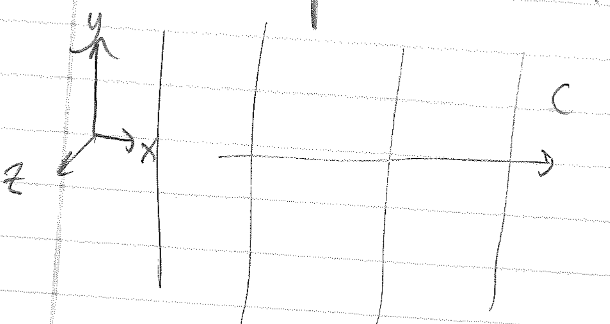
Lecture 23

Nov 28, 2018

Compressible flows: acoustic waves

small perturbation w.r.t. ambient



plane wave along x

→ neglect viscous effects (good for short distances)

→ neglect gravity

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

→ no heat addition (elastic collisions between molecules)
i.e. adiabatic, no friction due to viscosity
∴ isentropic

→ assume ideal gas

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \qquad \gamma = \frac{c_p}{c_v}$$

$p = p(\rho)$ barotropic

(a) Rest state: $u=0$, $\rho=\rho_0$, $p=p_0$

In motion state: $u=u'$, $\rho=\rho_0+\rho'$, $p=p_0+p'$

small perturbation $\rho' \ll \rho_0$

$$\frac{\partial}{\partial t}(\cancel{\rho_0}+\rho') + u'\frac{\partial(\cancel{\rho_0}+\rho')}{\partial x} + (\rho_0+\rho')\frac{\partial u'}{\partial x} = 0$$

$$\frac{\partial \rho'}{\partial t} + \cancel{u'\frac{\partial \rho'}{\partial x}} + \rho_0\frac{\partial u'}{\partial x} + \cancel{\rho'\frac{\partial u'}{\partial x}} = 0$$

small terms

① $$\frac{\partial \rho'}{\partial t} = -\rho_0\frac{\partial u'}{\partial x}$$

$$\frac{\partial u'}{\partial t} + \cancel{u'\frac{\partial u'}{\partial x}} = -\frac{1}{\rho}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x}$$

$$\frac{1}{\rho} = \frac{1}{\rho_0+\rho'} = \frac{1}{\rho_0}\frac{1}{1+\rho'/\rho_0}$$

$$= \frac{1}{\rho_0}\left(1-\frac{\rho'}{\rho_0}+\cdots\right)$$

$$\frac{\partial p}{\partial \rho} = f(\rho_0+\rho') = f(\rho_0) + \left.\frac{\partial f}{\partial \rho}\right|_{\rho_0}\rho' + \cdots$$

$$\frac{\partial \rho}{\partial x} = \frac{\partial(\rho_0+\rho')}{\partial x} = \frac{\partial \rho'}{\partial x}$$

② $$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0}\left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}\frac{\partial \rho'}{\partial x}$$

In medium, sound speed = $c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}}$

air $\rightarrow c \approx 340$ m/s
water $\rightarrow c \approx 1500$ m/s

② $\frac{\partial u'}{\partial t} = -\frac{c^2}{\rho_0}\frac{\partial \rho'}{\partial x}$

$\frac{\partial}{\partial t}$ ① $\Rightarrow \frac{\partial^2 \rho'}{\partial t^2} = -\rho_0 \frac{\partial}{\partial t}\frac{\partial}{\partial x} u'$

Substituting ②:

$$= -\rho_0 \frac{\partial}{\partial x}\left(\frac{-c^2}{\rho_0}\frac{\partial \rho'}{\partial x}\right)$$

$$\boxed{\begin{gathered}\frac{\partial^2 \rho'}{\partial t^2} = c^2 \frac{\partial^2 \rho'}{\partial x^2} \\ \frac{\partial^2 u'}{\partial t^2} = c^2 \frac{\partial^2 u'}{\partial x^2}\end{gathered}}$$

linear acoustic wave eqn

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$$

$$c^2 = \frac{\partial p}{\partial \rho} = p_0 \gamma \left(\frac{\rho^{\gamma-1}}{\rho^\gamma}\right)_{\rho=\rho_0}$$

$$= \gamma \frac{p_0}{\rho_0} = \gamma R T_0$$

$c = \sqrt{\gamma R T_0}$ $\qquad \gamma = 1.4$
$c \sim 340$ m/s