in}}{\text{in}}$ for ε and $\frac{\text{in}}{\text{in}}$ or radian for γ

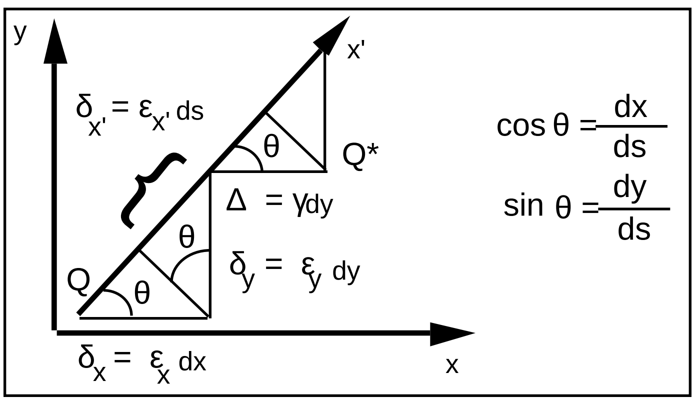
Strain Transformation

For the plane strain condition (e.g., strain at a surface where no deformation occurs normal to the surface), strains exist only in the plane of the surface ($\varepsilon_x ; \varepsilon_y ; \gamma_{xy}$)

The plane strain state at a point is uniquely represented by three components acting on a element that has a specific orientation (e.g., x, y) at the point. The strain transformation relation for any other orientation (e.g., x', y') is found by summing displacements in the appropriate directions keeping in mind that $\delta = \varepsilon L_o$ and $\Delta = \gamma h$



Rotated coordinate axes and displacements for x and y directions



Displacements in the x' direction for strains/ displacements in the x and y directions

$\sum$ displacements in x 'direction for Q to Q * gives

$$\varepsilon_{x'} = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\cos\theta\sin\theta \;\; or \;\; \varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$\sum$ rotation of dx ' and dy' gives

$$\frac{\gamma_{x'y'}}{2} = (\varepsilon_x - \varepsilon_y)\cos\theta\sin\theta + \frac{\gamma_{xy}}{2}(\cos^2\theta + \sin^2\theta) \text{ or } \frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Similarly, $\sum$ displacements in y 'direction for Q to Q * gives

$$\varepsilon_{y'} = \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy}\cos\theta\sin\theta \;\; or \;\; \varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

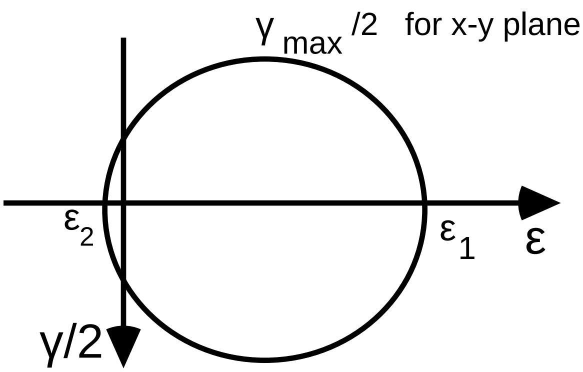
Principal Normal Strain - maximum or minimum normal strains acting in principal directions on principal planes on which no shear strains act.
Note that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$

For the plane strain case $\varepsilon_{1,2} = \dfrac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\dfrac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\dfrac{\gamma_{xy}}{2}\right)^2}$ and $\tan 2\theta_p = \dfrac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$

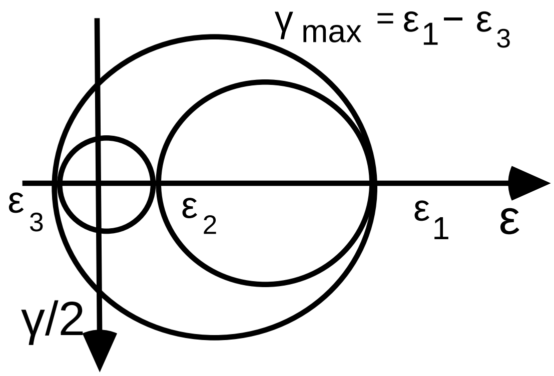
and $\dfrac{\gamma_{max}}{2} = \sqrt{\left(\dfrac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\dfrac{\gamma_{xy}}{2}\right)^2}$, $\varepsilon_{ave} = \dfrac{\varepsilon_x + \varepsilon_y}{2}$ and $\tan 2\theta_s = \dfrac{-(\varepsilon_x - \varepsilon_y)}{\gamma_{xy}}$

Mohr's Circles for Strain States - graphical representation of strain

Examples of Mohr's circles



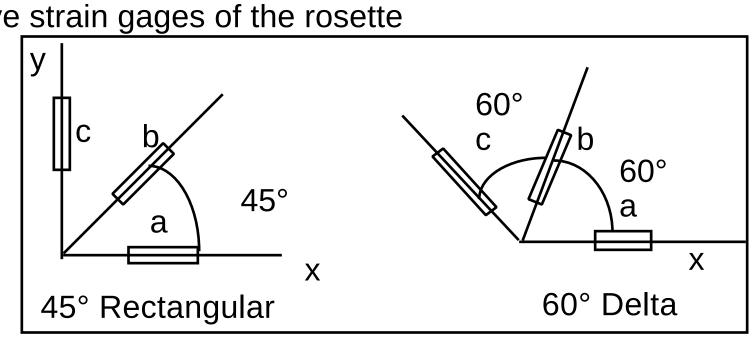
Mohr's circle for strains in x-y plane



Mohr's circle for strains in x-y-z planes

Strain Gage Rosettes

Rosette orientations and equations relating x-y coordinate strains to the respective strain gages of the rosette



45° Rectangular	60° Delta
$\varepsilon_x = \varepsilon_a$	$\varepsilon_x = \varepsilon_a$
$\varepsilon_y = \varepsilon_c$	$\varepsilon_y = \dfrac{1}{3}(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a)$
$\gamma_{xy} = 2\varepsilon_b - (\varepsilon_a + \varepsilon_c)$	$\gamma_{xy} = \dfrac{2}{\sqrt{3}}(\varepsilon_b - \varepsilon_c)$

Continuum Mechanics and Constitutive Relations

Equations which relate stress and strain (a.k.a., Generalized Hooke's Law)

$$\{\sigma\} = [C]\{\varepsilon\}$$

$$\sigma_x = \frac{E}{(1+\nu)}\varepsilon_x + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\sigma_y = \frac{E}{(1+\nu)}\varepsilon_y + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\sigma_z = \frac{E}{(1+\nu)}\varepsilon_z + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\tau_{xy} = G\gamma_{xy}$$

$$\tau_{yz} = G\gamma_{yz}$$

$$\tau_{xz} = G\gamma_{xz}$$

$$\{\varepsilon\} = [S]\{\sigma\}$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\gamma_{xz} = \frac{1}{G}\tau_{xz}$$

$[C] = [S]^{-1}$ *and* $[S] = [C]^{-1}$

Elastic relation (1-D Hooke's Law) $\sigma = E\varepsilon$

Plastic relation (Strain -Hardening) $\sigma = K\varepsilon^n$

Poisson's ratio, $\nu = -\dfrac{\varepsilon_{transverse}}{\varepsilon_{longitudinal}}$

Plane stress : $\sigma_z = 0,\ \varepsilon_z \neq 0 = \dfrac{-\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)$

Plane strain : $\varepsilon_z = 0,\ \sigma_z \neq 0 = \nu(\sigma_x + \sigma_y)$

Stress − strain relations for plane stress (x − y plane)

$$\sigma_x = \frac{E}{(1-\nu^2)}\left(\varepsilon_x + \nu\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{(1-\nu^2)}\left(\varepsilon_y + \nu\varepsilon_x\right)$$

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

$$\tau_{xy} = G\gamma_{xy}$$

Elastic Modulus, $E = \dfrac{\Delta\sigma}{\Delta\varepsilon}$

Poisson's ratio, $\nu = -\dfrac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}}$

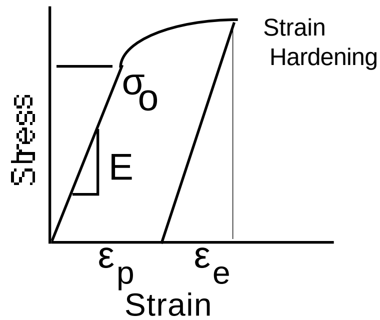
Shear Modulus, $G = \dfrac{\Delta\tau}{\Delta\gamma} = \dfrac{E}{2(1+\nu)}$

Bulk Modulus, $k = \dfrac{\left(\sigma_x + \sigma_y + \sigma_x\right)}{3\left(\varepsilon_x + \varepsilon_y + \varepsilon_x\right)} = \dfrac{E}{3(1-2\nu)}$

PLASTIC DEFORMATION

Non recoverable deformation beyond the point of yielding where Hooke's law (proportionality of stress and strain) no longer applies. Flow curve is the true stress vs. true strain curve describing the plastic deformation.

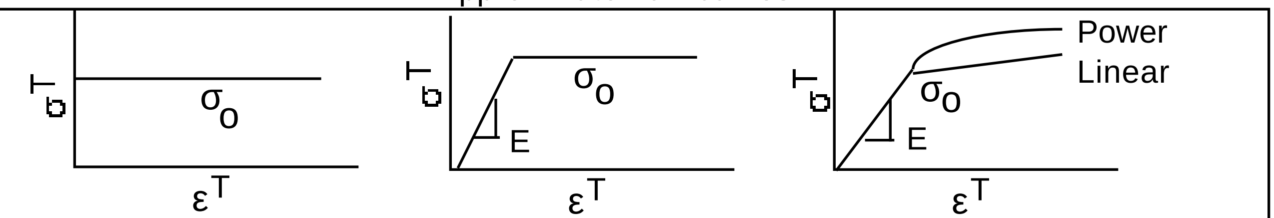
Simple Power Law



Elastic: $\sigma = E\varepsilon \qquad (\sigma \le \sigma_o)$

Plastic: $\sigma = H\varepsilon^n \qquad (\sigma \ge \sigma_o)$

Approximate flow curves



Rigid-Perfectly Plastic | Elastic-Perfectly Plastic | Elastic-Linear Hardening, Elastic-Power Hardening

Ramberg-Osgood Relationship

Total strain is sum of elastic and plastic $\varepsilon = \varepsilon_e + \varepsilon_p = \dfrac{\sigma}{E} + \varepsilon_p \qquad \sigma = H\,\varepsilon_p^n \qquad \varepsilon = \dfrac{\sigma}{E} + \left(\dfrac{\sigma}{H}\right)^{\frac{1}{n}}$

Deformation Plasticity

$$\bar{\sigma}_{eff} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \text{ and } \bar{\varepsilon}_{eff} = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

Effective stress-effective strain curve is independent of the state of stress and is used to estimate the stress-strain curves for other states of stress.

Failure Theories

Two types: Fracture and Yield Criteria. Generally used to predict the safe limits of a material/component under combined stresses.

Factor of Safety, $FS = \dfrac{\text{Material Strength}}{\text{Component Stress}}$, Failure occurs if FS<1

Maximum Normal Stress Criterion

Fracture criterion generally used to predict failure of **brittle** materials.

$$FS = \frac{S_{UTS}}{MAX(|\sigma_1|,|\sigma_2|,|\sigma_3|)}$$

Maximum Shear Stress (Tresca) Criterion

Yield criterion generally used to predict failure in materials which yield in shear (i.e. **ductile** materials)

$$FS = \frac{(\tau_o = S_y / 2 = \sigma_0 / 2)}{MAX\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}\right)}$$

Von Mises (Distortional Energy)
or Octahedral Shear Stress Criterion

Yield criterion generally used to predict failure in materials. which yield in shear (i.e. **ductile** materials)

$$FS = \frac{(\sigma_o = S_y)}{\sigma'}$$

$$\sigma' = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\sigma' = \frac{1}{\sqrt{2}}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yx}^2 + \tau_{zx}^2)}$$

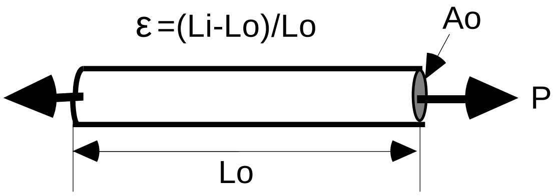
Mechanical Testing

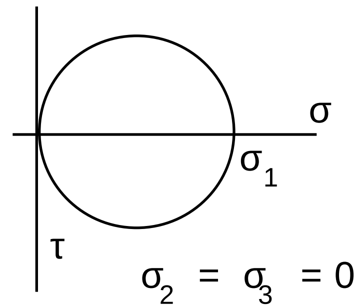
The results of materials tests (e.g. tensile, compressive, torsional shear, hardness, impact energy, etc.) are used for a variety of purposes including to obtain values of material properties for use in engineering design and for use in quality control to ensure materials meet established requirements

Tensile Testing

$\sigma_1 = P/Ao$

$\varepsilon = (Li-Lo)/Lo$





Mohr's Circle for Uniaxial Tension

Elastic Modulus : $E = \dfrac{d\sigma}{d\varepsilon}$ of the linear part of the stress-strain curve.

Yielding : Proportional limit, σ_p; elastic limit; offset yield (S_{ys} at 0.2% strain) where σ_o is used to generally designate the stress at yielding.

Ductility : % elongation $= \dfrac{L_f - L_o}{L_o} x\,100 = \varepsilon_f x100$ or $\%RA = \dfrac{A_o - A_f}{A_o} x\,100$

Necking is geometric instability at S_{UTS} and ε_U

Strain hardening ratio = $\dfrac{S_{UTS}}{\sigma_o}$ where ≥1.4 is high and ≤1.2 is low.

Energy absorption (energy/volume):

Modulus of Resilience	Modulus of Toughness
= measure of the ability to store elastic energy	= measure of the ability to absorb energy without fracture
= area under the linear portion of the stress-strain curve	= area under the entire stress-strain curve

$$\boxed{U_R = \int_o^{\varepsilon_o} \sigma\, d\varepsilon \approx \frac{\sigma_o \varepsilon_o}{2} \approx \frac{\sigma_o^2}{2E}}$$

$$U_T = \int_o^{\varepsilon_f} \sigma\, d\varepsilon \approx \frac{(S_{UTS} + \sigma_o)\varepsilon_f}{2} \quad (\text{"flat" } \sigma - \varepsilon \text{ curves})$$

$$\text{or} \quad \int_o^{\varepsilon_f} \sigma\, d\varepsilon \approx \frac{2 S_{UTS} \varepsilon_f}{3} \quad (\text{parabolic } \sigma - \varepsilon \text{ curves})$$

Strain-hardening: $\sigma^T = K(\varepsilon^T)^n = H(\varepsilon^T)^n$

H=K=strength coefficient and n = strain hardening exponent (0≤n≤1)

Representative stress-strain curves for tensile tests of brittle and ductile materials

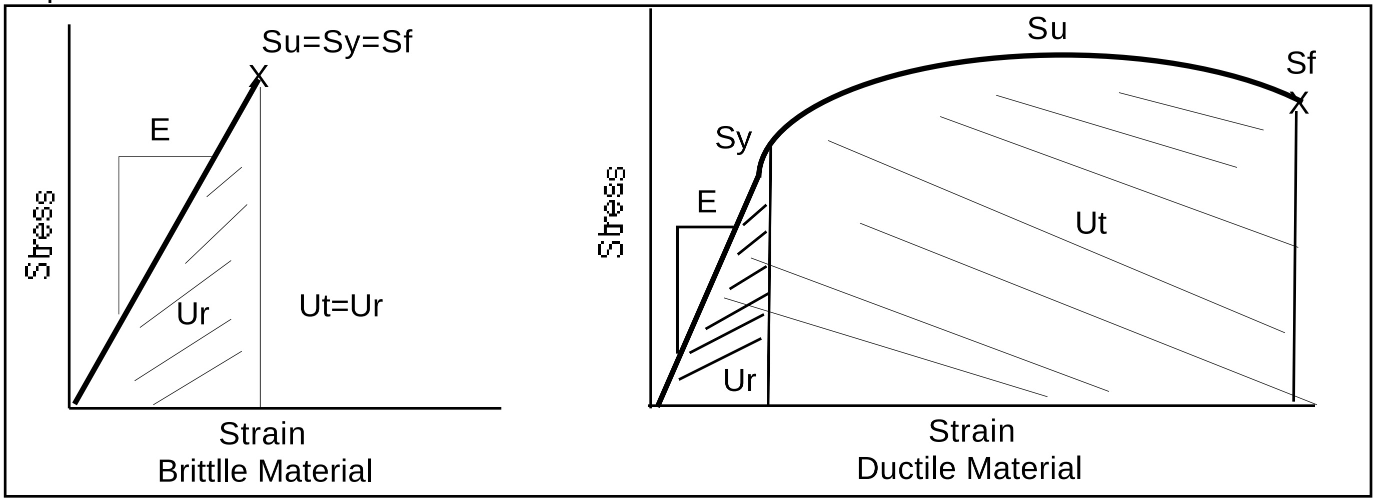


Table: Stress-strain definitions for tensile testing

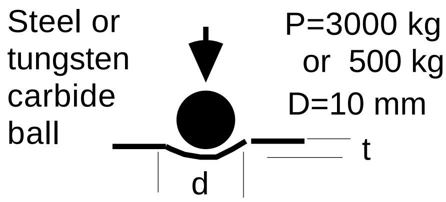
PARAMETER	**FUNDAMENTAL DEFINITION**	**PRIOR TO NECKING**	**AFTER NECKING**
Engineering Stress (σ^E)	$\sigma^E = \frac{P_i}{A_o}$	$\sigma^E = \frac{P_i}{A_o}$	$\sigma^E = \frac{P_i}{A_o}$
True Stress (σ^T)	$\sigma^T = \frac{P_i}{A_i}$	$\sigma^T = \frac{P_i}{A_i}$ $\sigma^T = \sigma^E(1+\varepsilon^E)$	$\sigma^T = \frac{P_i}{A_{neck}}$
Engineering Strain (ε^E)	$\varepsilon^E = \frac{\Delta L}{L_o} = \frac{L_i - L_o}{L_o}$	$\varepsilon^E = \frac{\Delta L}{L_o} = \frac{L_i - L_o}{L_o}$	$\varepsilon^E = \frac{\Delta L}{L_o} = \frac{L_i - L_o}{L_o}$
True Strain (ε^T)	$\varepsilon^T = \ln\frac{L_i}{L_o}$ $\varepsilon^T = \ln\frac{A_o}{A_i}$	$\varepsilon^T = \ln\frac{L_i}{L_o}$ $\varepsilon^T = \ln\frac{A_o}{A_i}$ $\varepsilon^T = \ln(1+\varepsilon^E)$	$\varepsilon^T = \ln\frac{A_o}{A_{neck}}$

Note: Subscripts: i=instantaneous, o=original; Superscripts: E=engineering, T=true

Hardness Testing

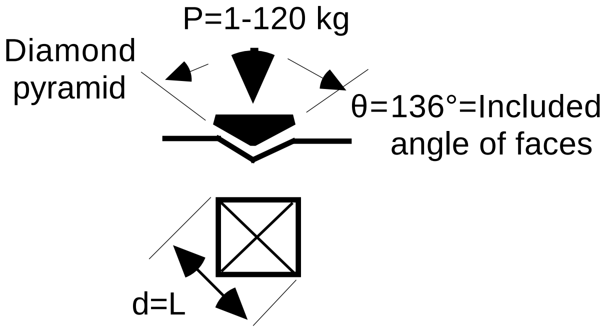
Resistance of material to penetration

Brinell



$$BHN = HB = \frac{P}{\pi D t} = \frac{2P}{\pi D\left[D - \sqrt{\left(D^2 - d^2\right)}\right]}$$

Vickers



$$VHN = HV = \frac{2P}{L^2}\sin\frac{\theta}{2}$$

Rockwell

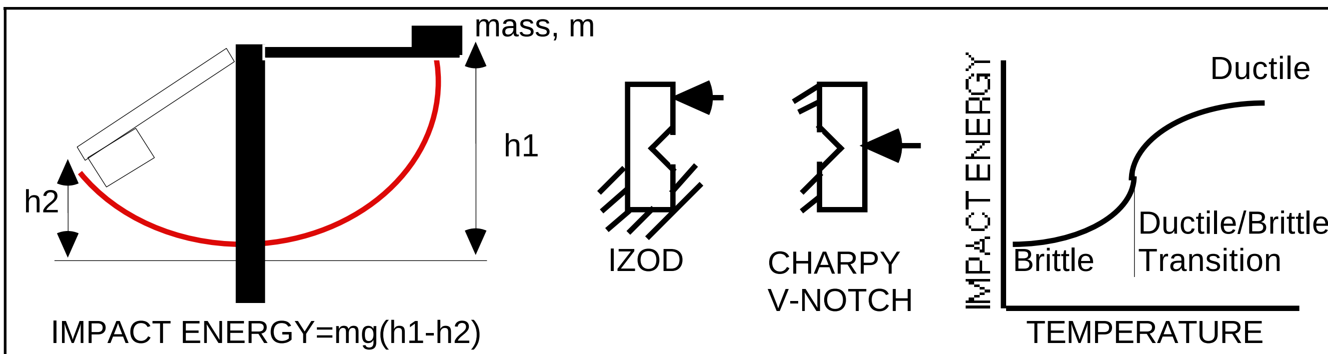
Requires Rockwell subscript to provide meaning to the Rockwell scale.

Examples of Rockwell Scales

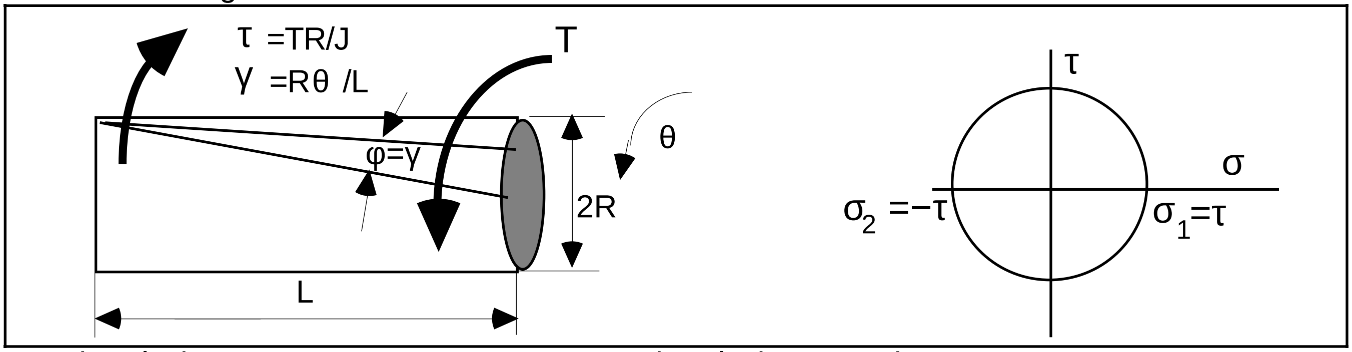
Rockwell Hardness	Indentor	Load (kg)
A	Diamond point	60
B	1.588 mm dia. ball	100
C	Diamond point	150
D	Diamond point	100
E	3.175 mm dia. ball	100
M	6.350 mm dia. ball	100
R	12.70 mm dia. ball	60

Notch-Impact Testing

Resistance of material to sudden fracture in presence of notch



Torsion Testing



Torsional Shear Stress

$$\tau = \frac{TR}{J} \qquad J = \frac{\pi D^4}{32} \text{ for solid shaft}$$

$$J = \frac{\pi(D^4_{outer} - D^4_{inner})}{32} \text{ for tube}$$

Torsional Shear Strain

$$\gamma = \frac{R\theta}{L}$$

Shear Modulus : $G = \dfrac{\tau}{\gamma} = \dfrac{E}{2(1+\nu)}$

• For linear elastic behaviour, plane sections remain plane, so $\gamma = \dfrac{R\theta}{L}$ *and* $\tau = \dfrac{TR}{J}$

Modulus of Rupture (maximum shear stress) : $\tau_u = \dfrac{T_{max}R}{J}$

• For nonlinear behaviour, plane sections remain plane, so

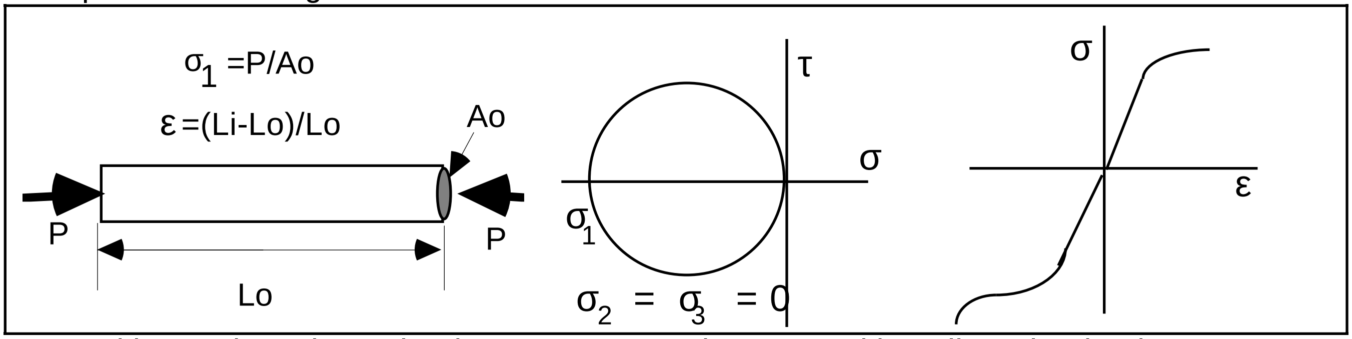
$\gamma = \dfrac{R\theta}{L}$ *but* $\tau \neq \dfrac{TR}{J}$ beyond linear region . Instead $\tau = \dfrac{1}{2\pi R^3}\left((\theta/L)\dfrac{dT}{d(\theta/L)} + 3T\right)$

Modulus of Rupture (maximum shear stress) when dT/d(θ/L) = 0 so $\tau_u \approx \dfrac{3T_{max}}{2\pi R^3}$

Table: Comparison of stresses and strains for tension and torsion tests

Tension Test	**Torsion Test**
$\sigma_1 = \sigma_{max}; \sigma_3 = \sigma_2 = 0$	$\sigma_1 = -\sigma_3;\ \sigma_2 = 0$
$\tau_{max} = \dfrac{\sigma_1}{2} = \dfrac{\sigma_{max}}{2}$	$\tau_{max} = \dfrac{2\sigma_1}{2} = \sigma_{max}$
$\varepsilon_{max} = \varepsilon_1;\ \varepsilon_2 = \varepsilon_3 = -\dfrac{\varepsilon_1}{2}$	$\varepsilon_{max} = \varepsilon_1 = -\varepsilon_3;\ \varepsilon_2 = 0$
$\gamma_{max} = \dfrac{3\varepsilon_1}{2}$	$\gamma_{max} = \varepsilon_1 - \varepsilon_3 = 2\varepsilon_1$
effective stress $\overline{\sigma}_{eff} = \dfrac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}$	
effective strain $\overline{\varepsilon}_{eff} = \dfrac{\sqrt{2}}{3}\sqrt{(\varepsilon_1-\varepsilon_2)^2 + (\varepsilon_2-\varepsilon_3)^2 + (\varepsilon_3-\varepsilon_1)^2}$	
$\overline{\sigma} = \sigma_1$	$\overline{\sigma} = \sqrt{3}\sigma_1$
$\overline{\varepsilon} = \varepsilon_1$	$\overline{\varepsilon} = \dfrac{2}{\sqrt{3}}\varepsilon_1 = \dfrac{\gamma}{\sqrt{3}}$

Compression Testing



No necking and maximum load may not occur since pancaking allows load to keep increasing. For many metals and polymers, the compressive stress and strain relations are similar to those in tension (including elastic constants, ductility, and yield). For other materials, such as ceramics, glasses, and composites (often at elevated temperatures), compression behavior may be quite different than tensile behavior.

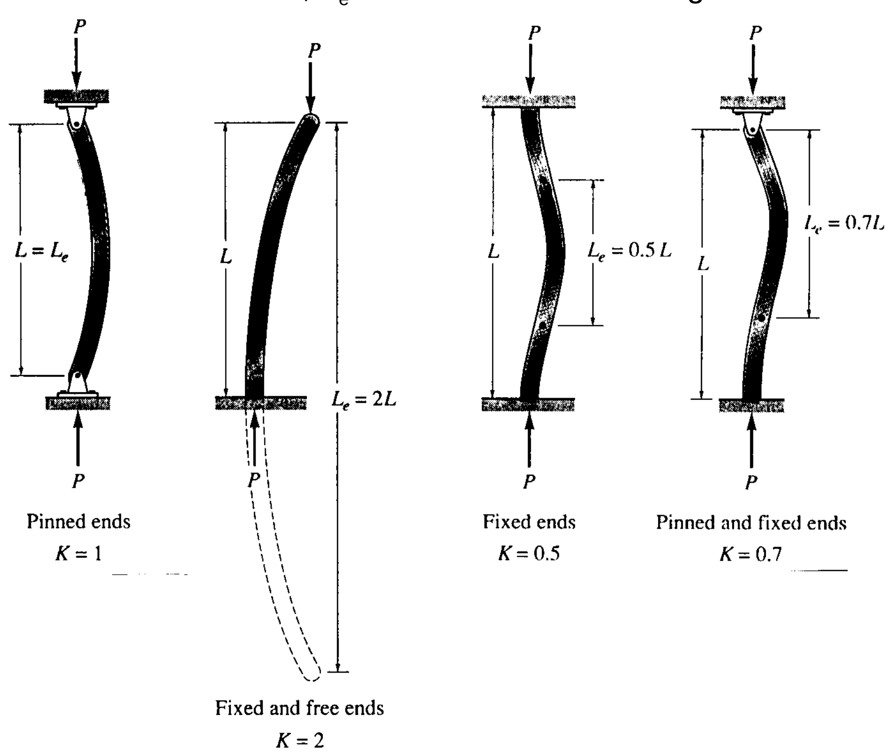
In an ideal column (no eccentricity) the axial load, P, can be increased until failure occurs wither by fracture, yielding or buckling. Buckling is a geometric instablity related only to the elastic modulus (stiffness) of the material and not the strength.

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \quad or \quad \sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2}$$

where (L/r) is the slenderness ratio

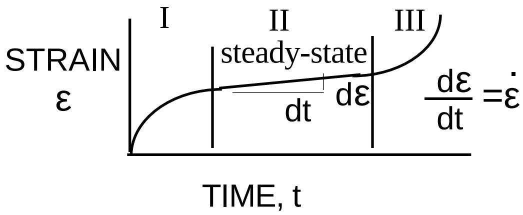
and (KL/r) is the effective slenderness ratio

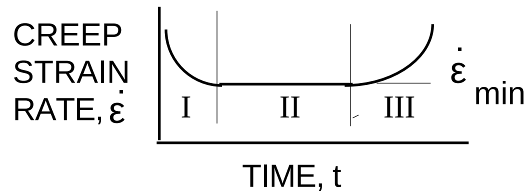
Sometimes, L_e =KL is the effective length.



Creep and Time Dependent Deformation

Time dependent deformation under constant load or stress at temperatures greater than 30 and 60% of the melting point (i.e. homologous temperatures, T/T_{mp} >0.3-0.6)



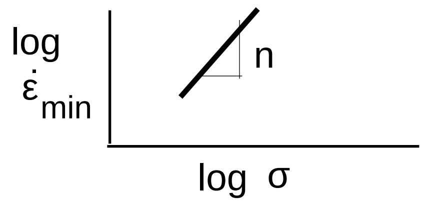


$$\dot{\varepsilon}_{min} = A\sigma^n \exp(-Q/RT)$$

<u>Stress exponent, n, from isothermal tests:</u>

$\dot{\varepsilon}_{min} = B\sigma^n$ so that $\log\dot{\varepsilon}_{min} = \log B + n \log \sigma$

or $n = \dfrac{\log\dot{\varepsilon}_{min,1} - \log\dot{\varepsilon}_{min,2}}{\log\sigma_1 - \log\sigma_2}$

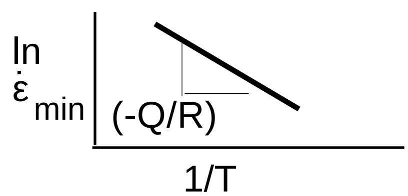


<u>Activation energy, Q, from isostress tests:</u>

$\dot{\varepsilon}_{min} = C\exp(-Q/RT)$ so that

$\ln\dot{\varepsilon}_{min} = \ln C + (-Q/R)\,(1/T)$

or $Q = \dfrac{-R(\ln\dot{\varepsilon}_{min,1} - \ln\dot{\varepsilon}_{min,2})}{\dfrac{1}{T_1} - \dfrac{1}{T_2}}$

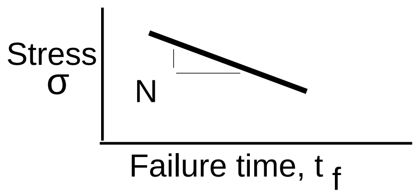


Long term predictions from short term results - valid only if the creep/creep rupture mechanism does not change over time. Rule-of-thumb: short-time test lives should be at least 10% of the required long-term design life. Creep rupture occurs by the coalescence of the diffusional damage (creep cavitation by inter or intragranular diffusion and oxidation) which is manifested during secondary (steady-state creep).

Stress-rupture

Empirical relation $\sigma = A\, t_f^{N}$

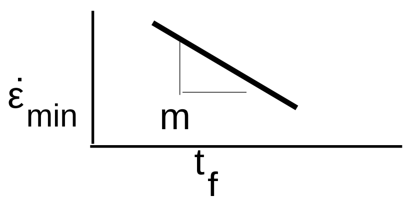
Important where creep deformation is tolerated but rupture is to be avoided.



Monkman-Grant

Empirical relation $\dot{\varepsilon}_{min} t_f = C$ or $\dot{\varepsilon}_{min} = C t_f^{m}$ where $m = -1$ if the relation is applicable .

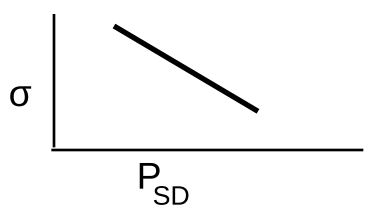
Important where total creep deformation (i.e. $\dot{\varepsilon}_{min} t_f$) is of primary concern.



Sherby-Dorn

Assumes that Q≠f(σ or T) and suggests that the creep strains for a given stress form a unique curve if plotted versus the temperature compensated time, $\theta = t\ \exp(-Q/RT)$.

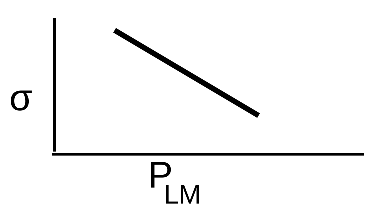
A common physical mechanism is assumed to define the time-temperature paramter such that the Sherby-Dorn parameter $P_{SD} = \log\theta = \log t_f - \left(\frac{\log(e)}{R}\right) Q \left(\frac{1}{T}\right)$



Larson-Miller

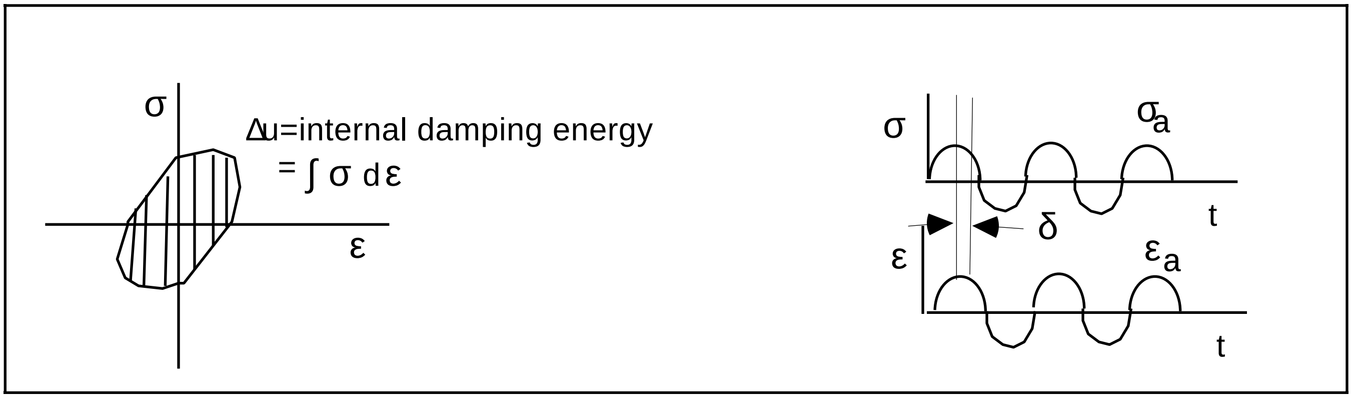
Assumes that Q=f(σ) and suggests that the creep strains for a given stress form a unique curve if plotted versus the temperature compensated time, $\theta_f = t_f\ \exp(-Q/RT)$.

A common physical mechanism is assumed to define the time-temperature parameter such that the Larson-Miller parameter $P_{LM} = \left(\frac{\log(e)}{R}\right) Q = T(\log t_f + C)$



Material Damping

Energy dissipation during cyclic loading - internal friction which is material, frequency, temperature dependent.



Dynamic Modulus : $E^* = \frac{\sigma_a}{\varepsilon_a}$ Phase Angle : $\varphi = \delta$

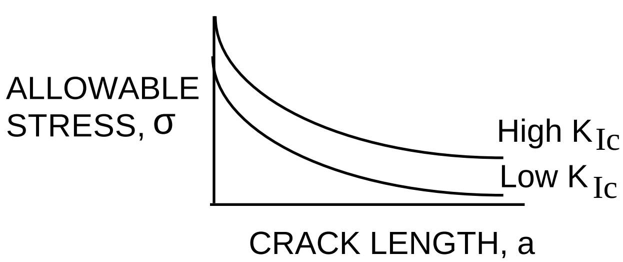
Loss Coefficient : $Q^{-1} = \tan\delta = \frac{\Delta u}{2\pi U_e}$

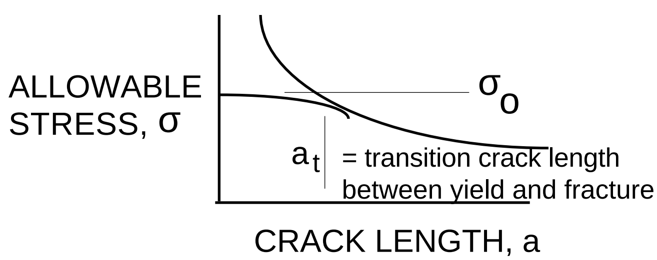
Storage Modulus: $\frac{\sigma'}{\varepsilon_a} = E^* \cos\delta$ (where $\sigma' = \sigma$ at ε_a)

Elastic Energy: $U_e = \frac{1}{2}\sigma'\varepsilon_a$ at ε_a maximum extension

Fracture

Fracture is the separation (or fragmentation) of a solid body into two or more parts under the action of stress (crack initiation and crack propagation) Presence of cracks may weaken the material such that fracture occurs at stresses much less than the yield or ultimate strengths. Fracture mechanics is the methodology used to aid in selecting materials and designing components to minimize the possibility of fracture from cracks.

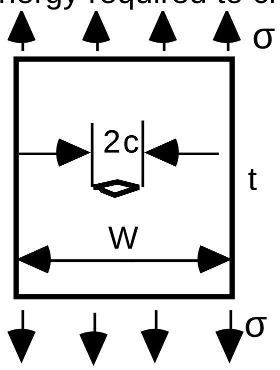


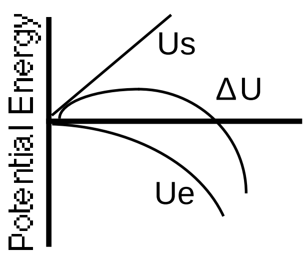


Cracks lower the material's tolerance (allowable stress) to fracture.

Griffith Theory of Brittle Fracture

A crack will propagate when the decrease in elastic strain energy is at least equal to the energy required to create the new fracture surfaces





For completely brittle material :

Elastic strain energy with no crack , $U_e = \dfrac{\pi c^2 \sigma^2 t}{E}$

Energy required to produce crack surfaces , $U_s = 2(2c\gamma_s)t$

Energy balance , $\Delta U = U_s - U_e = 4c\gamma_s t - \dfrac{\pi c^2 \sigma^2 t}{E}$

At critical crack length fracture will occur , $\dfrac{d\Delta U}{dc} = 0 = 4\gamma_s t - \dfrac{2\pi c \sigma^2 t}{E}$

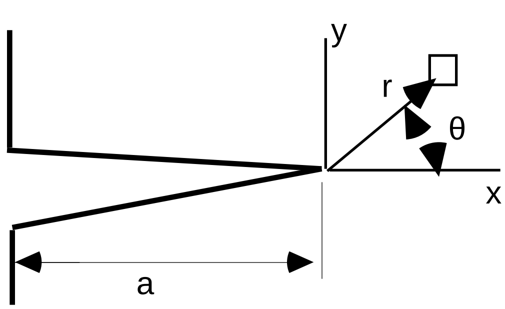
Such that $\sigma_f = \sqrt{\dfrac{E\,2\gamma_s}{\pi c}}$ for plane stress and $t = 1$

If plastic deformation occurs $\sigma_f = \sqrt{\dfrac{E\,2(\gamma_s + \gamma_p)}{\pi c}} \approx \sqrt{\dfrac{E\gamma_p}{c}}$

Strain Energy Release Rate

If $\sigma_f = \sqrt{\dfrac{E\,2\gamma_s}{\pi c}}$ let $G = 2\gamma_s$ then $G = \dfrac{\sigma^2 \pi c}{E}$ where G is the linear elastic strain energy release rate.

The stress intensity factor, K, uniquely defines the stress state at a crack tip in a linear-elastic, isotropic material.



$$\sigma_x = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + ...$$

$$\sigma_y = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + ...$$

$$\tau_{xy} = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + ...$$

$\sigma_z = 0$ for plane stress or $\sigma_z = \nu(\sigma_x + \sigma_y)$

$\tau_{yz} = \tau_{zx} = 0$

In general

$K = F\sigma\sqrt{\pi a} = Y\sigma\sqrt{\pi a} = \alpha\sigma\sqrt{\pi a}$

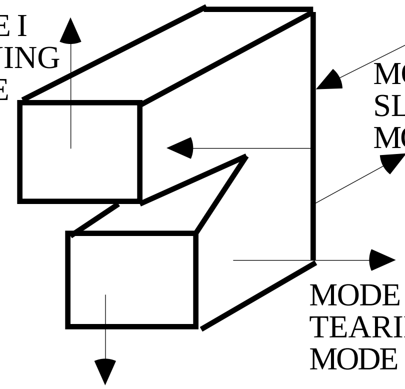
where F, Y, and α are geometry correction factors

Subscripts on K refer to fracture mode :

K_I = Mode I, opening mode

K_{II} = Mode II, sliding mode

K_{III} = Mode III, tearing mode



Note: $\mathcal{G} = \frac{K^2}{E'}$ where E' = E (plane stress)

and E' = E/(1-ν^2) (plane strain)

<u>Plane strain fracture toughness</u>
K_{IC} is the critical stress intensity factor in plane strain conditions at stress intensity factors below which brittle fracture will not occur. The plane strain fracture toughness, K_{IC}, is a material property and is independent of geometry (e.g. specimen thickness).

<u>Fracture toughness in design</u>
Fracture occurs when

$K_{Ic} = K_I = F\sigma\sqrt{\pi a}$

where F is the geometry correction factor for the particular crack geometery.

Designer can choose a material with required K_{Ic},

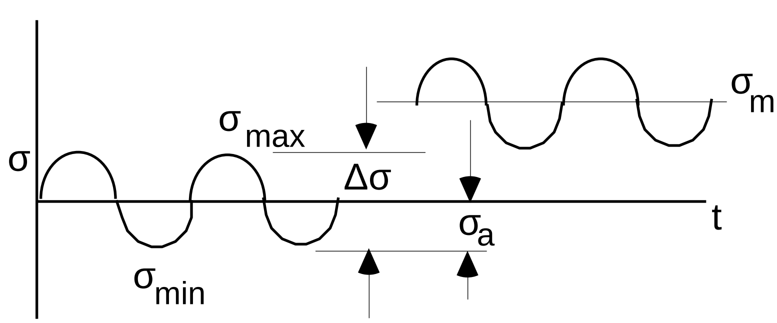
OR design for the stress, σ, *to prevent fracture*,

OR choose a critical crack length, a, *which is* det*ectable* (*or tolerable*).

Cyclic Fatigue

Fatigue is failure due to cyclic (dynamic) loading including time-dependent failure due to mechanical and/or thermal fatigue. Fatigue analysis may be stress-based, strain based, or fracture mechanics based.

<u>Stress-based analysis</u>



σ_{max} = Maximum stress

σ_{min} = Minimum stress

σ_m = Mean stress $= \dfrac{\sigma_{max} + \sigma_{min}}{2}$

$\Delta\sigma$ = Stress range $= \sigma_{max} - \sigma_{min}$

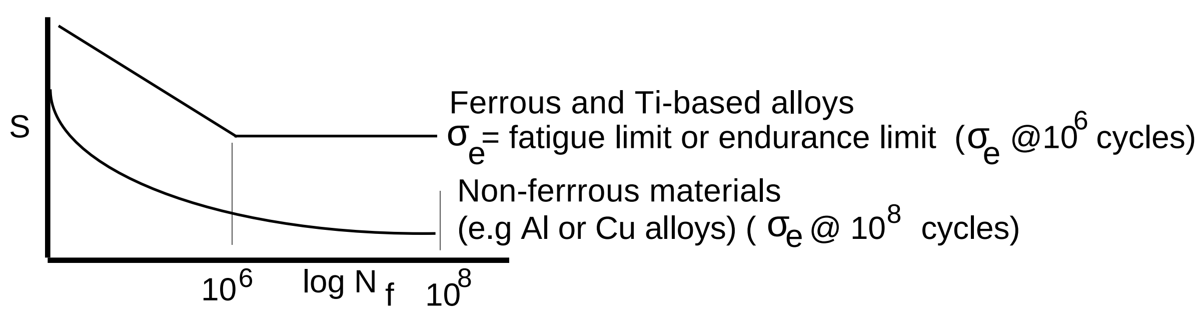
σ_a = Stress amplitude $= \dfrac{\Delta\sigma}{2} = (\sigma_{max} - \sigma_m) = (\sigma_m - \sigma_{min})$

Note: tension = $+\sigma$ and compression = $-\sigma$. Completely reversed $R = -1$, $\sigma_m = 0$.

R = Stress ratio $= \dfrac{\sigma_{min}}{\sigma_{max}}$

A = Amplitude ratio $= \dfrac{\sigma_a}{\sigma_m} = \dfrac{1-R}{1+R}$

<u>S-N Curves</u>
Stress (S)-fatigue (N_f) life curve where gross stress, S, may be presented as $\Delta\sigma$, σ_a, σ_{max}, or σ_m. High cycle $N_f > 10^5$ (sometimes 10^2-10^4) with gross stress elastic. Low cycle $N_f < 10^2$-10^4 with gross elastic plus plastic strain.

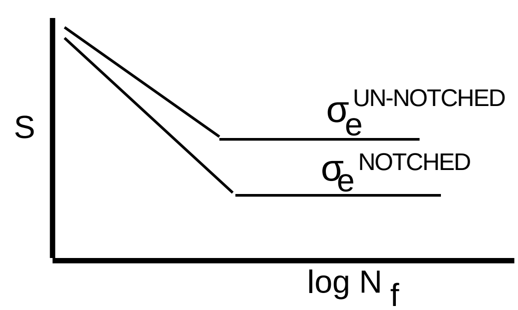


<u>Fatigue factors</u>

Recall stress concentration factor: $k_t = \dfrac{\sigma_{LOCAL}}{\sigma_{REMOTE}}$

Fatigue strength reduction factor: $k_f = \dfrac{\sigma_e^{UN-NOTCHED}}{\sigma_e^{NOTCHED}}$

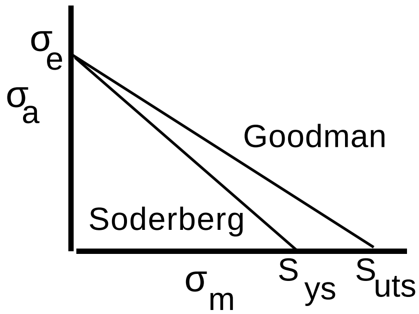
Notch sensitivity factor, $q = \dfrac{k_f - 1}{k_t - 1}$ where q=0 for no notch sensitivity, q=1 for full sensitivity.



q" as notch radius, ρ, " and q" as S_{UTS}"
Generally, $k_f \ll k_t$ for ductile materials and sharp notches but $k_f \approx k_t$ for brittle materials and blunt notches. This is due to i) steeper $d\sigma/dx$ for sharp notch so average stress in fatigue process zone is greater for the blunt notch, ii) volume effect of fatigue which is tied to average stress over larger volume for blunt notch, iii) crack cannot propagate far from a sharp notch because steep stress gradient lowers K_I quickly. In design, avoid some types of notches, rough surfaces, and certain types of loading. Compressive residual stresses at surfaces (from shot peening, surface rolling, etc.) can increase fatigue lives.
Endurance limit, σ_e is also lowered by factors such as surface finish (m_a), type of loading (m_t), size of specimen (m_d), miscellaneous effects (m_o) such that: $\sigma_e' = m_a m_t m_d m_o \sigma_e$

Note that σ_e can be estimated from the ultimate tensile strength of the material such that:
$\sigma_e \approx m_e S_{UTS}$ where m_e=0.4-0.6 for ferrous materials.

For design purposes:
Effect of mean stress for constant amplitude completely reversed stress.



Goodman: $\sigma_a = \sigma_e \left(1 - \dfrac{\sigma_m}{S_{UTS}}\right)$ Soderberg, use S_{YS} instead of S_{UTS}.

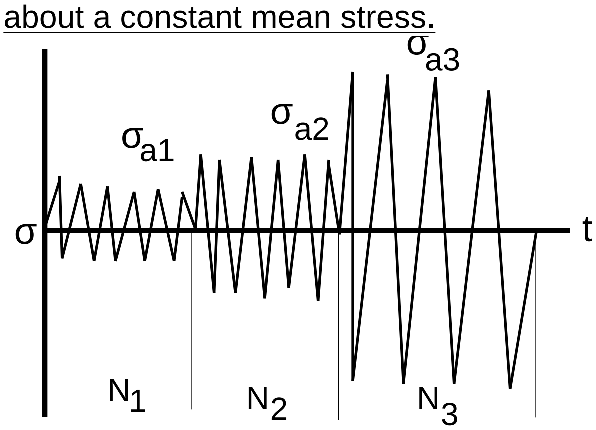
If factor of safety and /or fatigue factor are used:
For brittle materials, apply k_f to σ_e, $k_f \approx k_t$ to S_{UTS}, and FS to S_{UTS} and σ_e.

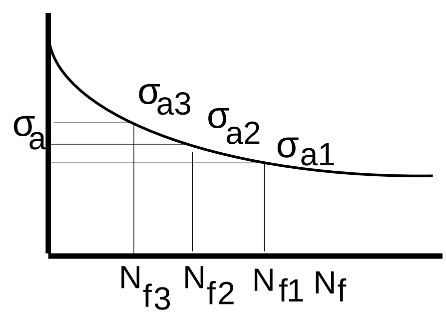
$$\sigma_a = \frac{\sigma_e}{FS \bullet k_f}\left(1 - \frac{\sigma_m}{(S_{UTS}/(k_f \approx k_t) FS)}\right)$$

For ductile materials, apply k_f to σ_e and FS to S_{UTS} and σ_e.

$$\sigma_a = \frac{\sigma_e}{FS \bullet k_f}\left(1 - \frac{\sigma_m}{(S_{UTS}/FS)}\right)$$

Effect of variable amplitude
about a constant mean stress.





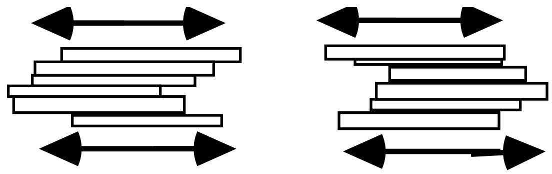
Palmgren-Miner Rule (Miner's Rule)

$$\frac{N_1}{N_{f1}} + \frac{N_2}{N_{f2}} + \frac{N_3}{N_{f3}} = \sum \frac{N_j}{N_{fj}} = 1$$

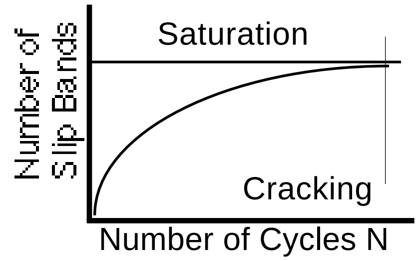
Fatigue crack growth
The fatigue process consists of 1) crack initiation, 2) slip band crack growth (stage I crack propagation) 3) crack growth on planes of high tensile stress (stage II crack propagation) and 4) ultimate failure.

Fatigue cracks initiate at free surfaces (external or internal) and initially consist of slip band extrusions and intrusions. Fatigue striations (beach marks) on fracture surfaces represent successive crack extensions normal to tensile stresses when 1 mark≈1N but $\sum$marks$\neq N_f$.

Slip bands on first loading



Slip band Intrusions and Extrusions under cyclic loading



During fatigue crack propagation (stage II may dominate) such that crack growth analysis can be applied to design: a) cracks are inevitable, b) minimum detectable crack length can be used to predict total allowable cycles, c) periodic inspections can be scheduled to monitor and repair growing cracks, d) damage tolerant design can be applied to allow structural survival in presence of cracks.

Most important advance in fatigue crack propagation was realizing the dependence of crack propagation on the stress intensity factor. Paris power law relation: $\frac{da}{dN} = C(\Delta K)^m$

For constant stress range such that $\Delta K = F(\Delta\sigma)\sqrt{\pi a}$ and F can be approximated as nearly constant over the range of crack growth. Assume m and C are constant, then:

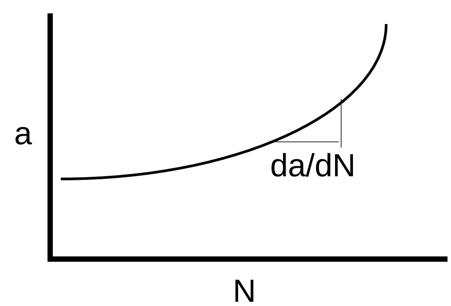
$$\int_{N_i}^{N_f} dN = \int_{a_i}^{a_f} \frac{da}{C(\Delta K)^m} = \int_{a_i}^{a_f} \frac{da}{C(F\Delta\sigma\sqrt{\pi a})^m}$$

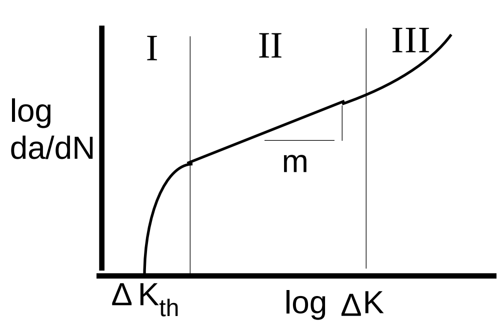
OR

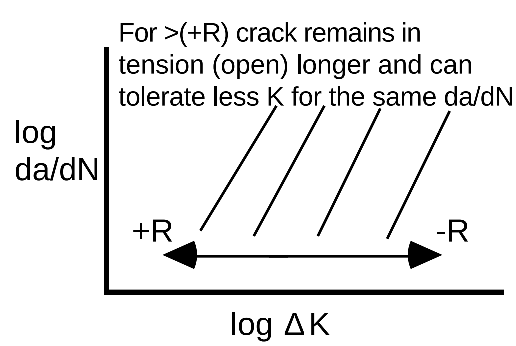
$$N_f = \frac{a_f^{(1-(m/2))} - a_i^{(1-(m/2))}}{C\left[F(\Delta\sigma)\sqrt{\pi}\right]^m\left[1-(m/2)\right]}$$

where a_i is the initial crack length which is either assumed or determined from non destructive evaluation (NDE) and

$$a_f = \frac{1}{\pi}\left(\frac{K_{Ic}}{F\sigma_{max}}\right)^2$$



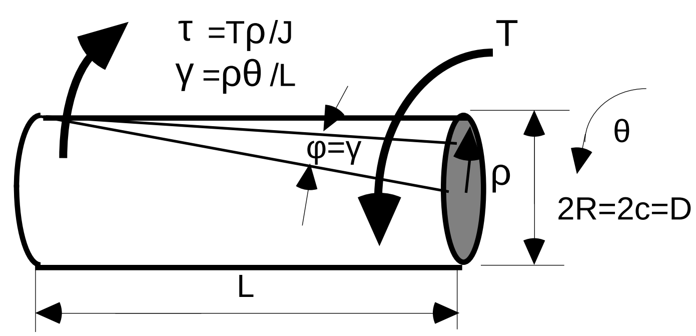


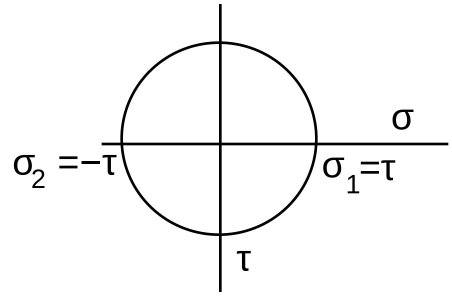


If F is a function of crack length, i.e. F(a,W, etc.), then numerical integration must be used.

$$\int_{N_i}^{N_f} dN = \int_{a_i}^{a_f} \frac{da}{C(\Delta K)^m} = \int_{a_i}^{a_f} \frac{da}{C\left[F(a,W,etc)\Delta\sigma\sqrt{\pi a}\right]^m}$$

Shafts in Torsion





2-D Mohr's Circle for Pure Torsion

Torsional Shear Stress

$$\tau = \frac{T\rho}{J} \quad \text{where } J = \text{ polar moment of intertia} = \int_{dA} \rho^2 \, dA$$

$$\tau_{max} = \frac{Tc \text{ (or } c_o)}{J}$$

Torsional Shear Strain

$$\gamma = \frac{\rho\theta}{L}$$

$$\gamma_{max} = \frac{c\theta}{L}$$

Shear Modulus : $G = \frac{d\tau}{d\gamma} = \frac{E}{2(1+\nu)}$

For linear elastic behaviour, plane sections remain plane, so $\gamma = \frac{\rho\theta}{L}$ *and* $\tau = \frac{T\rho}{J}$

Special cases

$$J = \frac{\pi D^4}{32} = \frac{\pi c^4}{2} \text{ for solid shaft}$$

$$J = \frac{\pi(D_{outer}^4 - D_{inner}^4)}{32} = \frac{\pi(c_o^4 - c_i^4)}{2} \text{ for tube}$$

Power transmission

$P = T\omega$

P =power (S.I. units, P= W = N•m/s, US Customary, P =HP = 550 ft•lb/s

T = torque

$\omega = \frac{d\theta}{dt}$ =angular velocity, rad/s (ω =RPM $\frac{2\pi}{60}$)

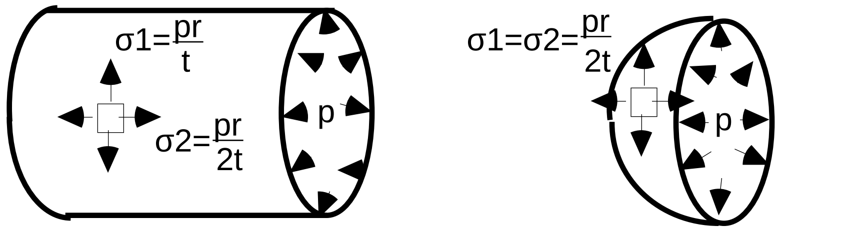
Angle of twist

$$\varphi = \int_0^L \frac{T(x)dx}{J(x)G} \text{ (in general)}$$

$$\varphi = \frac{TL}{JG} \text{ (at x =L for constant T, J, G)}$$

$$\varphi = \sum \frac{TL}{JG} \text{ (for multiple segments for different T, J, G)}$$

Pressure Vessels



Thin wall cylindrical pressure vessel

Thin wall spherical pressure vessel

Thin wall refers to a vessel with inner radius to wall thickness ratio, r/t, of greater than 10.

For cylindrical vessel with internal gage pressure only,

At outer wall, $\sigma_1 = \frac{pr}{t}$ (hoop); $\sigma_2 = \frac{pr}{2t}$ (longitudinal); $\sigma_3 = 0$ (radial),

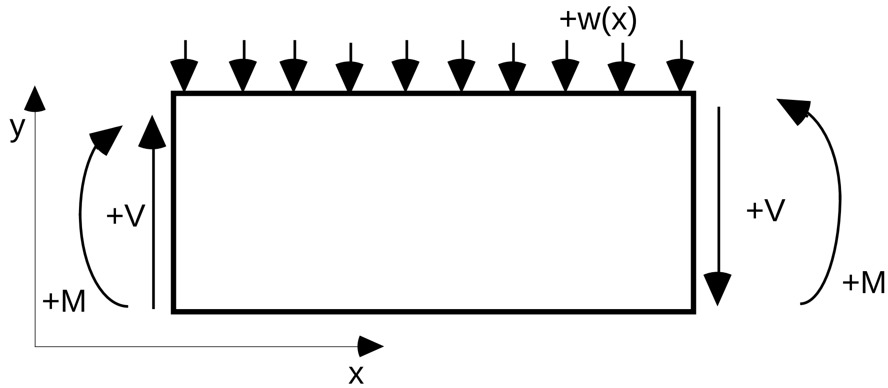
At inner wall, $\sigma_1 = \frac{pr}{t}$ (hoop); $\sigma_2 = \frac{pr}{2t}$ (longitudinal); $\sigma_3 = -p$ (radial)

For spherical vessel with internal gage pressure only,

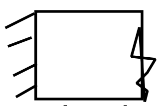
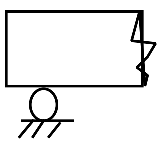
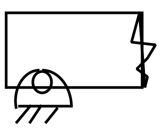
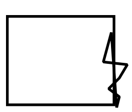
At outer wall, $\sigma_1 = \frac{pr}{2t}$ (hoop); $\sigma_2 = \frac{pr}{2t}$ (longitudinal); $\sigma_3 = 0$ (radial),

At inner wall, $\sigma_1 = \frac{pr}{2t}$ (hoop); $\sigma_2 = \frac{pr}{2t}$ (longitudinal); $\sigma_3 = -p$ (radial)

Beams



Beam Sign Convention

Support Condition	Force Reaction	Boundary Condition
 Fixed	M R	$v=0$, $\frac{dv}{dx}=0$
 Roller	M=0 R	$v=0$, $\frac{dv}{dx} \neq 0$
 Pinned	M=0 R	$v=0$, $\frac{dv}{dx} \neq 0$
 Free	R=0 M=0	$v \neq 0$, $\frac{dv}{dx} \neq 0$

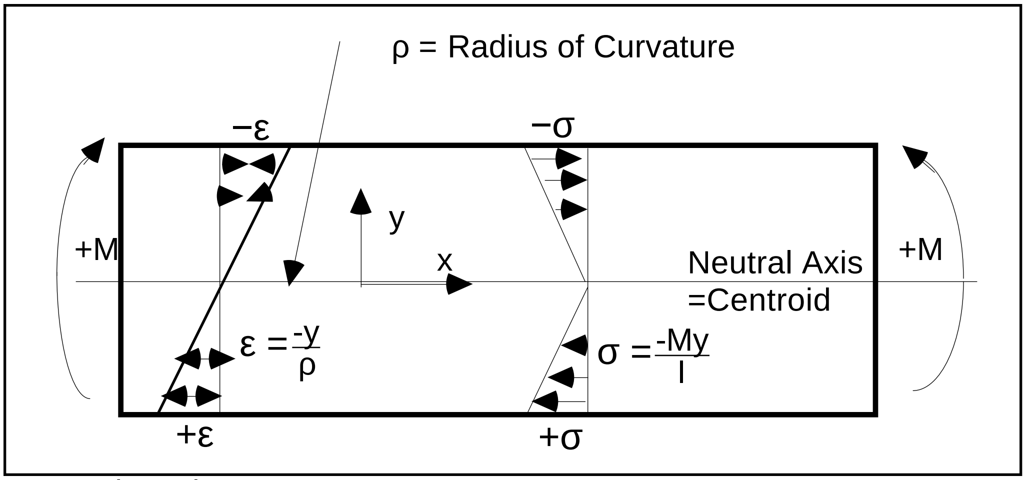
FBD, Shear Diagram and Moment Diagram

FBD: $\sum F = 0, \quad \sum M = 0$

Shear Diagram (V): $\frac{dV}{dx} = -w(x)$

Moment Diagram (M): $\frac{dM}{dx} = V$

Bending strain and stress



Normal Stress and Strain

$$\varepsilon = \frac{-y}{\rho} = -\left(\frac{y}{c}\right)\varepsilon_{max} \text{ where } \varepsilon_{max} = \frac{-c}{\rho}$$

$$\sigma = -\frac{My}{I} \text{ and } \sigma_{max} = \frac{Mc}{I}$$

y = distance from neutral axis

ρ = radius of curvature of neutral axis

c = distance from neutral axis to point furthest from neutral axis

M = bending moment

I = moment of inertia of cross section= $\int_{dA} y^2 dA$

Shear Stress

$$\tau = \frac{VQ}{It}$$

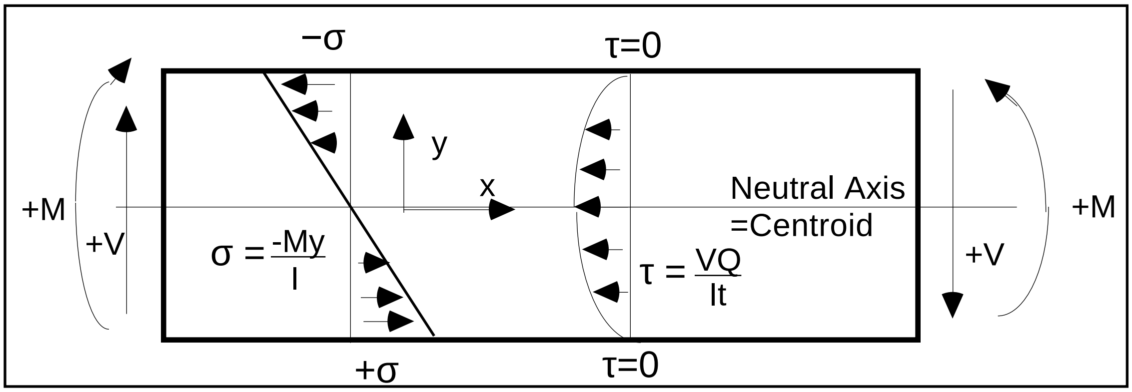
V = shear force

$Q = \int_{A'} y dA' = \bar{y}'A'$ where A'= portion of cross section

I = moment of inertia of entire cross section

t = thickness of cross section at point of interest

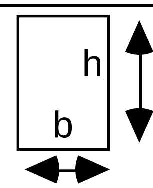
Compare normal and shear stress distributions



Special cases

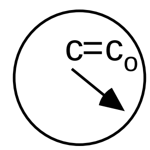
Rectangular Cross Section

$$I = \frac{bh^3}{12} \qquad \sigma_{max} = \frac{6M}{bh^2} \qquad \tau_{max} = \frac{3V}{2A} = \frac{3V}{2(bh)}$$



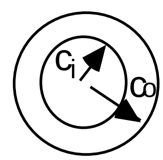
Circular Cross Section

$$I = \frac{\pi c^4}{4} \qquad \sigma_{max} = \frac{2M}{\pi c^3} \qquad \tau_{max} = \frac{4V}{3A} = \frac{4V}{3(\pi c^2)}$$



Tubular Cross Section

$$I = \frac{\pi(c_o^4 - c_i^4)}{4} \qquad \sigma_{max} = \frac{2Mc_o}{\pi(c_o^4 - c_i^4)} \qquad \tau_{max} = \frac{2V}{A} = \frac{2V}{\pi(c_o^2 - c_i^2)}$$



Beam Deflections

Moment Curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

Equations for Elastic Curve

$$EI\frac{d^4v}{dx^4} = -w(x)$$

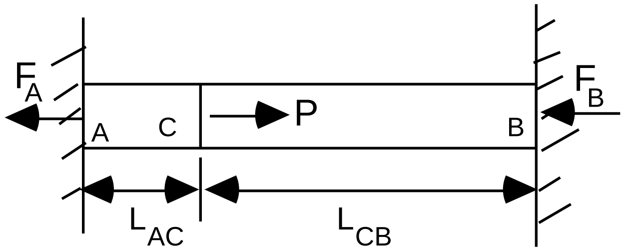
$$EI\frac{d^3v}{dx^3} = V(x)$$

$$EI\frac{d^2v}{dx^2} = M(x)$$

Need to integrate equations for elastic curve for find v(x) and dv(x)/dx in terms of M(x), V(x), w(x), and constants of integration. The specific solution for the elastic curve is then found by applying the boundary conditions. Note that v=dv/dx=0 for fixed support, v=0 but dv/dx≠0 for simple support, and v=max or min when dv/dx=0 at maximum moment (i.e. inflection point).

Statically Indeterminate

Axially-Loaded Members



$\sum F = 0$ so $-F_A - F_B + P = 0$

But F_A and F_B are unknown

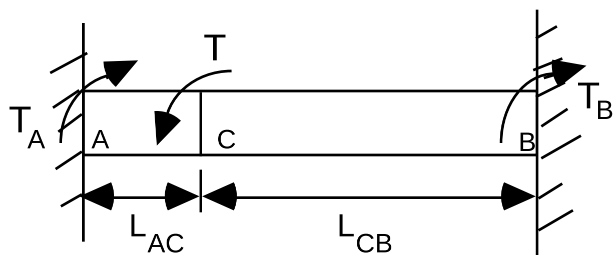
so

Use load-displacement relation and compatibility

at the common point C

$$\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0$$

Torsionally-Loaded Members



$\sum M = 0$ so $-T_A - T_B + T = 0$

But T_A and T_B are unknown

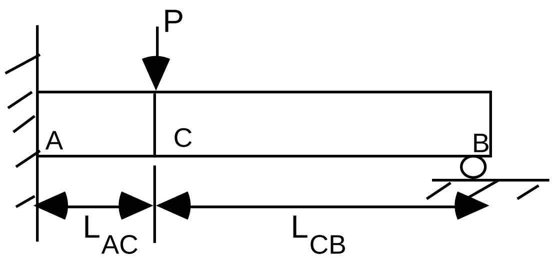
so

Use torque-twist relation and compatibility

at the common point C

$$\frac{T_A L_{AC}}{JG} - \frac{T_B L_{CB}}{JG} = 0$$

Beams



$\sum M = 0$ and $\sum F = 0$

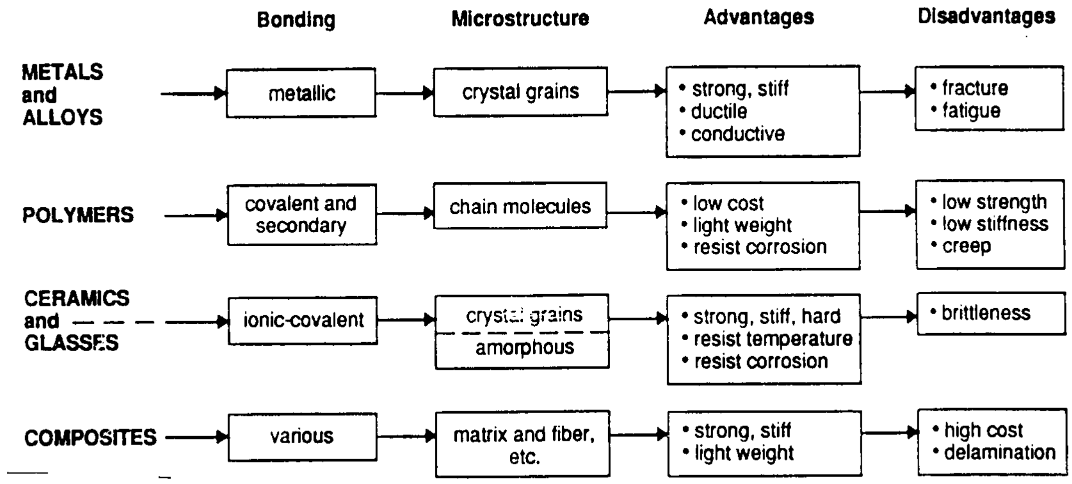
But there are additional supports not needed for stable equilibrium which are redundants and determine the degree of indeterminacy so

First determine redundant reactions, then use compatibility conditions to determine redundants and apply these to beam to solve for the remaining reactions using equilibrium

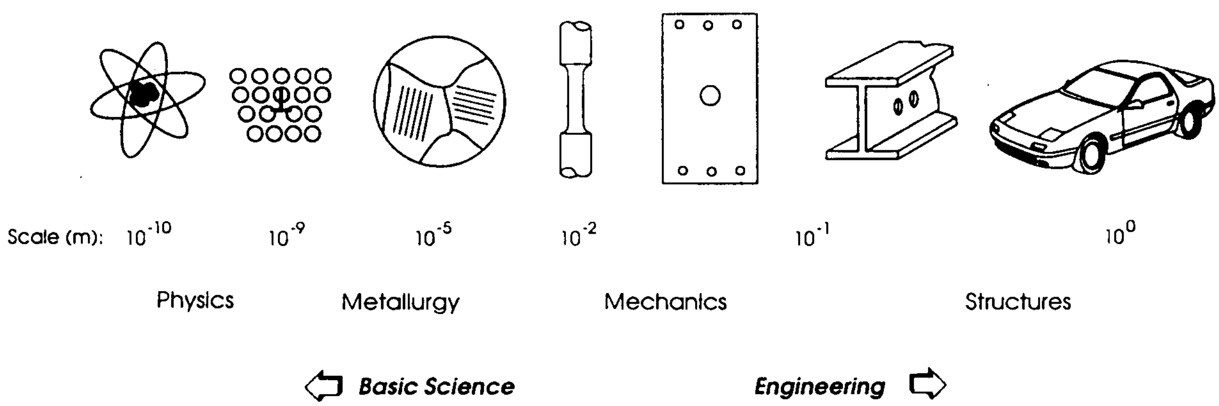
If use method of integration, integrate the differential equation, $\frac{d^2v}{dx^2} = \frac{M}{EI}$ twice to find the internal moment in terms of x (i.e., M(x)).

The redundants and constants of integration are found from the boundary conditions.

Engineering Materials

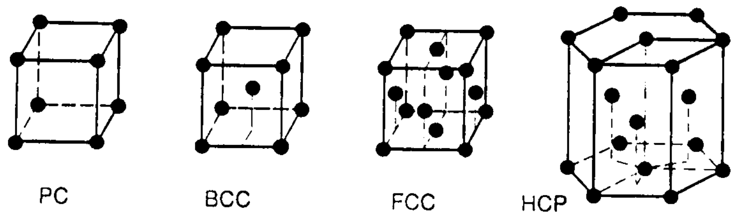


Classes and various aspects of engineering materials.

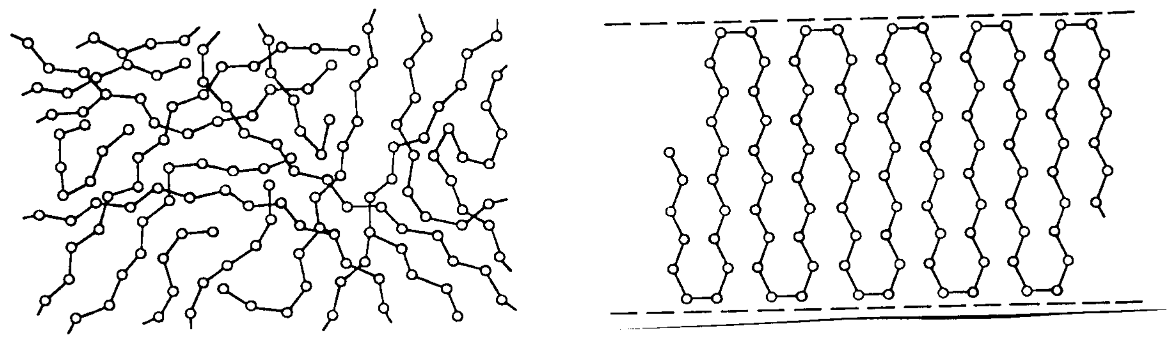


Size scales and disciplines involved in the study of engineering materials.

Crystals, structures, defects and dislocations, theoretical strength

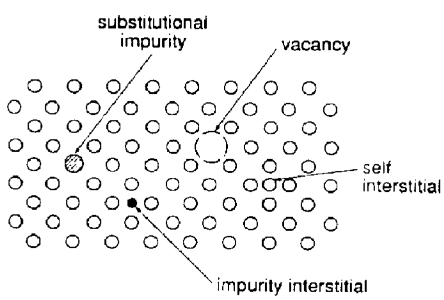


Four common crystal structures: (primitive) cubic, body-centered cubic, face-centered cubic, and hexagonal close packed.

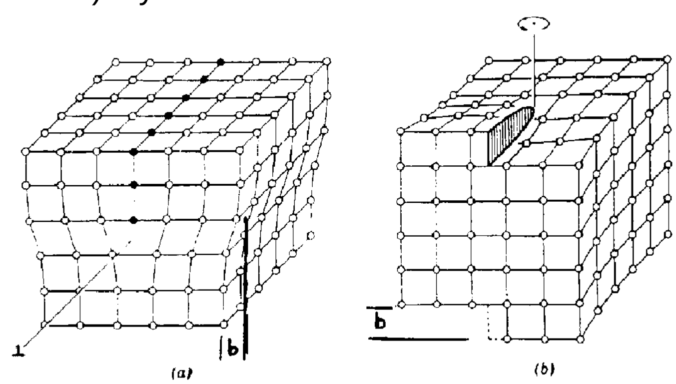


a) amorphous **b) crystalline**

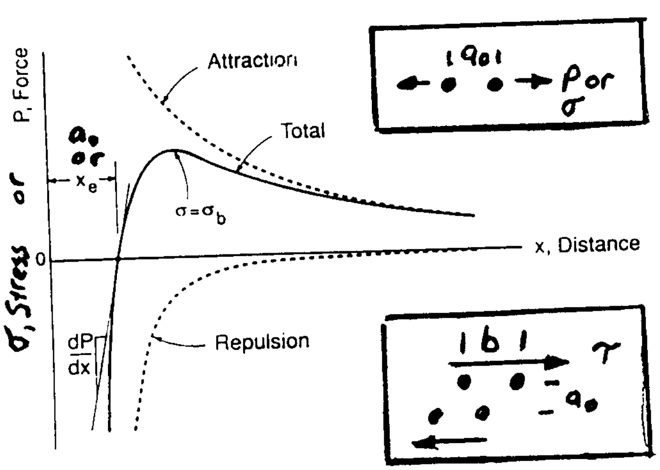
Examples of a) amorphous (without form) and b) crystalline structures



Types of point defects



Types of line defects (dislocations)
[a) edge dislocations and b) screw dislocations]



Maximum Cohesive Strength $\sigma_{max} = \frac{E}{\pi} \approx \frac{E}{10}$ $\frac{\text{Upper}}{\text{Bound}}$

$\sigma_{max} = \sqrt{\frac{E\gamma_s}{a_0}}$ $\frac{\text{Lower}}{\text{Bound}}$

Maximum Shear Stress at Slip $\tau_{max} = \frac{Gb}{2\pi a_o}$

Strengthening Mechanisms

Grain Boundary Strengthening

Mechanism: GB is region of disturbed lattice with steep strain gradients

High angle = high fracture energy plus diffusion sites

Low angle = edge dislocations climb

T_{eq} is equicohesive temp where GB is weaker than grain and d is the grain diameter.

Result: At R.T. As d then H and S_{UTS} AND as d then H and S_{UTS} such that

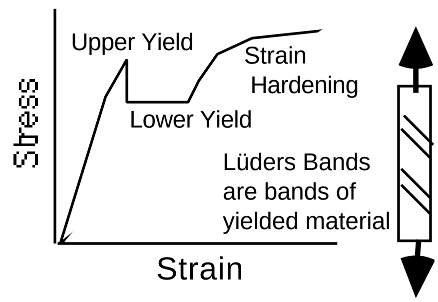
$\sigma_o = \sigma_i + kd^{-1/2}$ (Hall-Petch Eq. where σ_o is yield stress, σ_i is friction stress and k is the"locking" parameter

At H.T. If $T>T_{eq}$ as d then S_{UTS} BUT if $T<T_{eq}$ as d then S_{UTS}

Yield Point Phenomenon

Mechanism: Lüders bands of yielded and unyielded material with C and N atoms forming atmospheres (interstitials) to pin dislocations and forcing new dislocations to form.

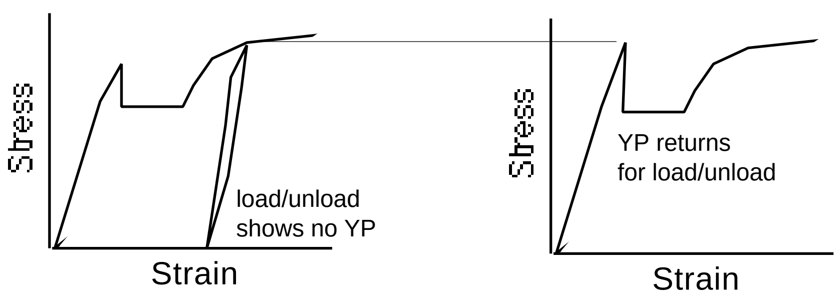
Result: Upper yield point followed by lower yield point before strain hardening.



Strain Aging

Mechanism: C and N atoms form atmospheres (interstitials) to pin dislocations and forcing new dislocations to form BUT diffusion of interstitials can repin dislocations.

Result: Upper yield point and lower yield point return even if material is strain hardening.



At R.T., No strain age and no YP

Aged at T or after days at R.T., YP returns

Solid Solution Strengthening

Mechanism: Atomic-level interstitial and substitutional solute atoms provide resistance to dislocation motion as dislocations bend around regions of high energy.

Result: Level of stress strain curve increases and yield strength increases.

Two Phase Aggregates

Mechanism: Microstructural-level solid solution (dispersed structure) or particulate additions (aggregated structures). Super saturation of particles in a matrix where hard particles block slip in a ductile matrix and localized strain concentration raise yield strength due to plastic constraint.

Result: Yield strength increases, hardness increases

Bounds on properties: Isostrain: $\varepsilon_m=\varepsilon_p=\varepsilon_c$ so $\sigma_c=V_p\sigma_p+V_m\sigma_m$

Isostress: $\sigma_m=\sigma_p=\sigma_c$ so $\varepsilon_c=V_p\varepsilon_p+V_m\varepsilon_m$

Strengthening Mechanisms (cont'd.)

Fiber Strengthening

Mechanism: Discrete fibers carry load and directional properties "toughen" composite. Discrete matrix transmits load to fibers and protects fibers.

Result: High strength to weight ratio, directional properties

Bounds on properties: Isostrain: $\varepsilon_m=\varepsilon_p=\varepsilon_c$ so $\sigma_c=V_p\sigma_p+V_m\sigma_m$

Isostress: $\sigma_m=\sigma_p=\sigma_c$ so $\varepsilon_c=V_p\varepsilon_p+V_m\varepsilon_m$

Martensite Strengthening

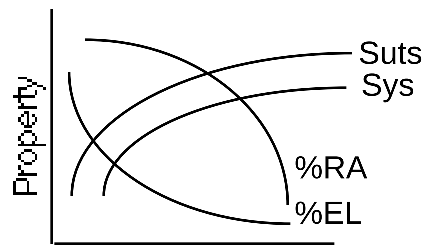
Mechanism: Fine structure and high dislocation density provide effective barriers to slip with C atoms strongly bound to dislocations and restrict dislocation motion.

Result: Hardness and strength increase

Cold Working

Mechanism: Strain hardening due to dislocations interacting with barriers and other dislocations to impede slip. As number of dislocations increase the resistance to slip increases (toughness increases)

Result: Energy required for plastic deformation increases with increasing cold work. Strength increases and ductility decreases.



Strain Hardening

Mechanism: Mutual obstruction of dislocations on intersecting slip systems through interaction of stress field aid interpenetration of slip systems both of which produce higher internal energy.

Result: Hardens alloys which do not heat treat harden. The "rate" of strain hardening is the slope of the flow curve (true stress - true strain curve). Tensile behaviour increases, density decreases (~0.2%), electrical conductivity decreases, thermal coefficient increases, chemical reactivity increases.

Annealing of Cold Work

Mechanism: Hold at elevated temperature to cause annealing.

Recovery - short time - restores physical properties without change in microstructure.

Recrystallization - longer time - cold worked microstructure is replaced with new sets of strain free grains.

Grain growth - longest time - progressive increase in size of strain free grains.

Result: High internal energy due to cold work is relieved - material reverts to strain free condition. Cold working is mechanically stable (shape) but not thermodynamically so annealing restores ductility while retaining shape changes of part.

Texture (Preferred Orientation)

Mechanism: Crystallographic fibering with reorientation of grains during deformation (e.g. extrusion, rolling, etc.) Mechanical fibering with alignment of inclusions, cavities, and secondary phases.

Result: Anisotropy of mechanical properties (generally enhanced in texture direction)