For convenience, drop $(\bar{\ })$ notation.

(1) $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{\nu_T}{P_{r_T}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$

(2) $\frac{T(r,x)}{\mathcal{T}(x)} = I(\eta) \quad \eta = \frac{r}{l(x)}, \quad \frac{u(r,x)}{U(x)} = F(\eta), \quad \frac{v(r,x)}{U(x)} = G(\eta)$

From the class notes,

(3) $\mathcal{T}(x) = S x^{-1}$, and

(4) $\frac{H}{\rho} = 2\pi\rho c\, U(x)\mathcal{T}(x) l^2(x)\int_0^\infty \eta F(\eta) I(\eta) d\eta = \text{Const}$

This will ultimately be used to determine S.

(5) $\frac{\partial T}{\partial x} = \mathcal{T}' I + \mathcal{T} I'\left(-\frac{r}{l^2}\right)l' = \mathcal{T}' I - \mathcal{T}\frac{l'}{l}\eta I'$

(6) $\frac{\partial T}{\partial r} = \mathcal{T} I'\frac{1}{l}$

(7) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{l^2}\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta\mathcal{T} I') = \frac{\mathcal{T}}{l^2}\frac{1}{\eta}(\eta I')'$

From the previous notes (6 Mar 08),

(8) $\frac{u(r,x)}{U} = \frac{1}{\eta}f' = \frac{1}{\left(1+\frac{a}{8}\eta^2\right)^2} \qquad a \doteq 3.31$

(9) $\frac{v(r,x)}{U} = l'\left(f' - \frac{f}{\eta}\right) \qquad l' = 0.0848$

Plugging (5) to (9) into (1) results in

$U\frac{1}{\eta}f'\left(\mathcal{T}' I - \mathcal{T}\frac{l'}{l}\eta I'\right) + Ul'\left(f' - \frac{f}{\eta}\right)\mathcal{T} I'\frac{1}{l} = \frac{\nu_T}{P_{r_T}}\frac{\mathcal{T}}{l^2}\frac{1}{\eta}(\eta I')'$

$U\mathcal{T}'\frac{1}{\eta}f' I - U\mathcal{T}\frac{l'}{l}\frac{1}{\eta}f'\eta I' + U\mathcal{T}\frac{l'}{l}f' I' - U\mathcal{T}\frac{l'}{l}\frac{f}{\eta}I'$

$= \frac{\nu_T}{P_{r_T}}\frac{\mathcal{T}}{l^2}\frac{1}{\eta}(\eta I')'$, or

(10) $U\mathcal{T}'\frac{1}{\eta}f' I - U\mathcal{T}\frac{l'}{l}\frac{f}{\eta}I' = \frac{\nu_T}{P_{r_T}}\frac{\mathcal{T}}{l^2}\frac{1}{\eta}(\eta I')'$

$= \frac{c_\mu}{P_{r_T}}\frac{Ul\mathcal{T}}{l^2}\frac{1}{\eta}(\eta I')' = \frac{c_\mu}{P_{r_T}}\frac{U\mathcal{T}}{l}\frac{1}{\eta}(\eta I')'$

(11) with $\nu_T = c_\mu U(x) l(x)$ from the previous notes.

Multiplying (10) through by $\frac{\eta l}{U\mathcal{T}}\frac{1}{l'}$ gives:

$$\frac{\gamma'}{\gamma}\frac{l}{l'}\,f'I - fI' = \frac{1}{P_{r_T}}\frac{cu}{l'}(\eta I')'$$

But $\frac{\gamma'}{\gamma}\frac{l}{l'} = \frac{-Sx^{-2}}{Sx^{-1}}\left(\frac{ax}{a}\right) = -1$, so

$$-f'I - fI' = \frac{1}{P_{r_T}}\frac{cu}{l'}(\eta I')',$$

or, with $\frac{cu}{l'} \equiv \frac{1}{a}$ as in the previous notes.

(12) $$-(fI)' = \frac{1}{P_{r_T}}\frac{1}{a}(\eta I')'$$

Integrating this once gives:

(13) $$-(fI) = \frac{1}{P_{r_T}}\frac{1}{a}\eta I'$$

using the axisymmetry conditions that

(14) $$\left.\frac{\partial T}{\partial r}\right|_{r=0} = \left.\frac{\partial T}{\partial r}\right|_{\eta=0} = \left.\gamma^{\circ} I'\frac{1}{l}\right|_{\eta=0} = 0, \text{ or } I'|_0 = 0$$

$v|_{r=0} = 0$ implying $\frac{f}{\eta} \to 0$ as $\eta \to 0$ from the previous notes.

Therefore

(15) $$\frac{I'}{I} = -P_{r_T}\,a\,\frac{f}{\eta}$$

From the previous notes,

(16) $$f(\eta) = \frac{1}{2}\frac{\eta^2}{\left(1+\frac{a}{8}\eta^2\right)}, \text{ so}$$

$$\frac{dI}{I} = -P_{r_T}\,a\,\frac{1}{2}\,\frac{\eta^2\,d\eta}{\eta\left(1+\frac{a}{8}\eta^2\right)} = -\frac{1}{2}P_{r_T}\,a\,\frac{\eta\,d\eta}{\left(1+\frac{a}{8}\eta^2\right)}$$

Integrating this gives

$$\ln I = -\frac{1}{2}P_{r_T}\,a\,\frac{\ln\left(1+\frac{a}{8}\eta^2\right)}{2\frac{a}{8}} = -2P_{r_T}\ln\left(1+\frac{a}{8}\eta^2\right)$$

$$= \ln\left(1+\frac{a}{8}\eta^2\right)^{-2P_{r_T}}, \text{ so}$$

(17) $$I(\eta) = \frac{1}{\left(1+\frac{a}{8}\eta^2\right)^{2P_{r_T}}}$$

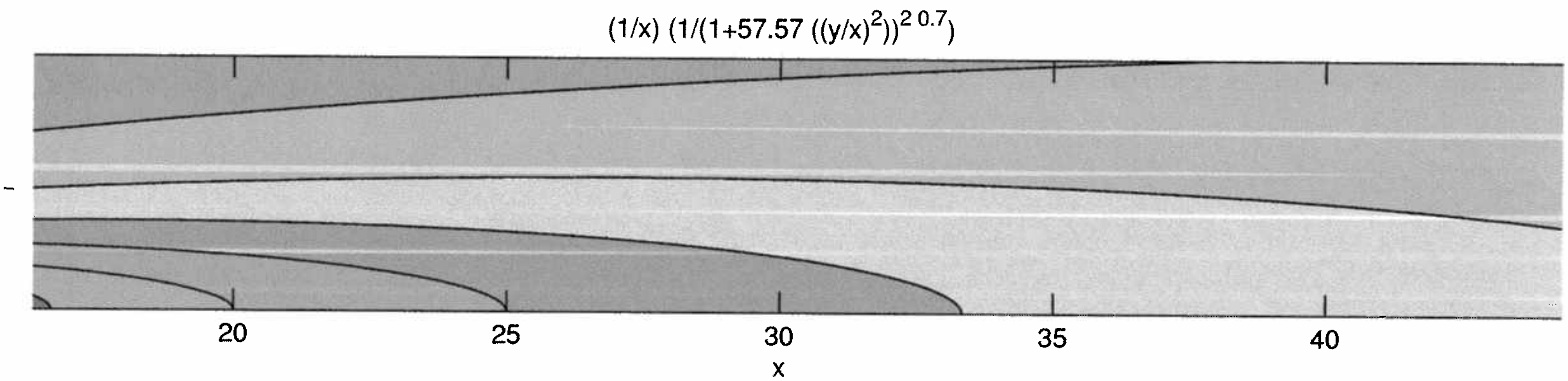
(Note - in the last integration, the constant of integration was used to insure that $I(0) = 1$.)

To determine S, plug (17) into (4), along with expressions for $U(x)$, $l(x)$ and $F(\eta)$ to give

$$(18)\quad H_f = 2\pi\rho c\ 7.41\sqrt{\frac{M}{\rho}}\,\frac{1}{x}\,S x^{-1} A^2 x^2 \underbrace{\int_0^\infty \eta \frac{1}{(1+\frac{a}{8}\eta^2)^2}\cdot\frac{d\eta}{(1+\frac{a}{8}\eta^2)^{2P_T}}}_{\frac{0.208}{a/8} \doteq 0.503},\ \text{so}$$

$$(19)\quad S = \sqrt{\frac{\rho}{M}}\,\frac{H_f}{2\pi\rho c} \times 37.33,\ \text{so}$$

$$(20)\quad T(r,x) = 37.33\,\frac{H_f}{2\pi\rho c}\sqrt{\frac{\rho}{M}}\,\frac{1}{x}\,\frac{1}{(1+0.412\eta^2)^{2P_T}} \qquad \eta = \frac{r}{l(x)}$$



Color contour plot of $\bar{T}(r,x)$ for the case of a self-similar round jet