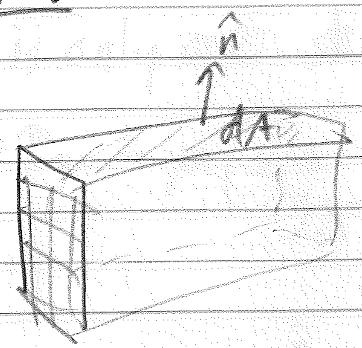
lecture 8

Oct 11, 2018



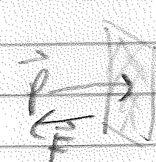
shear force:

tensor rank 2

vector

$$\vec{F}_v = -\iint_A \underline{\underline{\tau}} \cdot \hat{n}\, dA$$ gives a vector

$$\vec{F}_p = \left(\iint -p\, dA\right) \hat{n}$$



$\sum F = 0 \Rightarrow$ no acceleration, steady state

$$Q = \iint_A \vec{v} \cdot \hat{n}\, dA$$

Channel: $Q = \int_{-h}^{h} u\, dy$

$$= V_m \int_{-h}^{h} \left(1 - \frac{y^2}{h^2}\right) dy$$

$s = \frac{y}{h}$

$h s = dy$

$$= V_m h \int_{-h}^{h} (1 - s^2)\, dy$$

$$= V_m h \left[s^2 - \frac{s^3}{3} \right]_{-h}^{h} = V_m h \left(1 - (-1) + \frac{1}{3} - \left(\frac{-1}{3}\right)\right)$$

$$= \frac{4}{3} V_m h = \boxed{\frac{2}{3} \frac{h^3}{\mu} \left|\frac{dP}{dx}\right|}$$

$$\left|\frac{dP}{dx}\right| = \frac{3}{2} Q \frac{\mu}{h^3}$$

$\frac{dP}{dx} \propto Q$ $\qquad$ $\frac{dP}{dx} \uparrow$ $Q \uparrow$

$\frac{dP}{dx} \propto \mu$ $\qquad$ $\mu \uparrow$ $\frac{dP}{dx} \uparrow$

$\frac{dP}{dx} \propto \frac{1}{h^3}$ $\qquad$ $h \downarrow$ $\frac{dP}{dx} \uparrow$

average velocity: $\langle u \rangle = \frac{Q}{2h} = \frac{1}{3} \frac{h^2}{\mu} \left|\frac{dP}{dx}\right| = \frac{2U_m}{3}$

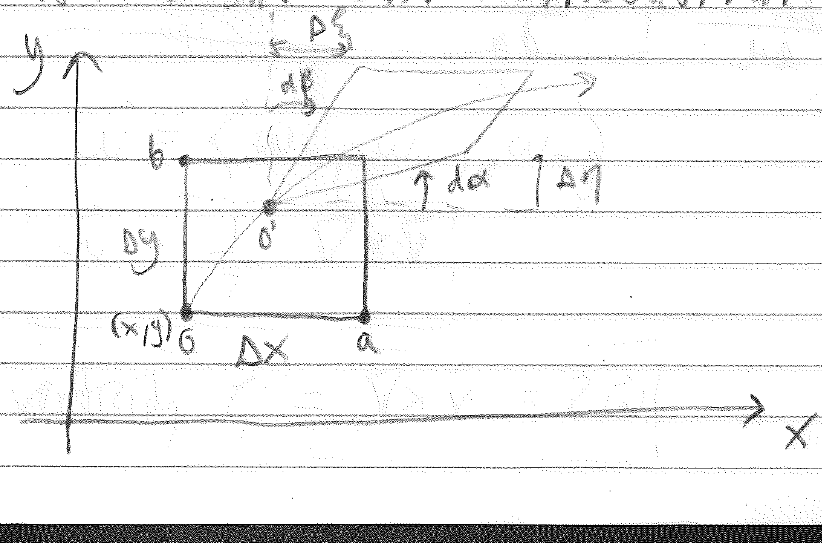
$$= \frac{\int_{-h}^{h} u \, dy}{h-(-h)}$$

Vorticity $\qquad$ Smits 6.1

→ describes rotation of a fluid element (vortex)

→ can be used in turbulent flow

→ if it doesn't exist → irrotational



1. rotation
2. translation
3. deformation (strain rate)

At time t, velocity at $(x,y) = (u_0, v_0)$

During Δt, the fluid particle moves from O to O' with coordinates $(x + u_0 \Delta t,\ y + v_0 t)$

Simultaneously, the lines $\overline{OA}$ and $\overline{OB}$ rotates around O

The rotation rate of $\overline{OA}$ is

$$\omega_{OA} = \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t}$$

$$\frac{\Delta \eta}{\Delta x} = \tan(\Delta \alpha) \approx \Delta \alpha \qquad \text{(small angle)}$$

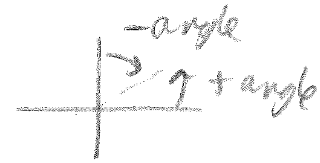
$$V_A = V_0 + \frac{\partial v}{\partial x} \Delta x$$

$$\Delta \eta = \frac{\partial v}{\partial x} \Delta x \Delta t$$

$$\Delta \eta = (V_A - V_0) \Delta t$$

$$\Delta \alpha = \frac{\Delta \eta}{\Delta t} = \frac{\partial v}{\partial x} \Delta t$$

$$\omega_{OA} = \lim_{\Delta t \to 0} \frac{\frac{\partial v}{\partial x} \Delta t}{\Delta t} = \frac{\partial v}{\partial x}$$



$$\omega_{OB} = \lim_{\Delta t \to 0} \frac{-\Delta \beta}{\Delta t}$$

$$\frac{\Delta \xi}{\Delta y} = \tan(\Delta \beta) \approx \Delta \beta$$

$$u_p = u_0 + \frac{\partial u}{\partial y} \Delta y$$

$$\Delta \xi = (u_p - u_0)\Delta t$$

$$\Delta \xi = \frac{\partial u}{\partial y} \Delta y \Delta t$$

$$\lim_{\Delta t \to 0} \frac{-\frac{\partial u}{\partial y}\Delta t}{\Delta t} = -\frac{\partial v}{\partial y}$$

$$\boxed{\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$$ average of two terms
xy plane → z comp. ω

$$\boxed{\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}$$ yz plane → x comp. ω

$$\boxed{\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}$$ xz plane → y comp. ω

$\partial/\partial z$	$\partial/\partial y$	$\partial/\partial z$
u	v	u

$$\vec{\omega} = \langle \omega_x, \omega_y, \omega_z \rangle$$

$$\boxed{\vec{\omega} = \frac{1}{2} \nabla \times \vec{V}}$$

$$\boxed{\text{vorticity } \zeta = \nabla \times \vec{V} = 2\vec{\omega}}$$

The fluid element

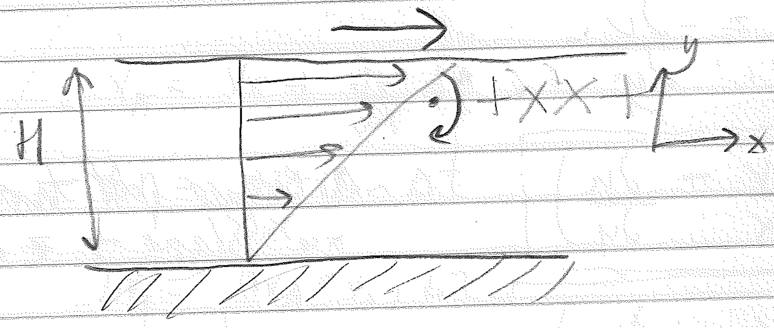
1) advects (u_0, v_0)

2) rotates $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

3) deforms $\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

Examples

Plane Couette flow

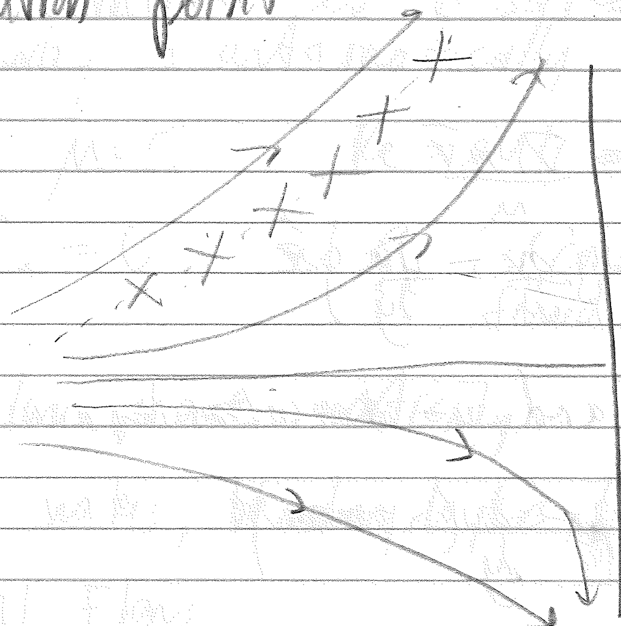


$$u = \frac{U}{h} y \qquad v = 0$$

$$\zeta = \zeta_z = \cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y} = -\frac{U}{h}$$

clockwise rotation at a rate of $\frac{1}{2}\frac{U}{h}$

stagnation point



$u = -ax$ $\quad \zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ irrotational

$v = ay$

$$-\frac{\partial}{\partial y}\left(\rho\frac{Du}{Dt} = \rho g + \mu\left(\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 v}{\partial y^2}\right) - \cancel{\frac{\partial p}{\partial x}}\right)$$

$$\frac{\partial}{\partial x}\left(\rho\frac{Dv}{Dt} = \rho g + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \cancel{\frac{\partial p}{\partial y}}\right)$$

Lecture 9

Oct 15, 2018

$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$

3D flow: $\zeta = \nabla \times \vec{V}$

2D flow: $\zeta_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{z}$

In 2D, for x- and y- momentum eqn:

$$\frac{\partial}{\partial x}(y\text{-mom.}) - \frac{\partial}{\partial y}(x\text{-mom.})$$

Continuity for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \boxed{\rho\frac{D\zeta_z}{Dt} = \mu\left(\frac{\partial^2 \zeta_z}{\partial x^2} + \frac{\partial^2 \zeta_z}{\partial y^2}\right) = \mu\nabla^2\zeta_z}$$

heat equation!

Stream function:

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\boxed{\zeta_z = -\nabla^2\Psi}$$

vorticity can be determined by Ψ and vice versa

these combined can be solved instead of N-S.
Continuity is automatically satisfied.

Assume $\mu = 0$ $\quad Re = \frac{\rho U L}{\mu} \to \infty$

$\frac{D \xi_z}{Dt} = 0$ so ξ_z is conserved following a fluid particle

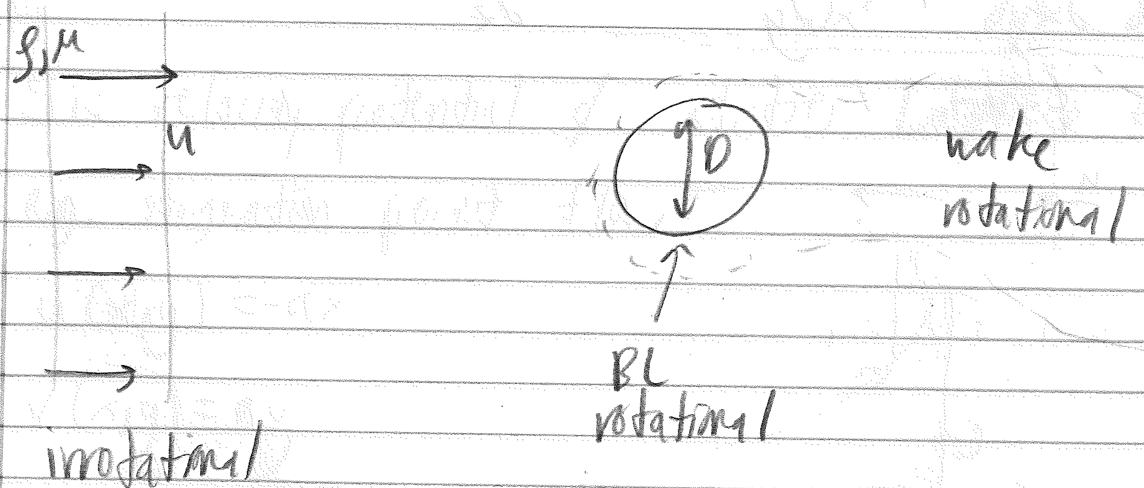
Many flows have vorticity

e.g. wakes, boundary layers, shear layers (jets)

Potential Flows

Smits 6.2

Consider the flow past a cylinder (Smits Chap 6)
$Re \gg 1$



$\nabla \times \vec{v} = 0$

Vorticity is generated by viscous effects i.e. boundary layer, and swept into wake.

The rest of the flow is irrotational.

Advantageous to split into two parts:
rotational + irrotational

$$\vec{\zeta} = \nabla \times \vec{v} = \vec{0}$$

$$\nabla \times (\nabla \phi) = 0 \qquad \text{from calculus}$$

ϕ = any scalar
$\phi = (\vec{x}, t)$

Suggest $\vec{v} = \nabla \phi$ gradient of "velocity potential"

$$\vec{\zeta} = \nabla \times (\nabla \phi) = 0$$

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$