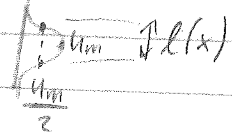
Lecture 21 Nov 21, 2018

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\overline{u'v'}\right)$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}) = 0$$

$$\frac{\bar{u}(r,x)}{U(x)} = f\left(\frac{r}{\ell(x)}\right) = f(\eta) \qquad \ell(x) = \text{half width}$$



$$\frac{\bar{v}(r,x)}{U(x)} = g\left(\frac{r}{\ell(x)}\right) = g(\eta)$$

$$\frac{\overline{u'v'}(x,r)}{U^2(x)} = h\left(\frac{r}{\ell(x)}\right) = h(\eta)$$

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial U}{\partial x} f - U f' \frac{\ell'}{\ell}\eta \qquad \eta = \frac{r}{\ell}$$

$$\frac{\partial \bar{u}}{\partial r} = \frac{U}{\ell} f'$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\overline{u'v'}\right) = \frac{U^2}{\ell}\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta h)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v) = \frac{U}{\ell}\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta g)$$

Plug into x-momentum:

$$\frac{U'\ell}{U} f^2 - \ell'\eta f f' + f' g = -\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta h)$$

Continuity:

(b) $$\left(\frac{U'\ell}{U}\right) f - \ell'\eta f' + \frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta g) = 0$$

Look for a solution, assume power law:

$$U(x) = A x^{-n} \qquad \ell(x) = B x^{m} \qquad n, m > 0$$

$x\uparrow \; U\downarrow \qquad x\uparrow \; \ell(x)\uparrow$

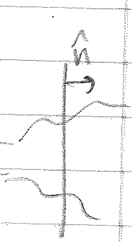
$\frac{U'\ell}{U}$ must be constant since RHS is a function of η

$$\frac{U'\ell}{U} = \frac{-nAx^{-n-1}}{Ax^{-n}} Bx^{m} = nBx^{m-1} = \text{const.}$$

$$m-1=0$$
$$m=1$$

$\boxed{\ell(x) = Bx}$ $\rightarrow$ universal cone angle

Momentum flux conservation for jets



$$M(x) = \int_{\text{vertical plane}} \rho\, \bar{u} \underbrace{(\vec{V}\cdot\hat{n})}_{\bar{u}} dA$$

$$= \rho \int_{\theta=0}^{2\pi}\int_{0}^{\infty} \bar{u}^2 \, r\,dr\,d\theta$$

$$= 2\pi\rho \int_0^\infty r\bar{u}^2(x,r)\,dr$$

Momentum + continuity

$$\star \quad \frac{\partial \bar{u}^2}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{u}\bar{v}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\overline{u'v'}\right)$$

$$\frac{\partial M(x)}{\partial x} = -2\pi\rho \int_0^\infty \frac{\partial}{\partial r}(r\bar{u}\bar{v})\,dr - 2\pi\rho \int_0^\infty \frac{\partial}{\partial r}(\overline{u'v'})\,dr$$

$$= -2\pi\rho\left[r\bar{u}\bar{v}\Big|_0^\infty + r\overline{u'v'}\Big|_0^\infty \right]$$

at $r=0 \rightarrow r\bar{u}\bar{v} = r\overline{u'v'} = 0$

at $r \rightarrow \infty \rightarrow \bar{u} \rightarrow 0,\ \bar{v} \rightarrow 0,\ \overline{u'v'} \rightarrow 0$

faster than r approaches ∞

$\frac{\partial M}{\partial x} = 0$

$\therefore M(x) = \text{constant} = M_0$

$$M = 2\pi\rho \int_0^\infty r\bar{u}^2(x,r)\,dr = \text{const.}$$

$$= 2\pi\rho\, l^2 U^2 \underbrace{\int_0^\infty \eta f^2\, d\eta}_{\text{const.}} = \text{const.}$$

$l^2 U^2 = \text{const.}$

$= B^2 x^2 A^2 x^{-2n} = \text{const.}$

$n = 1$

$$\boxed{U(x) = \frac{A}{x}}$$ max velocity decreases linearly

$$\frac{U'\ell}{U} = B$$

(a) x-momentum

$$-Bf^2 - B\eta f f' + f' g = -\frac{1}{\eta}\frac{\partial(\eta h)}{\partial \eta}$$

(b) continuity

$$-Bf - B\eta f' + \frac{1}{\eta}\frac{\partial(\eta g)}{\partial \eta} = 0$$

2 eqns, 3 unknowns (f, g, h)

<u>Mass flux</u>

$$m(x) = \int_{\substack{\text{vertical}\\ \text{plane}}} \rho(\vec{v}\cdot\hat{n})\, dA$$

$$= 2\pi\rho \int_0^\infty r\,\bar{u}(x,r)\, dr$$

$$= 2\pi\rho\, U(x)\,\ell^2(x) \underbrace{\int_0^\infty \eta f(\eta)\, d\eta}_{\text{const.}}$$

$$m(x) \propto U(x)\,\ell^2(x)$$

$$\propto \frac{1}{x}\, x^2$$

more mass moving (not more mass)

$$m(x) \propto x$$

$m(x) \uparrow$ with x — entrainment

Turbulence modelling

$$-\overline{u'v'} = -\nu_T \frac{\partial \overline{u}}{\partial r} \qquad \nu_T = c\,U_T L_T$$

$$U_T = U$$
$$L_T = \ell$$

$$\overline{u'v'} = U^2 h = c\,U\ell \frac{\partial}{\partial r}(Uf)$$

$$\overline{u'v'} = -c\,U^2 \frac{\partial f}{\partial \eta}$$

$$h = -c \frac{\partial f}{\partial \eta}$$