**Time-Averaged Equations of Motion**

The objective here is to derive the equations for the time-averaged quantities, $\bar{u}$, $\bar{v}$, $\bar{w}$, and $\bar{p}$. This will be done by time-averaging the equations for $u$, $v$, $w$, and $p$. In class and in the text (Sections 5.2 and 5.3 of the text by Smits) it was shown that, for the incompressible flow of a Newtonian fluid, the conservation of mass and the momentum balance are, in vector form:

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \frac{\partial}{\partial t}\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{v}\,.$$

In component form in Cartesian coordinates, the conservation of mass is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \tag{1}$$

while the $x$-component of the momentum balance is

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{2}$$

For the moment it is convenient to write this last equation in viscous stress form, i.e.,

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\,, \tag{3}$$

where

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}\,, \quad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \tag{4}$$

Make sure to convince yourself that Equation (3) with Equations (4) is consistent with Equation (2) (you can do this as a self-test, and, hint: you will need to use continuity); Equation (1) will be needed to show this.

We now make Reynolds decompositions for $u$, $v$, $w$, and $p$, i.e.,

$$u = \bar{u} + u'\,, \quad v = \bar{v} + v'\,. \quad w = \bar{w} + w'\,, \quad p = \bar{p} + p'\,, \tag{5}$$

where $\overline{(\cdot)}$ denotes a time average, and $(\cdot)'$ the fluctuation about the average. When these forms are substituted into Eqn. (1), averaged, and noting that, using the properties of the averaging operator,

$$\overline{\frac{\partial(\bar{u}+u')}{\partial x}} = \overline{\frac{\partial \bar{u}}{\partial x}} + \overline{\frac{\partial u'}{\partial x}} = \frac{\partial \bar{\bar{u}}}{\partial x} + \frac{\partial \overline{u'}}{\partial x} = \frac{\partial \bar{u}}{\partial x}\,,$$

and similarly for the other terms in Eqn. (1), the result is

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0\,, \tag{6}$$

the averaged form of the conservation of mass. Note that the averaged velocity then also satisfies Eqn. (1). Furthermore, if Eqn. (6) is subtracted from Eqn. (1), noting Eqn. (5), the result is

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0\,. \tag{7}$$

Therefore the fluctuating velocity also satisfies Eqn. (1).

Next consider the $x$-component of the momentum equation in viscous stress form, Eqn. (3). Plugging in Reynolds decompositions, Eqn. (5), will result in the following terms, again using the properties of the averaging operator.

$$\begin{aligned}
\overline{\frac{(\bar{u}+u')\partial(\bar{u}+u')}{\partial x}} &= \overline{\bar{u}\frac{\partial \bar{u}}{\partial x}} + \overline{u'\frac{\partial \bar{u}}{\partial x}} + \overline{\bar{u}\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial u'}{\partial x}} \\
&= \bar{u}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}} .
\end{aligned}$$

$$\begin{aligned}
\overline{\frac{(\bar{v}+v')\partial(\bar{u}+u')}{\partial y}} &= \overline{\bar{v}\frac{\partial \bar{u}}{\partial y}} + \overline{v'\frac{\partial \bar{u}}{\partial y}} + \overline{\bar{v}\frac{\partial u'}{\partial y}} + \overline{v'\frac{\partial u'}{\partial y}} \\
&= \bar{v}\frac{\partial \bar{u}}{\partial y} + \overline{v'\frac{\partial u'}{\partial y}} .
\end{aligned}$$

$$\begin{aligned}
\overline{\frac{(\bar{w}+w')\partial(\bar{u}+u')}{\partial z}} &= \overline{\bar{w}\frac{\partial \bar{u}}{\partial z}} + \overline{w'\frac{\partial \bar{u}}{\partial z}} + \overline{\bar{w}\frac{\partial u'}{\partial z}} + \overline{w'\frac{\partial u'}{\partial z}} \\
&= \bar{w}\frac{\partial \bar{u}}{\partial z} + \overline{w'\frac{\partial u'}{\partial z}} .
\end{aligned}$$

Furthermore,

$$\overline{\frac{\partial(\bar{p}+p')}{\partial x}} = \frac{\partial \bar{p}}{\partial x}$$

Finally,

$$\overline{\frac{\partial u}{\partial t}} = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \frac{\partial u}{\partial t}\, dt = \frac{1}{\mathcal{T}}(u(\mathcal{T}) - u(0)) \to 0 \quad \text{as} \quad \mathcal{T}\to\infty .$$

When these averaged quantities are plugged into Eqn. (3), the result is

$$\begin{aligned}
\rho\bar{u}\frac{\partial \bar{u}}{\partial x} + \rho\bar{v}\frac{\partial \bar{u}}{\partial y} + \rho\bar{w}\frac{\partial \bar{u}}{\partial z} &= \rho g_x - \frac{\partial \bar{p}}{\partial x} \\
&+ \frac{\partial}{\partial x}(\bar{\tau}_{xx} - \rho\overline{u'u'}) + \frac{\partial}{\partial y}(\bar{\tau}_{xy} - \rho\overline{u'v'}) + \frac{\partial}{\partial z}(\bar{\tau}_{xz} - \rho\overline{u'w'}) ,
\end{aligned} \tag{8}$$

since

$$\overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}} = \frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'} - \underbrace{\overline{u'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)}}_{=0} ,$$

where Eqn. (7) has been used, and

$$\bar{\tau}_{xx} = 2\mu\frac{\partial \bar{u}}{\partial x} , \quad \bar{\tau}_{xy} = \mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) , \quad \bar{\tau}_{xz} = \mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right) . \tag{9}$$

Self-test: Show that $\overline{u'^2}$ is the variance of $u$.
Equations (8) with (9) give the averaged $x$-component of the momentum balance. The $y$- and $z$-components can be obtained in a similar manner, and are the following:

$$\begin{aligned}
\rho\bar{u}\frac{\partial \bar{v}}{\partial x} + \rho\bar{v}\frac{\partial \bar{v}}{\partial y} + \rho\bar{w}\frac{\partial \bar{v}}{\partial z} &= \rho g_y - \frac{\partial \bar{p}}{\partial y} \\
&+ \frac{\partial}{\partial x}(\bar{\tau}_{yx} - \rho\overline{u'v'}) + \frac{\partial}{\partial y}(\bar{\tau}_{yy} - \rho\overline{v'v'}) + \frac{\partial}{\partial z}(\bar{\tau}_{yz} - \rho\overline{v'w'}) ,
\end{aligned} \tag{10}$$

$$\text{with } \bar{\tau}_{yx} = \mu\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right), \quad \bar{\tau}_{yy} = 2\mu\frac{\partial \bar{v}}{\partial y}, \quad \bar{\tau}_{yz} = \mu\left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right). \tag{11}$$

$$\begin{aligned} \rho\bar{u}\frac{\partial \bar{w}}{\partial x} + \rho\bar{v}\frac{\partial \bar{w}}{\partial y} + \rho\bar{w}\frac{\partial \bar{w}}{\partial z} &= \rho g_z - \frac{\partial \bar{p}}{\partial z} \\ &+ \frac{\partial}{\partial x}(\bar{\tau}_{zx} - \rho\overline{w'u'}) + \frac{\partial}{\partial y}(\bar{\tau}_{zy} - \rho\overline{w'v'}) + \frac{\partial}{\partial z}(\bar{\tau}_{zz} - \rho\overline{w'w'}), \end{aligned} \tag{12}$$

$$\text{with } \bar{\tau}_{zx} = \mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right), \quad \bar{\tau}_{zy} = \mu\left(\frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{v}}{\partial z}\right), \quad \bar{\tau}_{zz} = 2\mu\frac{\partial \bar{w}}{\partial z}. \tag{13}$$

Note that the equations for $\bar{u}$, $\bar{v}$, $\bar{w}$, and $\bar{p}$ are of the same form as those for $u$, $v$, $w$, and $p$, except for the addition of the Reynolds stress terms

$$-\rho\overline{u'^2}, \ -\rho\overline{v'^2}, \ -\rho\overline{w'^2}, \ -\rho\overline{u'v'}, \ -\rho\overline{u'w'}, \ -\rho\overline{v'w'}.$$

There are now four equations – three components of the averaged momentum equation, plus the averaged mass conservation equation, i.e., Equations (6), (8), (10), and (12). There are, however, ten unknowns – $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$, $-\rho\overline{u'^2}$, $-\rho\overline{v'^2}$, $-\rho\overline{w'^2}$, $-\rho\overline{u'v'}$, $-\rho\overline{u'w'}$, and $-\overline{v'w'}$. More information is needed to have a well-posed mathematical problem.

To obtain this information one would think first to develop equations for the new quantities, such as for $\rho\overline{u'^2}$. This can be done, as will be seen in the handout on the $\bar{k}$-$\epsilon$ modeling. However the equation for $\overline{u'^2}$ contains a number of additional unknowns, such as $\overline{v'u'^2}$ and $\overline{u'\dfrac{\partial p'}{\partial x}}$. Equations for these quantities lead to even further unknowns. The process never leads to a well-posed mathematical problem, there always being many more unknowns than equations. This is at the heart of the 'closure problem' in turbulence, and has as yet to be resolved. Instead researchers resort to hypotheses which are physically-based, but not totally justifiable. One such hypothesis used in much of turbulence modeling introduces a turbulence, or eddy, viscosity; this is discussed in the next handout.