ME 431/599F Handout 28 January 2008

Time-Averaged Equations of Motion

In class and in the text (Sections 6.2, 6.3, and 6.8), it was shown that, for the incompressible flow of a Newtonian fluid, the conservation of mass and the momentum balance are, in vector form:

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \frac{\partial}{\partial t}\mathbf{v} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{v}\,.$$

In component form in Cartesian coordinates, the conservation of mass is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \tag{1}$$

while the x-component of the momentum balance is

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{2}$$

For the moment it is convenient to write this last equation in viscous stress form, i.e.,

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\,, \tag{3}$$

where

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x}\,, \quad \tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \tau_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right). \tag{4}$$

We now make Reynolds decompositions for u, v, w, and p, i.e.,

$$u = \bar{u} + u'\,, \quad v = \bar{v} + v'\,. \quad w = \bar{w} + w'\,, \quad p = \bar{p} + p'. \tag{5}$$

When these forms are substituted into Eqn. (1), averaged, and noting that

$$\overline{\frac{\partial(\bar{u} + u')}{\partial x}} = \overline{\frac{\partial \bar{u}}{\partial x}} + \overline{\frac{\partial u'}{\partial x}} = \frac{\partial \bar{\bar{u}}}{\partial x} + \frac{\partial \overline{u'}}{\partial x} = \frac{\partial \bar{u}}{\partial x}\,,$$

and similarly for the other terms in Eqn. (1), the result is

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0\,, \tag{6}$$

the averaged form of the conservation of mass. Note that the averaged velocity then also satisfies Eqn. (1). Furthermore, if Eqn. (6) is subtracted from Eqn. (1), noting Eqn. (5), the result is

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0\,. \tag{7}$$

Therefore the fluctuating velocity also satisfies Eqn. (1).

Next consider the x-component of the momentum equation in viscous stress form, Eqn. (3). Plugging in Reynolds decompositions, Eqn. (5), will result in the following terms.

$$\begin{aligned}\overline{\frac{(\bar{u}+u')\partial(\bar{u}+u')}{\partial x}} &= \overline{\bar{u}\frac{\partial \bar{u}}{\partial x}} + \overline{u'\frac{\partial \bar{u}}{\partial x}} + \overline{\bar{u}\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial u'}{\partial x}} \\ &= \bar{u}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}}\,.\end{aligned}$$

$$\begin{aligned}\overline{\frac{(\bar{v}+v')\partial(\bar{u}+u')}{\partial y}} &= \overline{\bar{v}\frac{\partial \bar{u}}{\partial y}} + \overline{v'\frac{\partial \bar{u}}{\partial y}} + \overline{\bar{v}\frac{\partial u'}{\partial y}} + \overline{v'\frac{\partial u'}{\partial y}} \\ &= \bar{v}\frac{\partial \bar{u}}{\partial y} + \overline{v'\frac{\partial u'}{\partial y}}\,.\end{aligned}$$

$$\begin{aligned}\overline{\frac{(\bar{w}+w')\partial(\bar{u}+u')}{\partial z}} &= \overline{\bar{w}\frac{\partial \bar{u}}{\partial z}} + \overline{w'\frac{\partial \bar{u}}{\partial z}} + \overline{\bar{w}\frac{\partial u'}{\partial z}} + \overline{w'\frac{\partial u'}{\partial z}} \\ &= \bar{w}\frac{\partial \bar{u}}{\partial z} + \overline{w'\frac{\partial u'}{\partial z}}\,.\end{aligned}$$

Furthermore,

$$\overline{\frac{\partial(\bar{p}+p')}{\partial x}} = \frac{\partial \bar{p}}{\partial x}$$

Finally,

$$\overline{\frac{\partial u}{\partial t}} = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \frac{\partial u}{\partial t}\,dt = \frac{1}{\mathcal{T}}(u(\mathcal{T}) - u(0)) \to 0 \quad \text{as} \quad \mathcal{T} \to \infty\,.$$

When these averaged quantities are plugged into Eqn. (3), the result is

$$\begin{aligned}\rho\bar{u}\frac{\partial \bar{u}}{\partial x} + \rho\bar{v}\frac{\partial \bar{u}}{\partial y} + \rho\bar{w}\frac{\partial \bar{u}}{\partial z} &= g_x - \frac{\partial \bar{p}}{\partial x} && (8) \\ &+ \frac{\partial}{\partial x}(\bar{\tau}_{xx} - \rho\overline{u'u'}) + \frac{\partial}{\partial y}(\bar{\tau}_{xy} - \rho\overline{u'v'}) + \frac{\partial}{\partial z}(\bar{\tau}_{xz} - \rho\overline{u'w'})\,, && (9)\end{aligned}$$

where Eqn. (7) has been used, and

$$\bar{\tau}_{xx} = 2\mu\frac{\partial \bar{u}}{\partial x}\,, \quad \bar{\tau}_{xy} = \mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right), \quad \bar{\tau}_{xz} = \mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right). \tag{10}$$

Equations (9) and (10) give the averaged x-component of the momentum balance. The y- and z-components can be obtained in a similar manner.

Note that the equations for $\bar{u}$, $\bar{v}$, $\bar{w}$, and $\bar{p}$ are of the same form as those for u, v, w, and p, except for the addition of the Reynolds stress terms

$$-\rho\overline{u'^2}\,,\ -\rho\overline{v'^2}\,,\ -\rho\overline{w'^2}\,,\ -\rho\overline{u'v'}\,,\ -\rho\overline{u'w'}\,,\ -\rho\overline{v'w'}\,.$$

There are now four equations – three components of the averaged momentum equation, plus the averaged mass conservation equation. There are, however, ten unknowns – $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$, $-\rho\overline{u'^2}$, $-\rho\overline{v'^2}$, $-\rho\overline{w'^2}$, $-\rho\overline{u'v'}$, $-\rho\overline{u'w'}$, and $-\overline{v'w'}$. More information is needed to have a well-posed mathematical problem.