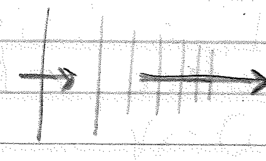
Lecture 12 Oct 22, 2018

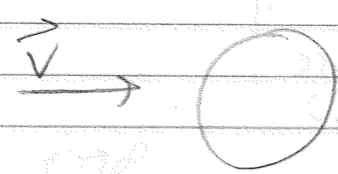
$\phi \perp \psi$

$\phi \to$ magnitude of $|\vec{v}|$



$\psi \to$ direction of $\vec{v}$

Pressure field



$$\frac{1}{2}\left(v_r^2 + v_\theta^2\right) + \frac{p}{\rho} = H_0$$

$$v_r^2 + v_\theta^2 = U^2\cos^2\theta\left[1 - \frac{2R^2}{r^2} + \frac{R^4}{r^4}\right] + U^2\sin^2\theta\left[1 + \frac{2R^2}{r^2} + \frac{R^4}{r^4}\right]$$

$$= U^2\left(1 + \frac{R^4}{r^4}\right) + 2U^2\underbrace{\left(\sin^2\theta - \cos^2\theta\right)}_{-\cos(2\theta)}\frac{R^2}{r^2}$$

$$= U^2\left[\left(1 + \frac{R^4}{r^4}\right) - 2\cos 2\theta\,\frac{R^2}{r^2}\right]$$

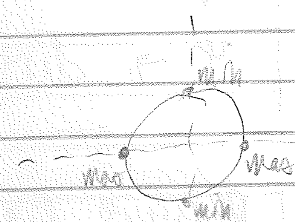
$$\frac{p}{\rho} = H_0 - \frac{U^2}{2}\left[\left(1 + \frac{R^4}{r^4}\right) - 2\cos 2\theta\,\frac{R^2}{r^2}\right]$$

$r \to \infty$, $p \to p_\infty$ $\quad H_0 = \text{const.} = \frac{p_\infty}{\rho} - \frac{U^2}{2}$

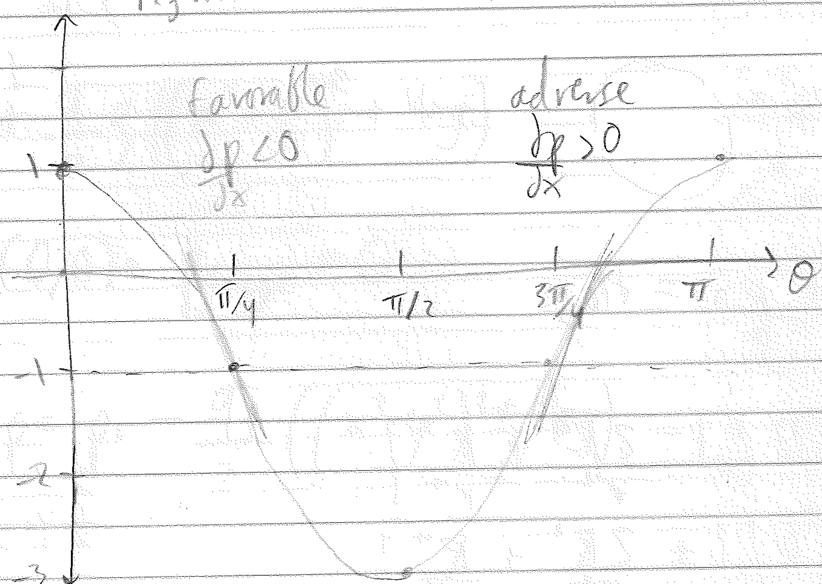
$$p = p_\infty + \frac{1}{2}\rho U^2 \left(2\cos(2\theta)\frac{R^2}{r^2} - \frac{R^4}{r^4}\right)$$

Cylinder surface, $r=R$

$$p(r,\theta)\Big|_{r=R} = p_\infty + \frac{1}{2}\rho U^2 \left(2\cos(2\theta) - 1\right)$$



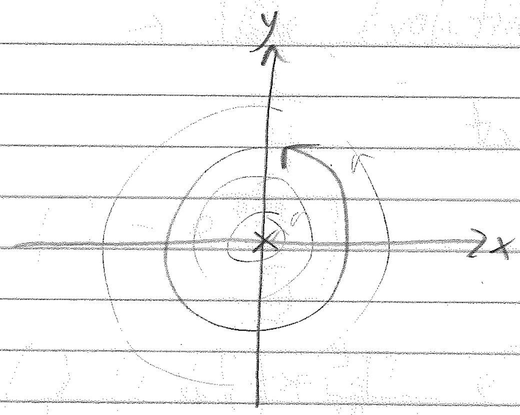
$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$



inviscid + symmetric $\rightarrow$ no force (d'Alembert's paradox)

Line vortex (a line in the z-direction)

↳ flow near vortex



$v_r = 0$

$v_\theta = \dfrac{\Gamma}{2\pi r}$

$v_\theta \to 0 \qquad r \to \infty$

$v_\theta \to \infty \qquad r \to 0$

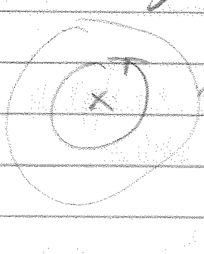
$$\zeta_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \underbrace{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}_{\to 0}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\Gamma}{2\pi r}\right) = 0 \quad \therefore \text{ irrotational}$$

All vorticity is concentrated at the origin (singularity)

Γ = strength of vortex (circulation)

$\Gamma = \oint \vec{v} \cdot d\vec{l}$ $\qquad \left[\frac{m}{s}\right][m] = \left[\frac{m^2}{s}\right]$



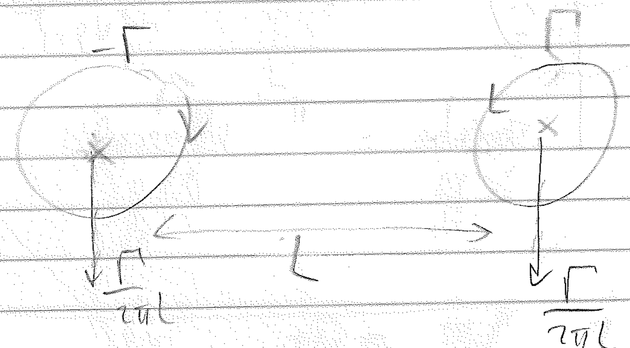
$d\vec{l} = r\,d\theta\,\hat{e}_\theta$

$$\Gamma = \int_0^{2\pi} v_\theta\, r\, d\theta = \int_0^{2\pi} \frac{\Gamma}{2\pi r}\, r\, d\theta = \Gamma \quad ✓$$

$\zeta = \nabla \times \vec{V}$

$\Gamma = \iint_S \zeta \cdot n \, dS$ } stokes thm

Vortices can interact...



vortices get advected at $v = \frac{\Gamma}{2\pi L}$

Superposition $\nabla^2 \phi = 0$

$$\nabla^2(\phi_1 + \phi_2) = \nabla^2 \phi_1 + \nabla^2 \phi_2 = 0$$