Lecture 10

Oct 17, 2018

* $\rho \dfrac{D\xi_z}{Dt} = \mu \nabla^2 \xi_z$ if 2D, incompressible

* $\xi_z = -\nabla^2 \psi$

* $\vec{v} = \nabla \phi$ velocity potential

$u = \dfrac{\partial \phi}{\partial x}$ $v = \dfrac{\partial \phi}{\partial y}$ $w = \dfrac{\partial \phi}{\partial w}$ irrotational

$\vec{\xi} = \nabla \times \vec{v} = \nabla \times (\nabla \phi) = \vec{0}$

For incompressible flow, continuity becomes

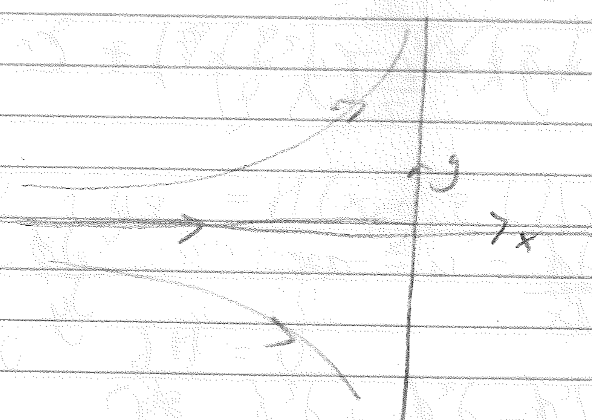
$$\nabla \cdot \vec{v} = \nabla \cdot (\nabla \phi) = \boxed{\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

The velocity potential ϕ satisfies Laplace eqn.

e.g. stagnation point flow

$u(x,y) = -ax$

$v(x,y) = ay$



$\nabla \cdot \vec{v} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = -a + a = 0$ ✓

$\xi_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$

Incompressible, irrotational $\rightarrow$ a velocity potential ϕ exists, satisfying: $u = \frac{\partial \phi}{\partial x} = -ax$

$$v = \frac{\partial \phi}{\partial y} = ay$$

$$\int u\,dx \Rightarrow \phi = \int u\,dx + f(y)$$

$$= \frac{-ax^2}{2} + f(y)$$

$$\int v\,dy \Rightarrow \phi = \int v\,dy + g(x)$$

$$= \frac{ay^2}{2} + g(x)$$

Solutions must be the same

$$f(y) = \frac{ay^2}{2} + C_1$$

$$g(x) = \frac{-ax^2}{2} + C_2$$

$$\phi(x,y) = \frac{a}{2}(y^2 - x^2) + C$$

Check

$$\frac{\partial \phi}{\partial x} = u = -ax \qquad \frac{\partial \phi}{\partial y} = v = ax$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Bernoulli's eqn Smits 4.2

Consider incompressible, irrotational flow

$$\nabla^2 \phi = 0$$

Consider momentum eqn. for incompressible, inviscid flow $(\mu = 0)$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \nabla p + g + \cancel{\frac{\mu}{\rho} \nabla^2 \vec{v}}$$

Irrotational + inviscid go hand in hand. $\Rightarrow \nabla^2 \vec{v} = 0$

Irrotational: $\vec{v} = \nabla \phi$

Vector identity: $\nabla \cdot \vec{v} = 0$

? $$(\vec{v} \cdot \nabla)\vec{v} = \frac{\nabla |\vec{v}|^2}{2} - \vec{v} \times \xi$$

Gravity: $\vec{g} = \nabla(-gz) = \langle 0, 0, -g \rangle$

$$\Rightarrow \frac{\partial (\nabla \phi)}{\partial t} + \frac{\nabla |v^2|}{2} - \vec{v} \times \xi = \nabla\left(\frac{p}{\rho}\right) - \nabla(gz)$$

$$H = \frac{\partial \phi}{\partial t} + \frac{|\vec{v}|^2}{2} + \frac{p}{\rho} + gz = H(x, y, z, t)$$

$$\frac{\partial H}{\partial x} = 0 \quad \frac{\partial H}{\partial y} = 0 \quad \frac{\partial H}{\partial z} = 0$$

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{|\vec{v}|^2}{2} + \frac{p}{\rho} + gz = H(t)}$$ unsteady Bernoulli

Steady flow $\frac{\partial \phi}{\partial t} = 0$

$H_0 = \text{constant}$

$$\frac{|\vec{v}^2|}{2} + \frac{p}{\rho} + gz = 0$$

<u>Notes</u>

→ assumed irrotational, incompressible, inviscid

→ dimensions of energy although it comes from momentum

→ other versions only require inviscid

→ Bernoulli's ⇒ p from $\vec{v}$ or ϕ

e.g. stagnation point

$u = -ax \qquad v = ay$

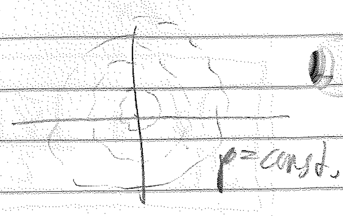
steady, incomp., neglect gravity

$$\frac{1}{2}(u^2+v^2) + \frac{p}{\rho} = \frac{1}{2}a(x^2+y^2) + \frac{p}{\rho} = H_0$$

Evaluate H_0 at stagnation point $(0,0)$
$p = p_0$

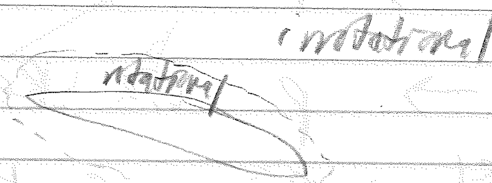
$$\frac{1}{2}a^2(x^2+y^2) + \frac{p}{\rho} = \frac{p_0}{\rho}$$

$$p(x,y) = p_0 - \frac{1}{2}\rho a^2(x^2+y^2)$$



Pressure is highest @ stagnation pt, decreases as $r^2 = x^2 + y^2$

A general approach to flows w/ substantial irrotational region

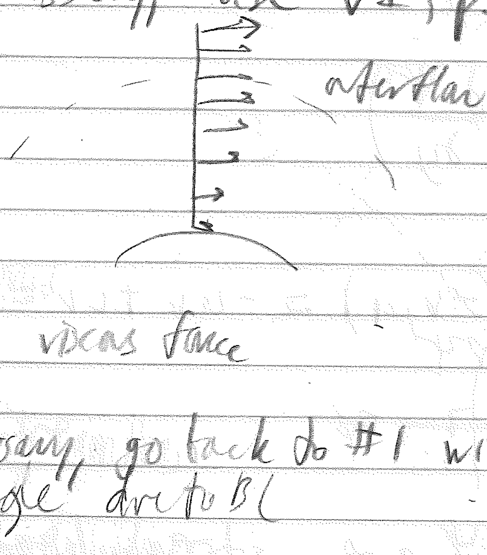


1) Solve for $\nabla^2 \phi = 0$ + BC → gives $\vec{v}_I = \nabla \phi_I$

2) Use Bernoulli's → $p_I(\vec{x})$

3) Compute pressure force on object

4) If necessary, use $\vec{v}_I, p_I$ to compute BL flow



5) Compute viscous force

6) If necessary, go back to #1 with the object 'change' due to BL