Lecture 18

Nov 9, 2018

Reynolds stress

$$R_{ij} = -\rho\,\overline{u_i' u_j'}$$

if $-\overline{u_i u_j} = \nu_T\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)$

$$NS(u) = NS(\overline{u}) \text{ with } \mu + \mu_T$$

$$\nu_T \rightarrow k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$ kinetic turbulent energy

$\overline{u}+u'$: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$\overline{u}$: $\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} + \overline{w}\frac{\partial \overline{w}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right)$

$\overline{u}+u'$

$-\overline{u} =$

u'

$$-\frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'w'}}{\partial z}$$

Reynolds decompose $(\overline{u}+u')$ → eqn for u'

$$f(\overline{u}+u') - g(\overline{u}) \rightarrow h(u')$$

$$\frac{\partial \overline{u}+u'}{\partial t} - 0 = \frac{\partial u'}{\partial t}$$

$$(\overline{u}+u')\frac{\partial(\overline{u}+u')}{\partial x} - \overline{u}\frac{\partial \overline{u}}{\partial x} = u'\frac{\partial \overline{u}}{\partial x} + \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}$$

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \overline{v}\frac{\partial u'}{\partial y} + \overline{w}\frac{\partial u'}{\partial z} + u'\frac{\partial \overline{u}}{\partial x} + v'\frac{\partial \overline{u}}{\partial y} + w'\frac{\partial \overline{u}}{\partial z}$$
$$+ u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right)$$
$$+ \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}$$

$\overline{\text{eqn} \times u'}$

$$\overline{u'\frac{\partial u'}{\partial t}} = \overline{\frac{\partial}{\partial t}\left(\frac{1}{2}u'^2\right)} = \frac{\partial}{\partial t}\left(\frac{1}{2}\underbrace{\overline{u'^2}}_{\text{constant}}\right) = 0$$

$$\overline{u'\bar{u}\frac{\partial u'}{\partial x}} = \bar{u}\frac{\partial}{\partial x}\left(\frac{1}{2}\overline{u'^2}\right)$$

$$\overline{u'u'\frac{\partial \bar{u}}{\partial x}} = \overline{u'^2}\frac{\partial \bar{u}}{\partial x}$$

$$\overline{u'v'\frac{\partial u'}{\partial x}} = \overline{v'\frac{\partial}{\partial y}\left(\frac{1}{2}u'^2\right)}$$

$$\overline{u'\frac{\partial^2 u'}{\partial x^2}} = \frac{\partial}{\partial x}\overline{\left(u'\frac{\partial u'}{\partial x}\right)} - \overline{\left(\frac{\partial u'}{\partial x}\right)^2}$$

$$= \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\overline{u'^2}\right) - \overline{\left(\frac{\partial u'}{\partial x}\right)^2}$$

with $\nabla \cdot \vec{v}' = 0$

$\frac{\partial k}{\partial t} + \dots \rightarrow$ $\bar{u}\frac{\partial}{\partial x}\left(\frac{1}{2}\overline{u'^2}\right) + \bar{v}\frac{\partial}{\partial y}\left(\frac{1}{2}\overline{u'^2}\right) + \bar{w}\frac{\partial}{\partial z}\left(\frac{1}{2}\overline{u'^2}\right)$

$P \rightarrow$ $+ \overline{u'^2}\frac{\partial \bar{u}}{\partial x} + \overline{u'v'}\frac{\partial \bar{u}}{\partial y} + \overline{u'w'}\frac{\partial \bar{u}}{\partial z}$

$D \rightarrow$ $+ \frac{\partial}{\partial x}\left(\overline{u'\frac{1}{2}u'^2}\right) + \frac{\partial}{\partial y}\left(\overline{v'\frac{1}{2}u'^2}\right) + \frac{\partial}{\partial z}\left(\overline{w'\frac{1}{2}u'^2}\right)$

$$= -\frac{1}{\rho}\overline{u'\frac{\partial p'}{\partial x}} + \nu\nabla^2\left(\frac{1}{2}\overline{u'^2}\right) - \underbrace{\nu\left[\overline{\left(\frac{\partial u'}{\partial x}\right)^2} + \overline{\left(\frac{\partial u'}{\partial y}\right)^2} + \overline{\left(\frac{\partial u'}{\partial z}\right)^2}\right]}_{\varepsilon}$$

sum u', v', w' equations

$$\frac{D\bar{k}}{Dt} = \boxed{\bar{u}\frac{\partial \bar{k}}{\partial x} + \bar{v}\frac{\partial \bar{k}}{\partial y} + w\frac{\partial \bar{k}}{\partial z} = P + D - \varepsilon}$$

production diffusion dissipation

P = production $\bar{k}\uparrow$

$$P = \overline{u'^2}\frac{\partial \bar{u}}{\partial x} + \overline{u'v'}\frac{\partial \bar{u}}{\partial y} + \overline{u'w'}\frac{\partial \bar{u}}{\partial z}$$
$$+ \overline{u'v'}\frac{\partial \bar{v}}{\partial x} + \overline{v'^2}\frac{\partial \bar{v}}{\partial y} + \overline{v'w'}\frac{\partial \bar{v}}{\partial z}$$
$$+ \overline{u'w'}\frac{\partial \bar{w}}{\partial x} + \overline{v'w'}\frac{\partial \bar{w}}{\partial z} + \overline{w'^2}\frac{\partial \bar{w}}{\partial z}$$

$R_{ij}\frac{\partial \bar{v}_i}{\partial x_j}$ turbulence × gradient of mean flow

D = turbulent diffusion

movement of $\bar{k}$ in space (high $\bar{k}$ area to low $\bar{k}$ area)
(lossless transfer)

$$D = -\frac{\partial}{\partial x}\left(\overline{p'u' + u'k}\right) - \frac{\partial}{\partial y}\left(\overline{p'v' + v'k}\right) - \frac{\partial}{\partial z}\left(\overline{p'w' + w'k}\right)$$

gradient

often ignore p' (hard to measure)

ε = kinetic energy dissipation rate > 0 (mech. energy → heat)

$$\varepsilon = \nu\left[\left(\frac{\partial u'}{\partial x}\right)^2 + \left(\frac{\partial u'}{\partial y}\right)^2 + \left(\frac{\partial u'}{\partial z}\right)^2 + \left(\frac{\partial v'}{\partial x}\right)^2 + \left(\frac{\partial v'}{\partial y}\right)^2 + \left(\frac{\partial v'}{\partial z}\right)^2 + \left(\frac{\partial w'}{\partial x}\right)^2 + \left(\frac{\partial w'}{\partial y}\right)^2 + \left(\frac{\partial w'}{\partial z}\right)^2\right]$$

If $\bar{k}$ computed

$$U_T = \sqrt{\bar{k}} \qquad \text{characteristic velocity}$$

$$L_T = \frac{\bar{k}^{3/2}}{\varepsilon} \qquad \text{characteristic length scale}$$

$$\nu_T = C_T \frac{\bar{k}^2}{\varepsilon} \qquad \nu = C_T U_T L_T$$

eqn for $\bar{k}$ is not closed (don't know u', v', w', p')

$$\overline{u'^2} = -\frac{2}{3}\bar{k} + 2\nu_T \frac{\partial \bar{u}}{\partial x}$$

$$\overline{v'^2} = -\frac{2}{3}\bar{k} + 2\nu_T \frac{\partial \bar{v}}{\partial y}$$

$$\overline{w'^2} = -\frac{2}{3}\bar{k} + 2\nu_T \frac{\partial \bar{w}}{\partial z}$$

$$-\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = 2\bar{k} - 2\nu_T \overbrace{\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right)}^{0}$$

$$-\overline{u'v'} = \nu_T\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)$$

$$-\overline{u'w'} = \nu_T\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)$$

$$-\overline{u'w'} = \nu_T\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)$$

D model

$$\overline{u'p'} + \overline{u'k} = -\frac{\nu_T}{\sigma_k}\frac{\partial \bar{k}}{\partial x} \qquad \sigma_k = \frac{\nu_T}{\nu_k} \quad \text{Schmidt \#}$$

$\nu_k \leftarrow$ turbulent diffusivity

ε model

$$-\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x}$$

$C_T, \sigma_k, \sigma_\varepsilon \rightarrow$ constants to be determined by experimental data