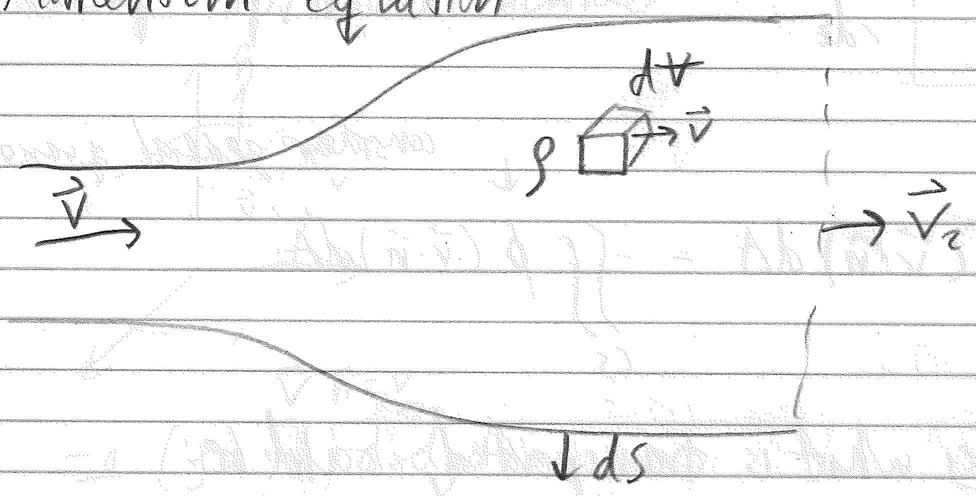
Smits 5.3

lecture 3 Oct 1, 2018

Review: $u = \frac{\partial \psi}{\partial y}$ $v = -\frac{\partial \psi}{\partial x}$

$\frac{dy}{dx} = \frac{v(x,y)}{u(x,y)}$ $\Rightarrow$ derivative of the streamline, which is tangent the velocity field

Momentum equation



Newton's 2nd law applied to fluids

$\begin{bmatrix} \text{rate of change of} \\ \text{momentum in a CV} \end{bmatrix} = \begin{bmatrix} \text{sum of the} \\ \text{forces applied to CV} \end{bmatrix} + \begin{bmatrix} \text{net rate of} \\ \text{momentum in} \\ \text{control surface} \end{bmatrix}$

$$\frac{d}{dt}\iiint_{CV} \rho \vec{v}\, d\forall = \sum_i F_i - \iint_{CS} \rho \vec{v} \underbrace{(\vec{v}\cdot\hat{n})}_{\text{scalar}} dA$$ 3 eqn $(\hat{i}, \hat{j}, \hat{k})$

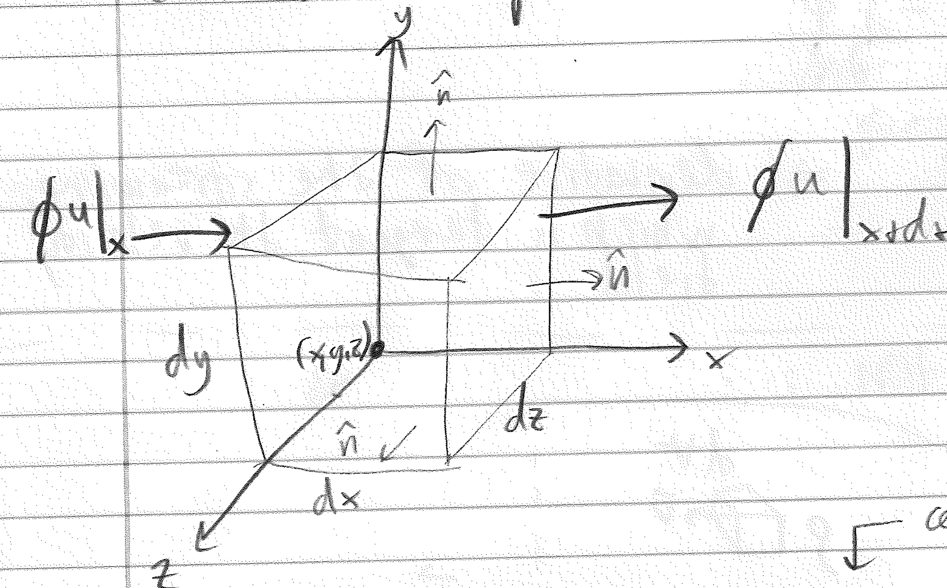
Evaluate for infinitesimal CV

$$\frac{d}{dt}\iiint \rho \vec{v}\, d\forall = \iiint \frac{d}{dt} \rho \vec{v}\, d\forall = \frac{\partial}{\partial t} \rho \vec{v}\, dx\,dy\,dz$$

CV indep. of time

$\frac{\partial(\rho \vec{v})}{\partial t}$ = constant in CV

Consider x-component:



consider general quantity ϕ

$$-\iint_{CS} \rho u (\vec{v}\cdot\hat{n})\, dA = -\iint_{CS} \phi\, (\vec{v}\cdot\hat{n})\, dA$$

ϕ describes what is transported, could be:

$\phi = \rho$ mass
$\phi = \rho u$ momentum
$\phi = \rho e$ energy where e = internal energy/vol.

at x: $-\phi(\vec{v}\cdot\hat{n}) = -\phi(-u) = \phi u$

at $x+dx$: $-\phi(\vec{v}\cdot\hat{n}) = -\phi(u) = -\phi u$

If $\phi u|_{x+dx} = \phi u|_x \Rightarrow$ no change w/in C.V.

Net effect: left/right $-\iint_{CS} \phi(\vec{v}\cdot\hat{n})\, dA = \phi u|_x\, dy\,dz - \phi u|_{x+dx}\, dy\,dz$

$f(x) = f(a) + \frac{df}{dx}(a+\Delta x)$

1st term: $\frac{df}{dx} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$

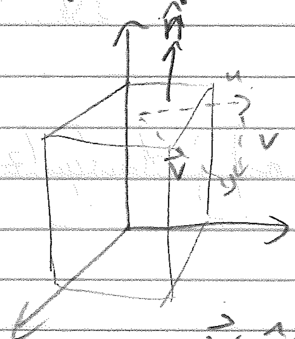
Taylor expansion

$$= -\frac{\partial(\phi u)}{\partial x}\, dx\,dy\,dz$$

x-comp. momentum $\phi = \rho u$

$$-\frac{\partial}{\partial x}(\rho u \cdot u)\, dx\,dy\,dz$$

x-comp top/bottom surfaces:



$\vec{V} \cdot \hat{n} = v$

$$-\frac{\partial}{\partial y}(\rho u \cdot v)\, dx\,dy\,dz$$

x-comp back + front surfaces:

$$-\frac{\partial}{\partial z}(\rho u \cdot w)\, dx\,dy\,dz$$

Total for 6 surfaces:

$$-\left[\frac{\partial}{\partial x}(\rho u \cdot u) + \frac{\partial}{\partial y}(\rho u \cdot v) + \frac{\partial}{\partial z}(\rho u \cdot w)\right] dx\,dy\,dz$$

y-momentum: $\phi = \rho v$

y comp, x surfaces:

$$-\frac{\partial}{\partial x}(\rho v \cdot u)\, dx\,dy\,dz$$

$$\Rightarrow -\left[\frac{\partial}{\partial x}(\rho v u) + \frac{\partial}{\partial y}(\rho v \cdot v) + \frac{\partial}{\partial z}(\rho v \cdot w)\right] dx\,dy\,dz$$

z-momentum: $\phi = \rho w$

$$\Rightarrow -\left[\frac{\partial}{\partial x}(\rho w \cdot u) + \frac{\partial}{\partial y}(\rho w \cdot v) + \frac{\partial}{\partial z}(\rho w \cdot w)\right] dx\,dy\,dz$$

In vector form, the final result is:

$$-\left[\frac{\partial}{\partial x}(\rho \vec{V} \cdot u) + \frac{\partial}{\partial y}(\rho \vec{V} \cdot v) + \frac{\partial}{\partial z}(\rho \vec{V} \cdot w)\right] dx\,dy\,dz$$

Moving this term to the LHS of mom. balance:

$$\left[\frac{\partial}{\partial t}(\rho \vec{V}) + \frac{\partial}{\partial x}(\rho \vec{V} \cdot u) + \frac{\partial}{\partial y}(\rho \vec{V} \cdot v) + \frac{\partial}{\partial z}(\rho \vec{V} \cdot w)\right] d\forall$$

chain rule

$$= \vec{V}\overbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]}^{\to 0} d\forall$$ ← conservation of mass

$$+ \rho\left[\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right] d\forall$$ ← material derivative

$$= \rho\frac{D\vec{V}}{Dt}\, d\forall \text{ where } \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Note! $\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\Big|_{\vec{x} \text{ fixed}}$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\Big|_{y, z, t, \text{ fixed}}$$

Momentum balance:

$$\rho \frac{D\vec{v}}{Dt} d\forall = \sum_i F_i$$