Lecture 11 — Oct 19, 2018

$$\vec{\zeta} = \vec{0}$$

$$\vec{v} = \nabla\phi$$

$$\nabla^2\phi = 0 \longrightarrow \vec{v}$$

$$\longrightarrow p \text{ via Bernoulli}$$

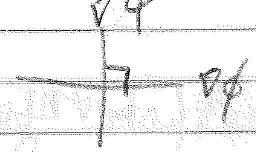
$$\frac{\partial\phi}{\partial t} + \frac{|\vec{v}|^2}{2} + \frac{p}{\rho} + gz = H_0$$

Geometrical interpretation of ϕ (2D, incompressible)

$$\vec{v} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$$

So
$$\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) = (-v, u)$$

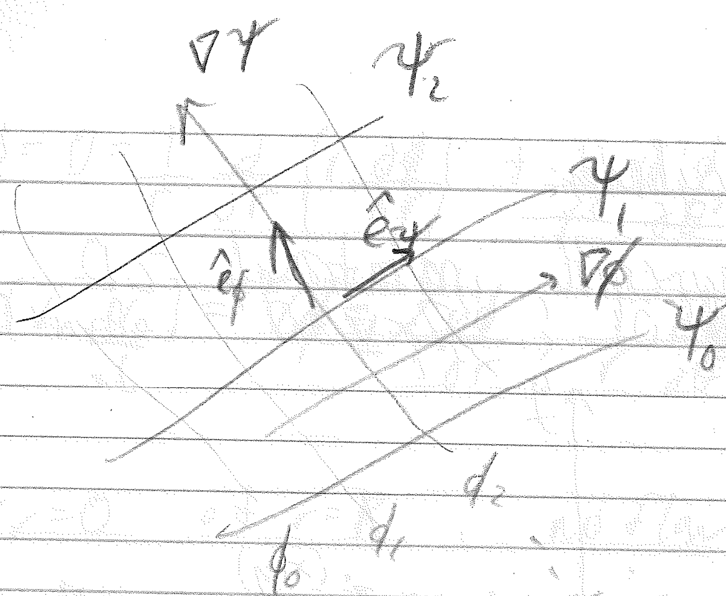
$$\nabla\psi\cdot\nabla\phi = (u, v)\cdot(-v, u) = -uv + uv = 0$$

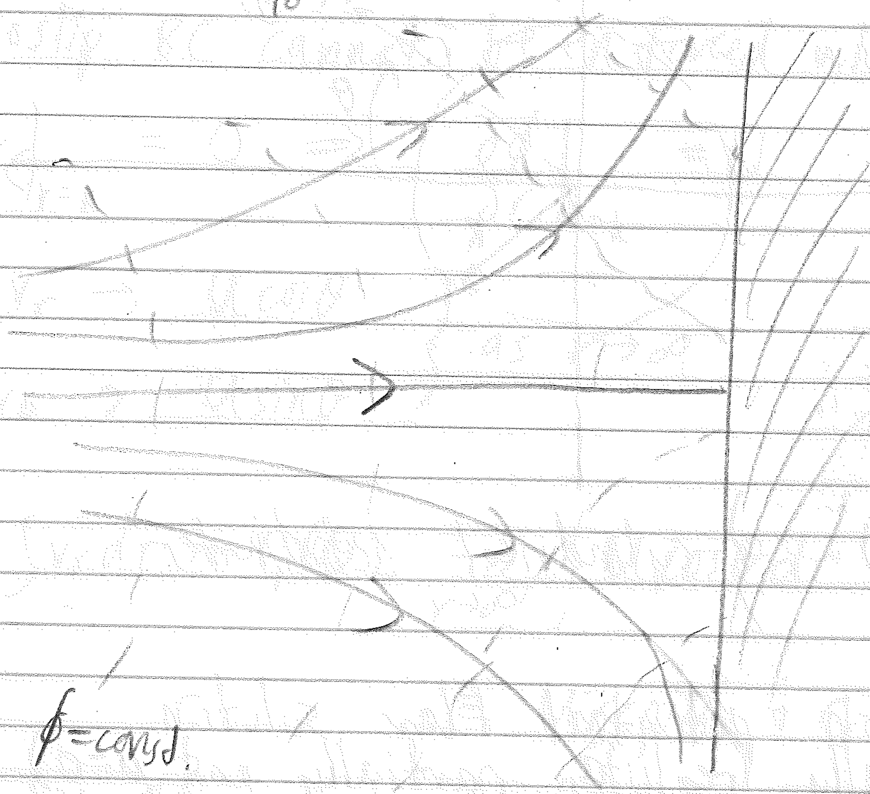


Lines of constant ψ are streamlines, and $\nabla\psi$ is $\perp$ to streamlines

$$\nabla\psi\cdot\hat{e}_\psi = 0 \qquad \therefore \hat{e}_\psi\cdot\hat{e}_\phi = 0$$

$$\nabla\phi\cdot\hat{e}_\psi = 0$$



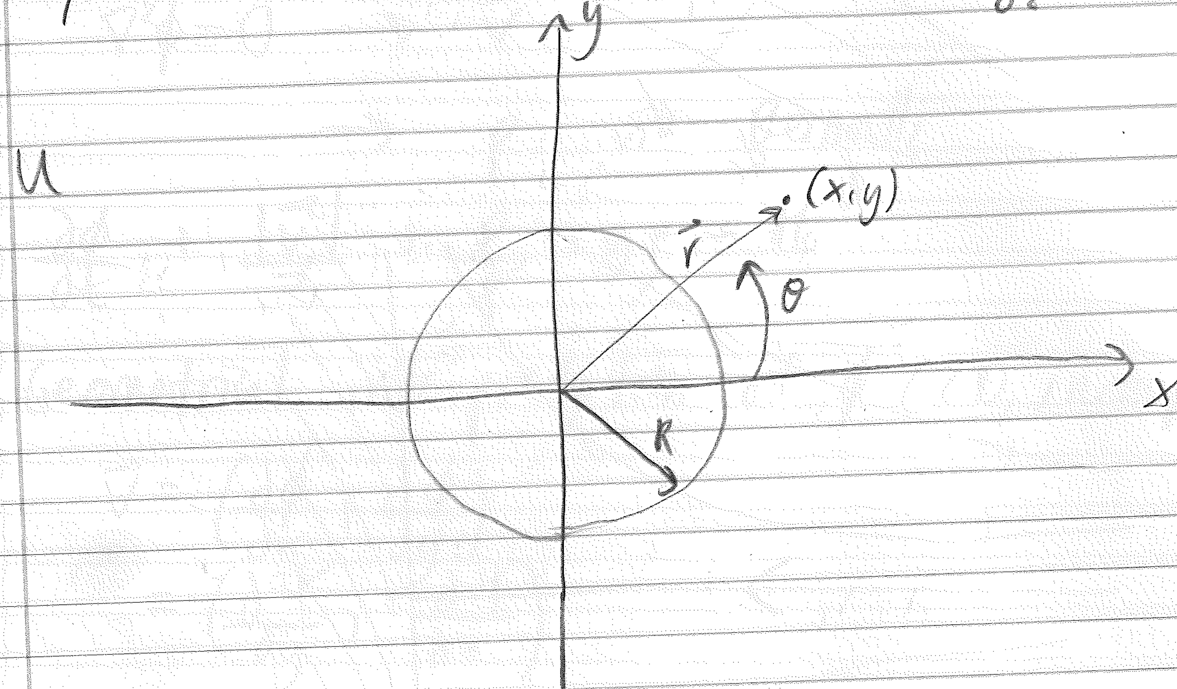


$\nabla\phi$ $\parallel$ streamlines

ϕ $\perp$ streamlines

Flow past a cylinder Smits 6.9

inviscid, incompressible, irrotational, 2D
$\mu=0$, $\nabla\cdot\underline{v}=0$, $\nabla\times\underline{v}=0$, $\frac{\partial}{\partial z}=0, w=0$



$r^2=x^2+y^2$
$\tan\theta=\frac{y}{x}$ } cylindrical coordinates

Work with potential flow solution
(partially realistic, partially not $\mu=0$)

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}$$

$$\nabla^2\phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\vec{V} = (v_r, v_\theta) = \nabla\phi = \left\langle \frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta} \right\rangle$$

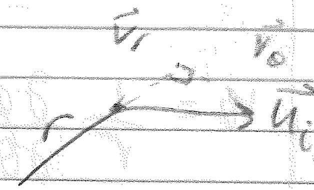
BC

① $v_r = 0$ at $r = R$ no flow $\perp$ cylinder

noslip BC cannot be enforced when $\mu = 0$

$$v_r\Big|_{r=R} = 0 \Rightarrow \frac{\partial\phi}{\partial r}\Big|_{r=R} = 0$$

② $\left.\begin{array}{l} v_r \to U\cos\theta \\ v_\theta \to -U\sin\theta \end{array}\right\}$ as $r \to \infty$



$\frac{\partial\phi}{\partial r} = v_r$

$$\phi = \int v_r\,dr + c_1(\theta) \xrightarrow[r\to\infty]{} \int U\cos\theta\,dr + c_1(\theta)$$

$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = v_\theta$

$$= Ur\cos\theta + \cancel{c_1(\theta)}$$

$$\phi = r\int v_\theta\,d\theta + c_2(r) \xrightarrow[r\to\infty]{} r\int -U\sin\theta\,d\theta + c_2(r)$$

$$= Ur\cos\theta + \cancel{c_2(r)}$$

$$\phi(r,\theta) = Ur\cos\theta\left[1 + \frac{R^2}{r^2}\right]$$

Show $\nabla^2\phi = 0$

Solve for velocity with ϕ

$$v_r = \frac{\partial \phi}{\partial r} = U\cos\theta\left(1 + \frac{R^2}{r^2}\right)$$

$$+ Ur\cos\theta\left(-\frac{2R^2}{r^3}\right)$$

$$= U\cos\theta\left(1 - \frac{R^2}{r^2}\right)$$

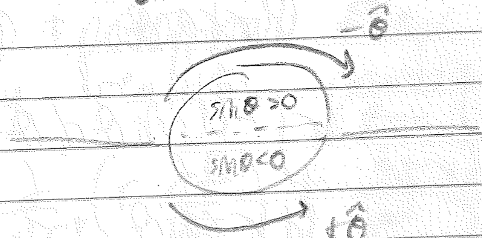
✓ $v_r = 0$ at $r = R$

$$v_r \xrightarrow[r\to\infty]{} U\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \left(-U\sin\theta\left(1 + \frac{R^2}{r^2}\right)\right)$$

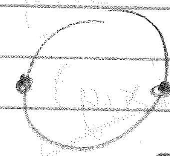
$$v_\theta \xrightarrow[r\to\infty]{} -U\sin\theta$$

$v_\theta = -2U\sin\theta$ at $r = R$



Stagnation point $(\vec{v}=\vec{0})$

$\theta=0,\pi$ $\quad r=R$



Max velocity

$\theta=\frac{\pi}{2},\frac{3\pi}{2}$ $\quad r=R$