Acoustic, or sound, waves are 'small' perturbations to the ambient medium, such as air or water. To understand acoustic waves, and acoustic wave propagation, consider the following problem.

- Assume a plane wave propagating in the x-direction (so that the wave properties are independent of y and z), as seen in Figure 1. Note that the assumption of a plane wave is for mathematical convenience, as it is easy to generalize the problem, e.g., to a spherical wave or wave of another shape.

- Neglect viscous effects. This is a good assumption unless acoustic wave propagation over a long distance is being considered. Also, neglect the force of gravity, since it will not directly affect the wave propagation. Therefore, the conservation of mass and the momentum balance are the following:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,. \tag{2}$$

- Assume that there is no heat addition, i.e., the flow is adiabatic. The adiabatic condition with no friction (viscosity) implies that the flow is isentropic.

- Assume an ideal gas. For isentropic conditions for an ideal gas, the relationship between pressure and density is:

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{3}$$

  where $\gamma = c_p/c_v$ is the ratio of specific heats, and (p_0, ρ_0) are (constant) ambient values. Note that this condition implies that p is a function of ρ alone, or that the process is barotropic. Also note that, in obtaining this relationship, both the conservation of energy and the fact that the fluid is an ideal gas have been used.

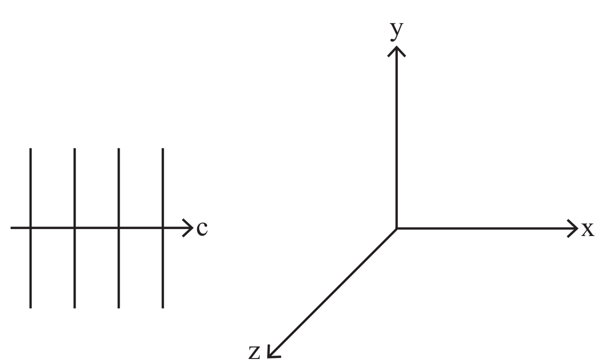

Figure 1: Plane wave propagation in the x-direction.

We next assume a state of rest and a state of motion, the latter in which the state of rest has been perturbed by an acoustic wave, given as follows:

- rest state: $u = 0$, $\rho = \rho_0$, $p = p_0$
- in motion: $u = u'$, $\rho = \rho_0 + \rho'$, $p = p_0 + p'$.

The superscript $(\cdot)'$ implies that the corresponding quantity is 'small', as for example, $\rho'/\rho_0 \ll 1$. Plugging in the expressions for the quantities in motion into the conservation of mass, Equation (1), gives the following:

$$\frac{\partial \rho'}{\partial t} + u'\frac{\partial}{\partial x}(\rho_0 + \rho') + (\rho_0 + \rho')\frac{\partial u'}{\partial x} = \frac{\partial \rho'}{\partial t} + \underbrace{u'\frac{\partial \rho'}{\partial x}}_{\text{neglect}} + \rho_0\frac{\partial u'}{\partial x} + \underbrace{\rho'\frac{\partial u'}{\partial x}}_{\text{neglect}} = 0\,,$$

or, neglecting the nonlinear terms, assuming that the perturbation fields are small,

$$\frac{\partial \rho'}{\partial t} = -\rho_0\frac{\partial u'}{\partial x}\,. \tag{4}$$

Now consider the momentum balance, Equation (2). Plugging in the perturbation velocity, the left-hand-side (LHS) of the equation is given by:

$$\frac{\partial u'}{\partial t} + \underbrace{u'\frac{\partial u'}{\partial x}}_{\text{neglect}}\,, \tag{5}$$

where the second term, a nonlinear term, will be neglected. On the right-hand-side (RHS), note that, from Equation (3), p is a function of ρ alone, or $p = p(\rho)$. Therefore, using the chain rule, the RHS becomes:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\underbrace{\frac{1}{\rho}}_{i}\underbrace{\frac{dp}{d\rho}}_{ii}\underbrace{\frac{\partial \rho}{\partial x}}_{iii}\,. \tag{6}$$

Each of the three terms in Equation (6) will be approximated separately.

$$\frac{1}{\rho} = \frac{1}{\rho_0 + \rho'} = \frac{1}{\rho_0(1 + \rho'/\rho_0)} = \frac{1}{\rho_0}[1 - \rho'/\rho_0 + \ldots]\,, \tag{7}$$

where the last step can be obtained by either a Taylor series expansion or a binomial series for $1/(1 + \rho'/\rho_0)$. Next, defining $\frac{dp}{d\rho} = f(\rho)$, then expanding f in a Taylor series about $\rho = \rho_0$,

$$f(\rho_o + \rho') = f(\rho_0) + \left.\frac{df}{d\rho}\right|_{\rho=\rho_0}\rho' + \ldots \tag{8}$$

Finally,

$$\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x}(\rho_0 + \rho') = \frac{\partial \rho'}{\partial x}\,. \tag{9}$$

Using Equations (7), (8), and (9) in Equation (6), neglecting nonlinear terms, and plugging the result along with Equation (5) into Equation (2) gives:

$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0}\left.\frac{dp}{d\rho}\right|_0\frac{\partial \rho'}{\partial x}\,, \tag{10}$$

where the subscript $(\cdot)_0$ signifies ambient conditions, and the definition of f has been used.

The sound speed c is defined by

$$c = \sqrt{\left.\frac{dp}{d\rho}\right|_0}\,, \tag{11}$$

where it must be remembered that this is for isentropic processes. So Equation (10) can be written as

$$\frac{\partial u'}{\partial t} = -\frac{c^2}{\rho_0}\frac{\partial \rho'}{\partial x}\,. \tag{12}$$

Equations (4) and (12) are two linear equations in two unknowns, u' and ρ'. An equation for ρ' alone can be obtained as follows. Differentiating Equation (4) with respect to t and using Equation 12 gives

$$\frac{\partial}{\partial t}\frac{\partial \rho'}{\partial t} = \frac{\partial^2 \rho'}{\partial t^2} = -\rho_0 \frac{\partial}{\partial x}\frac{\partial u'}{\partial t} = -\rho_0 \frac{\partial}{\partial x}\left(-\frac{c^2}{\rho_0}\frac{\partial \rho'}{\partial x}\right) = c^2 \frac{\partial^2 \rho'}{\partial x^2}\,, \text{ or}$$

$$\frac{\partial^2 \rho'}{\partial t^2} = c^2 \frac{\partial^2 \rho'}{\partial x^2}\,. \tag{13}$$

This is the linear wave equation for ρ'. In a similar manner, ρ' can be eliminated to give the same equation for u', i.e.,

$$\frac{\partial^2 u'}{\partial t^2} = c^2 \frac{\partial^2 u'}{\partial x^2}\,. \tag{14}$$

The sound speed can be obtained using Equation (3) in the derivative given in Equation (11), and evaluating it at $\rho = \rho_0$ to give:

$$c^2 = \left.\frac{dp}{d\rho}\right|_0 = p_0 \gamma \left.\frac{\rho^{\gamma-1}}{\rho^\gamma}\right|_0 = \gamma \frac{p_0}{\rho_0} = \gamma R T_0$$

using the ideal gas law, or

$$c = \sqrt{\gamma \frac{p_0}{\rho_0}} = \sqrt{\gamma R T_0}\,. \tag{15}$$

With $\gamma \doteq 1.4$ and standard conditions for p_0 and ρ_0, then

$$c \doteq 332\,\mathrm{m/sec}\,.$$

Note that an exact solution to Equation (13) is given by

$$\rho'(x,t) = g(x \pm ct)\,, \tag{16}$$

where g is any twice differentiable function. To prove that this is a solution, consider:

$$\frac{\partial \rho'}{\partial t} = \pm c g'(x \pm t)\,, \text{ and } \frac{\partial^2 \rho'}{\partial t^2} = c^2 g''(x \pm ct)\,,$$

where the chain rule has been used, and where $(\cdot)'$ here denotes the differentiation of a function with respect to its argument. Furthermore,

$$\frac{\partial \rho'}{\partial x} = g'(x \pm ct)\,, \text{ and } \frac{\partial^2 \rho'}{\partial x^2} = g''(x \pm ct)\,,$$

Plugging these back into Equation (13), one sees that it is identically satisfied.

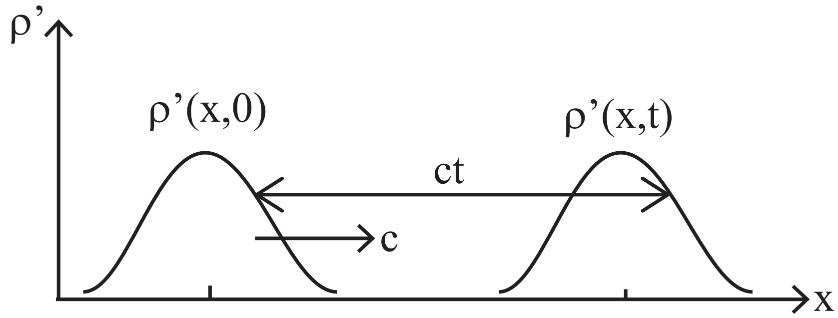

Figure 2: Acoustic wave propagation at speed c without change of shape.

Note that the argument $(x - ct)$ corresponds to a wave moving to the right, while $(x + ct)$ corresponds to a wave moving to the left (see Figure 2). Notice that the shape of the wave does not change as it propagates. Equations (13) and (14) have sine wave solutions as well, i.e.,

$$\rho'(x,t) = A \sin[k(x - ct)] \,, \tag{17}$$

where A is the wave amplitude, $\lambda = 2\pi/k$ is the wavelength, k is called the wave number, and $\omega = kc = 2\pi c/\lambda$ is the wave frequency. Note that a wave of any wavelength λ travels at the same wave speed c, i.e., the sound speed is independent of the wavelength. This is an important feature of sound waves. If it were not the case, then in listening to the spoken word, the different wavelengths of the speech would arrive at the ear at different times, producing a garbled speach. A wave whose wave speed is independent of wavelength is called a dispersionless, or nondispersive wave. Another example of a nondispersive wave is a light (or electromagnetic) wave. An example of a wave whose wave speed depends and the wavelength is a water wave. For a typical wave propagating over 'deep' water (the wavelength is much less than the depth of the water), the wave speed is proportional to the square root of the wavelength, i.e., $c \propto \sqrt{\lambda}$. Therefore, for deep water surface waves, the long waves travel faster than the short waves.