Lecture #1

user name: me431/599
password: fluid$

want to solve equations to get the flow field.

I/E Equation of motion

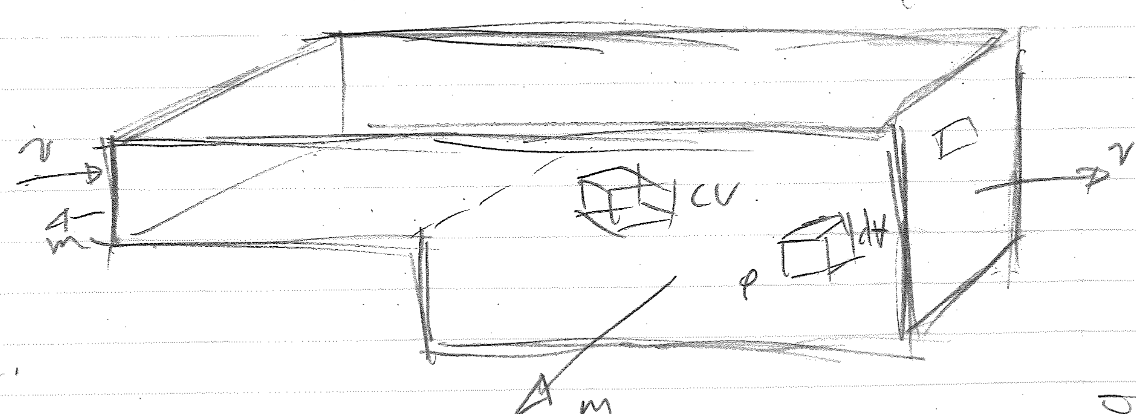
solve
1) conservation of mass
2) momentum balance
3) conservation of energy
4) fluid/gas law.

I) conservation of mass

Consider a control volume, fixed volume in space through which fluid flows

small v = velocity

big V = volume.



[rate of increase of mass in CV] = [net rate at which mass enters / leaves]

valid for any control volume

$$\Rightarrow \frac{d}{dt}\iiint_V \rho \, dV = -\iint_A \rho \, v \cdot n \, dA$$

minus negative to indicate outward normal

$d\forall$: differential volume
$dx\,dy\,dz$ or
$dr\,d\theta\,dz$

$dA = dx\,dy$

$\rho = \rho(\underbrace{x, y, z}_{\underset{\sim}{x}}, t) = \rho(\underset{\sim}{x}, t)$ fluid density

↳ can write with position vector

fluid velocity $\underset{\sim}{v} = [u(\underset{\sim}{x}, t), v(\underset{\sim}{x}, t), w(\underset{\sim}{x}, t)]$

$\underset{\sim}{n}$: unit normal vector.

⇒ Consider small differential volume element:

CV fixed in space ⇒ does <u>not</u> depend on time.

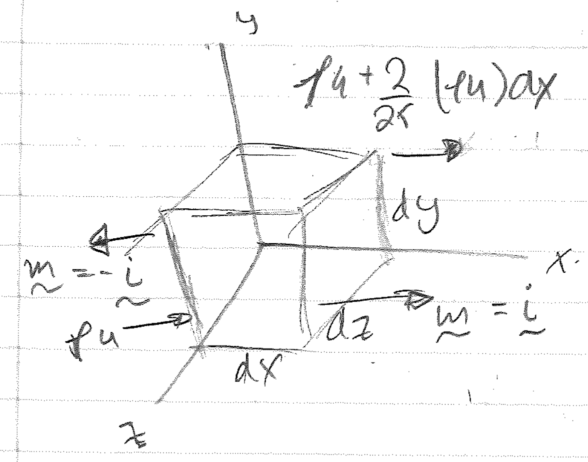
↳ $\frac{d}{dt}\iiint_{CV} \rho(\underset{\sim}{x}, t)\,dv = \iiint_{CV} \frac{\partial \rho}{\partial t}(\underset{\sim}{x}, t)\,d\forall$

(Le theorm)

$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t}\Big|_{\underset{\sim}{x}\ fix}$ time dependent
to point $\underset{\sim}{x}$ fixed

CV is very small ⇒ $\frac{\partial \rho}{\partial t}$ is constant.

$\iiint \frac{\partial \rho}{\partial t}\,d\forall = \frac{\partial \rho}{\partial t}\Big|_{xyz}$ $dx\,dy\,dz$



the net mass flux has contribution from all 6 faces.

Surfaces moved along the x-direction

left control surface.

$$-\iint_{cs} \rho(\underset{\sim}{v}\cdot\underset{\sim}{m})\,dA = +\rho u\Big|_x\, dz\, dy$$

R control surface

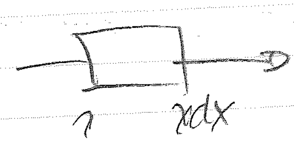
$$-\iint \rho(\underset{\sim}{v}\cdot\underset{\sim}{m})\,dA = -\rho u\Big|_{x+dx}\, dz\, dy.$$

↳ complete Taylor series expansion

$$(\rho u)\Big|_{x+dx} = (\rho u)\Big|_x + \frac{\partial}{\partial x}(\rho u)_x\, dx + HOT$$

The net contribution from L/R surfaces.

$$-\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz + HOT$$

e.g. 

$\frac{\partial}{\partial x}(\rho u) > 0$

$(\rho u)_{x+dx} > (\rho u)_x$

then

$\frac{d}{dt}(\ldots) = -\partial$

same argument for front + back faces.

$-\frac{\partial}{\partial y}(\rho u)\,dz\,dy\,dt$

top + bottom

$-\frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dt$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Differential form of the conservation of mass

$\equiv$ continuity equation

↳ for polar continuity see moodle pdf

introduce del operator $\nabla \equiv$

$$\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underset{\sim}{v} = 0$$

makes eq more compact.

(more modern) Material derivative: $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla$

$$\nabla \cdot (\rho \underset{\sim}{v}) = (\underset{\sim}{v} \cdot \nabla)\rho + \rho(\nabla \cdot \underset{\sim}{v})$$

show that $\boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot \underset{\sim}{v}) = 0}$

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consider incompressible flow $\rightarrow$ flow for which ρ = constant independent of x and time

$\Rightarrow \nabla \cdot \underset{\sim}{v} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Mach number $M = \frac{v}{a} < 0.2$

v = characteristic velocity
a = velocity of sound

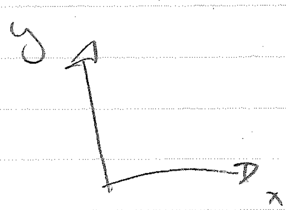
→ e.g. air $a = 356$ m/s $= 760$ mph

$M = 0.13$.

hurricane 150 mph $M = 0.2$ marginal

→ e.g. 2D incompressible flow

neglect z variation $\left(\frac{\partial}{\partial z} = 0\right)$

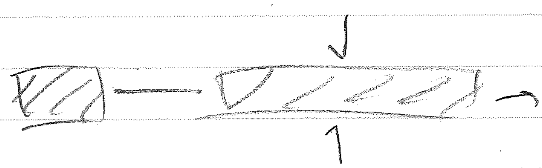


$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$ [illegible]

if $\frac{\partial u}{\partial x} > 0$ locally then $\frac{\partial v}{\partial y} < 0$

to conserve mass

$\rightarrow \;, \rightarrow \;\; \frac{\partial u}{\partial x} > 0$



→ e.g. $u = \alpha x^2$ ($\alpha > 0$, constant)

$$\frac{\partial u}{\partial x} = 2\alpha x = -\frac{\partial v}{\partial y} \rightarrow v = -2\alpha xy + f(x)$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underset{\sim}{v} = 0 \qquad \rho(x, y, z, t)$$

$$\nabla \cdot \underset{\sim}{v} = 0$$

For 2D flow (x, y) (r, θ)

Smits 6.3

also useful for stream lines

Incompressible.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

introducing stream function $\psi(x, y)$

definition $\to$ $u = \frac{\partial \psi}{\partial y}(x, y)$ $\qquad v = \frac{-\partial \psi}{\partial x}$ $\leftarrow$ definition

take continuity equation

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

as long as you are solving for ψ don't have to check that continuity holds.

if ψ exists:

1) Solve 1 unknown ψ instead of 2 (u, v)

2) Garuntee the conservation of mass.

Streamline: line which is tangent to the velocity vector.

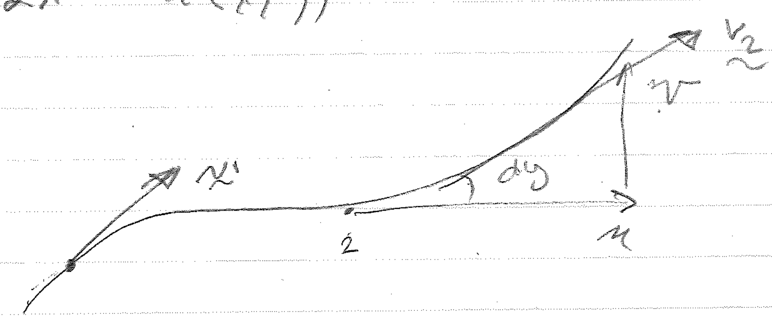
Streamline is at a fixed time (t).

line // to the local velocity

$$\frac{dy}{dx} = \frac{v(x,y)}{u(x,y)}$$

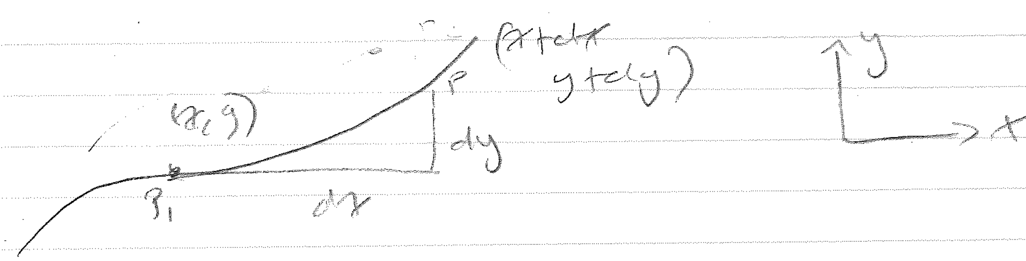
Smits 4.1.1

(chapter 4.1)



Consider 2D, incompressible such as ψ exist.

- 2 points close together along a streamline.



Taylor expansion

The value @ P_2 $\psi(x+dx, y+dy) =$

$$\psi(x,y) + \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$= \psi(x,y) - v\,dx + u\,dy$$

the streamline is such that

$$\frac{dy}{dx} = \frac{v}{u} \Rightarrow u\,dy = v\,dx$$

$\psi(x+dx, y+dy) = \psi(x,y)$ then

$\psi = \text{const}$ (if we can find this then we can just plot constant values of the streamline.)

if $\psi(x,y)$ const, $\psi = \text{const} \equiv$ streamline

↳ in Matlab could do countour plot

The velocity can change but the stream function doesn't change.

↳ this is only about the direction of the field NOT the magnitude e.g. $u = -\alpha x = \frac{\partial \psi}{\partial y}$

$v = +\alpha x = -\frac{\partial \psi}{\partial x}$

$\Rightarrow$ integrate u over y

$$\psi(x,y) = \int u\,dy = \int -\alpha x\,dy$$

$$= -\alpha x y + f(x)$$

$$\frac{\partial(-\alpha x y + f(x))}{\partial y} = -\alpha x + 0$$

$$\psi(x,y) = -\int v\,dx$$

$$= -\int \alpha y\,dx$$

$$= -\alpha x y + g(y)$$

$\downarrow$ put them together.

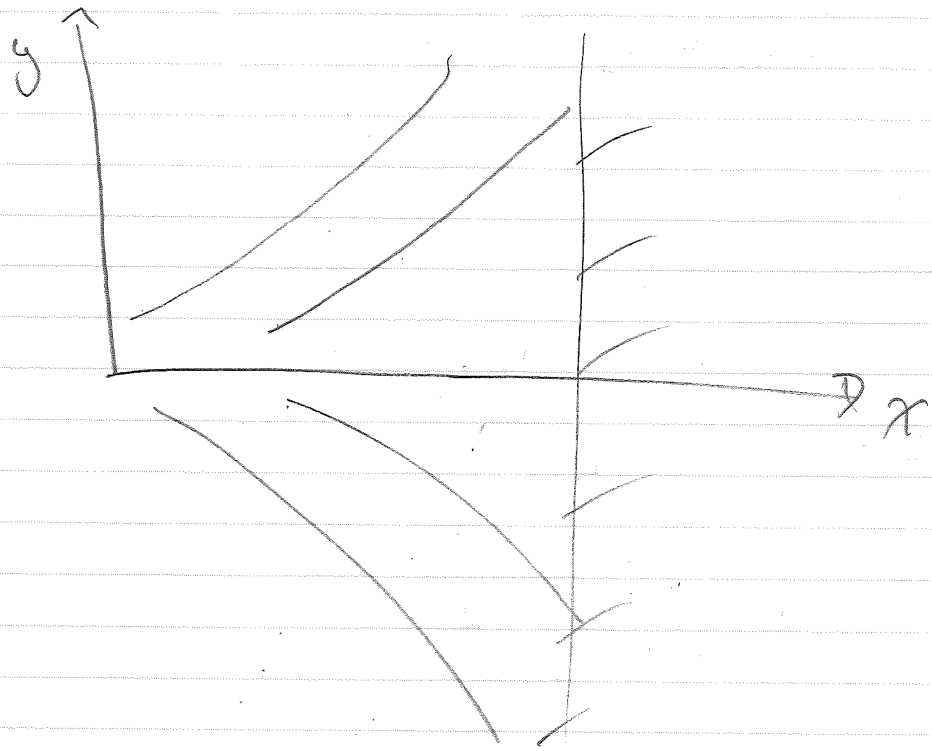
$$f(x) = g(y) = C_1$$

Streamline $\psi = \text{const} = C_2$

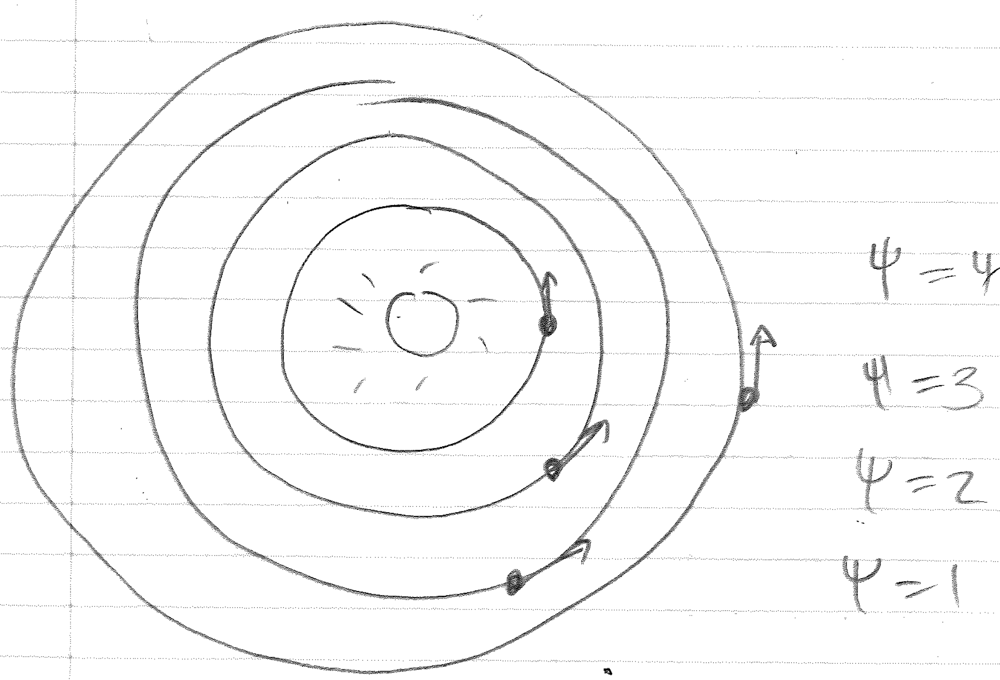
$$-\alpha x y + C_1 = C_2$$

$$\alpha x y = C_1 - C_2 = C \Rightarrow \boxed{y = \frac{C}{\alpha}\frac{1}{x}}$$

$y(x)$



recirculation



[Streamlines don't cross]