1 Similarity Analysis

1.1 Introduction

Consider the sketch of an axisymmetric, turbulent jet in Figure 1. Assume that measurements of the downstream average axial velocity profile $\bar{u}(x,r)$ are made downstream of the jet exit at various locations which satisfy $x/D > 10$ to 20, where D is the jet diameter. In particular, the velocity maxima $\bar{u}_m$ and the jet half-width ℓ are obtained from the measurements. For this axisymmetric jet, the maximum velocity $\bar{u}$ is found along the jet axis, $r = 0$. The half-width of the jet at a particular location in x is defined as the distance in y where the average jet velocity has dropped to one-half of the peak velocity; i.e., it is defined mathematically at the downstream location x as (refer to Figure 1):

$$\left.\frac{\bar{u}(x,r)}{u_m(x)}\right|_{r=\ell(x)} = \frac{1}{2}\,.$$

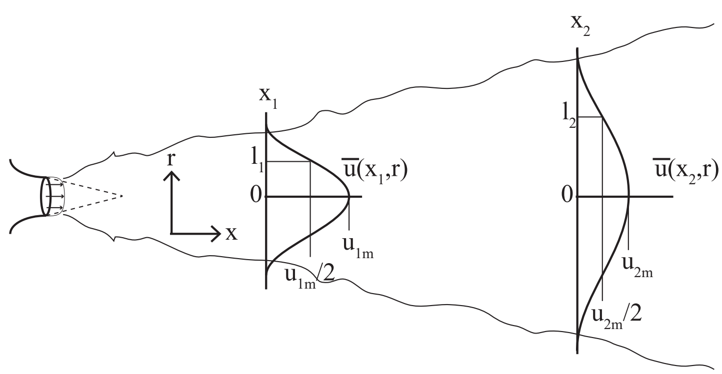

Figure 1: Sketch of an axisymmetric jet with measurements of the profiles of the average axial velocity $\bar{u}$ at the downstream locations x_1 and x_2.

Now assume that measurement of the mean axial velocity profile are made, in particular, at locations x_1 and x_2, i.e., $\bar{u}(x_1,r)$ and $\bar{u}(x_2,r)$ are measured. The peak velocities, $u_m(x_1) = u_{1m}$ and $u_m(x_2) = u_{2m}$, and the half-widths, $\ell(x_1) = \ell_1$ and $\ell(x_2) = \ell_2$, are determined. The mean velocities $\bar{u}$ normalized by u_m are then plotted on the same figure versus r normalized by ℓ, as seen in Figure 2. Plotted in this manner, the data collapse onto the same curve, i.e.,

$$\frac{\bar{u}(x_1,r)}{u_{1m}} = \mathcal{F}(r/\ell_1) = \frac{\bar{u}(x_2,r)}{u_{2m}} = \mathcal{F}(r/\ell_2)\,,$$

or, more generally,

$$\frac{\bar{u}(x,r)}{u_m} = \mathcal{F}(r/\ell)$$

for any downstream position x which is greater than $10D$ to $20D$. A flow which has this property is called 'self-similar' or 'self-preserving'. An important feature of such a flow is that, once $U_m(x)$, $\ell(x)$, and $\mathcal{F}$ are found, then $\bar{u}(x,r)$ is known. We will find that it is easier to determine these quantities, either from experiments or from theory and numerical simulations, than it is to find

$\bar{u}(x,r)$ itself. Furthermore, knowing $u_m(x)$ and $\ell(x)$ will allow us to make several inferences about the flows.

Note that wakes, mixing layers, and boundary layers also become self-similar. Furthermore, the concept of self-similarity is also useful in heat transfer and a number of other technical fields.

1.2 Self-similarity applied to the time-averaged equations

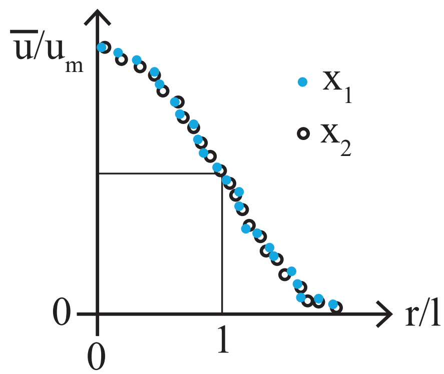

Figure 2: Sketch of an axisymmetric jet with measurements of the profiles of the average axial velocity $\bar{u}$ at the downstream locations x_1 and x_2.

As has been established in class, the boundary layer equations hold for the case of an axisymmetric, turbulent jet when $x/D > 10$ to 20, depending on the characteristics of the nozzle. These equations are:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'v'}) \tag{1}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}) = 0\,. \tag{2}$$

The similarity assumptions for the two components of the average velocity and for the Reynolds stress are:

$$\frac{\bar{u}(x,r)}{U(x)} = \mathcal{F}\left(\frac{r}{\ell(x)}\right) \tag{3}$$

$$\frac{\bar{v}(x,r)}{U(x)} = \mathcal{G}\left(\frac{r}{\ell(x)}\right)$$

$$\frac{\overline{u'v'}(x,r)}{U^2(x)} = \mathcal{H}\left(\frac{r}{\ell(x)}\right).$$

Here $U(x)$ and $\ell(x)$ are defined by:

$$U(x) = \bar{u}(x,r)\Big|_{\max} = \bar{u}(x,0)$$

$$\frac{\bar{u}(x,r)}{U(x)}\Big|_{r=\ell} = \frac{1}{2}\,,\ \text{i.e.,}$$

for a given downstream distance x, $U(x)$ is the maximum velocity (which is along the centerline), and ℓ is the radial distance at which $\bar{u}$ equals 1/2 of its maximum value (ℓ is often referred to as the half-width).

When these assumed forms are plugged into the various terms in the momentum and continuity equations, the following derivatives result:

$$\frac{\partial \bar{u}}{\partial x} = U'\mathcal{F} - U\frac{\ell'}{\ell}\eta\mathcal{F}' \,,$$

where $(\cdot)'$ denotes the derivative of a function with respect to its argument, and $\eta = r/\ell$. Also

$$\frac{\partial \bar{u}}{\partial r} = \frac{U}{\ell}\mathcal{F}'$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'v'}) = \frac{U^2}{\ell}\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta\mathcal{H})$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}) = \frac{U}{\ell}\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta\mathcal{G}) \,.$$

Finally, plugging these expressions into the original equations gives, after multiplying the momentum equation by ℓ/U^2 and the continuity equation by ℓ/U:

$$\left(\frac{U'\ell}{U}\right)\mathcal{F}^2 - \ell'\eta\mathcal{F}\mathcal{F}' + \mathcal{F}'\mathcal{G} = -\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta\mathcal{H}) \tag{4}$$

$$\left(\frac{U'\ell}{U}\right)\mathcal{F} - \ell'\eta\mathcal{F}' + \frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta\mathcal{G}) = 0 \,. \tag{5}$$

Next assume power law forms for U and ℓ, i.e.,

$$U(x) = Ax^{-n} \qquad \ell(x) = Bx^m \,,$$

where it is expected that $m, n > 0$. Note that the coefficients in Equations (4) and (5) must be constant for the equations to be consistent; i.e., since the last terms are only functions of η, then each term in the equations can only be a function of η and not x. This implies that

$$\frac{U'\ell}{U} = \frac{-nAx^{-n-1}}{Ax^{-n}}Bx^m = -nBx^{m-1} = \text{constant} \,,$$

so that $m = 1$ and

$$\ell(x) = Bx^1 \,. \tag{6}$$

Note that the coefficients of the first two terms in Equations (4) and (5) are $-nB$.

It has also been established in class that the momentum flux across any vertical plane perpendicular to the flow direction is constant, i.e.,

$$\mathcal{M} = 2\pi\rho\int_0^\infty r\bar{u}^2(x,r)dr = 2\pi\rho\ell^2U^2\int_0^\infty \eta\mathcal{F}^2 d\eta = \text{constant} \,.$$

Note that the last integral (over η) in this equation is constant, since it is an integral over a given function for fixed limits. Therefore,

$$\ell^2U^2 = B^2x^2A^2x^{-2n} = \text{constant} \,,$$

so $n = 1$, and

$$U(x) = Ax^{-1} \,. \tag{7}$$

With these constant coefficients, and the values found for n and m, Equations (4) and (5) reduce to:

$$-B\mathcal{F}^2 - B\eta\mathcal{F}\mathcal{F}' + \mathcal{F}'\mathcal{G} = -\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta\mathcal{H}) \tag{8}$$

$$-B\mathcal{F} - B\eta\mathcal{F}' + \frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta\mathcal{G}) = 0 \,. \tag{9}$$

Note that we still have two equations with three unknowns, the unknowns now being $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$. We still have to deal with the turbulence closure problem and eliminate one of the unknowns before solving the equations.

1.3 Self-similarity applied to the mass flux

Before considering turbulence modeling for this problem it is useful to consider the implications of Equations (6) and (7) on the mass flux $m(x)$. Using a definition similar to that for momentum flux, the mass flux through the vertical plane perpendicular to the flow direction is:

$$\begin{aligned} m(x) = \int_{\text{vertical plane}} \rho(\mathbf{v}\cdot\mathbf{n})\, d\mathcal{A} &= \rho \int_0^{2\pi}\int_0^{\infty} \bar{u}(x,r)\, \underbrace{dr\, r d\theta}_{d\mathcal{A}} \\ &= 2\pi\rho \int_0^{\infty} r\bar{u}(x,r)\, dr = 2\pi\rho\, U(x)\ell^2(x) \int_0^{\infty} \eta\mathcal{F}(\eta)\, d\eta, \end{aligned} \tag{10}$$

where the similarity form for $\bar{u}(x,r)$, Equation (3), has been used. Since the definite integral $\int_0^{\infty} \eta\mathcal{F}(\eta)\, d\eta$ must be a constant, then, using Equations (6) and (7),

$$m(x) \propto U(x)\ell^2(x) \propto x^{-1}x^2 = x \,,$$

that is, the mass is the jet increases proportional to x. This is a direct result of entrainment of fluid into the jet, is an essential feature of jets, and can be important in problems such as nonpremixed combustion.

This has implications regarding entrainment. As the flow goes from x_1 to x_2, the mass flux in the jet increases by an amount proportional to $x_2 - x_1$ due to entrainment (see Figure 3).

2 Turbulence modeling

The mixing length assumption for the Reynolds stress is:

$$-\overline{u'v'} = \nu_T \frac{\partial\bar{u}}{\partial r}$$

with the turbulent viscosity ν_T given by

$$\nu_T = c\,\mathcal{U}\mathcal{L} \,.$$

Here c is a constant, to be determined using theory, numerical simulations, or laboratory data; $\mathcal{U}$ is a characteristic velocity and $\mathcal{L}$ a characteristic length scale of the turbulence. Using the similarity

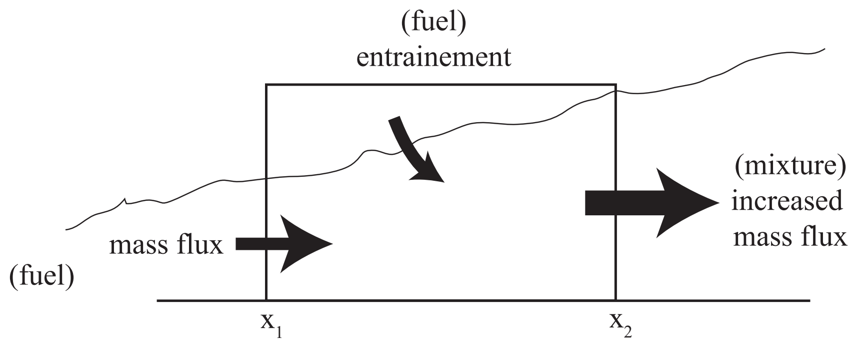

Figure 3: Sketch of the entrainment process in a jet flowing from the downstream location x_1 to x_2 as its mass increases by an amount proportional to $x_2 - x_1$.

assumption for the Reynolds stress, and identifying the velocity scale $\mathcal{U}$ as U and the length scale $\mathcal{L}$ as ℓ, then

$$\overline{u'v'} = U^2\mathcal{H} = c\,U\ell\frac{\partial}{\partial r}(U\mathcal{F}) = -c\,U^2\frac{\partial \mathcal{F}}{\partial \eta}\,,\text{or}$$

$$\mathcal{H} = -c\frac{\partial \mathcal{F}}{\partial \eta}\,.$$

With this in the momentum equation, Equation (8), it becomes, finally:

$$-B\mathcal{F}^2 - B\eta\mathcal{F}\mathcal{F}' + \mathcal{F}'\mathcal{G} = -\frac{c}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \mathcal{F}}{\partial \eta}\right). \tag{11}$$

The exact solution to these equations have been found to be:

$$\frac{\bar{u}(x,r)}{U(x)} = \mathcal{F}(\eta) = \frac{1}{\left(1+\frac{1}{8}\mathcal{A}\eta^2\right)^2} \quad \text{with} \quad \eta = \frac{r}{\ell(x)}$$

where

$$U(x) \equiv \bar{u}(x,r)\Big|_{r=0} \quad \text{and} \quad \frac{\bar{u}(x,r)}{U(x)}\Big|_{r=\ell(x)} = \frac{1}{2}$$

$$\frac{\bar{v}(x,r)}{U(x)} = \mathcal{G}(\eta) = \frac{d\ell}{dx}\frac{\eta}{2}\frac{\left[1-\frac{1}{8}\mathcal{A}\eta^2\right]}{\left[1+\frac{1}{8}\mathcal{A}\eta^2\right]^2}$$

$$\frac{\overline{u'v'}}{U^2(x)} = \mathcal{H}(\eta) = \frac{1}{2}\mathcal{A}\,c\frac{\eta}{\left[1+\frac{1}{8}\mathcal{A}\eta^2\right]^3}$$

In order for $\frac{\bar{u}(x,r)}{U(x)}\Big|_{r=\ell(x)} = \frac{1}{(1+\frac{1}{8}\mathcal{A})^2} = \frac{1}{2}$, then $\mathcal{A} \doteq 3.31$. Furthermore, to match laboratory data, then

$$c = 0.0256\,, \text{ and}$$

$$\ell(x) = Bx \quad \text{with} \quad B = 0.0848$$

$$U(x) = Ax^{-1} \quad \text{with} \quad A = 7.41\sqrt{\frac{\mathcal{M}}{\rho}}\,.$$

Figure 4 gives plots of $\bar{u}(x,r)/U$, $\bar{v}(x,r)/U$, and $\sqrt{\overline{u'v'}}/U$ as functions of $\eta = r/\ell$. A rough idea of the laboratory data is also given in the figure. Note that the agreement with the data is fairly good, partly since the constants have been adjusted to optimize agreement for this flow. The agreement of the model with the data begins to break down at around $\eta > 3$, where the data points drop below the model prediction, and where also the turbulent flow has become very intermittent. This intermittency is not accounted for in the modeling.

Some consistency checks can be made regarding the modeling assumptions. For example,

$$\frac{\partial \bar{u}}{\partial x} \sim \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(Ax^{-1}) = -\frac{1}{x}(Ax^{-1}) = -\frac{U}{x}\,.$$

So the differential scale for $\bar{u}(x,r)$ in x is just x itself, i.e., $\ell_x = x$. The differential scale in r was found to be $\ell_r = Bx$. Therefore,

$$\frac{\ell_r}{\ell_x} = \frac{Bx}{x} = B = 0.0848\,,$$

which is rather small and roughly consistent with the assumption that $\ell_r/\ell_x \ll 1$. Furthermore, with some calculus and algebra, it can be found that

$$\frac{\bar{v}}{U} \approx 0.0127\,.$$

This is consistent with the implied assumption that $\bar{v}_m/\bar{u}_m \ll 1$. Finally, some further calculus and algebra leads to

$$\frac{\overline{u'v'}_m}{U^2} = 0.017\,,$$

consistent with the conclusion that $\overline{u'v'}_m/U^2 \ll 1$.

3 Summary – Axisymmetric Jet

Results similar to those for the axisymmetric jet can be obtained for plane jets, axisymmetric and plane wakes, axisymmetric and plane mixing layers, and axisymmetric and plane buoyant plumes. What follows is a summary of the main results for the axisymmetric, turbulent jet.

- Principal assumptions:
  - the flow in 'thin', i.e., $\frac{\partial}{\partial r}(\cdot) \gg \frac{\partial}{\partial x}(\cdot)$
  - high Reynolds number: $Re = \frac{U\ell}{\nu} \gg 1$
  - the result is the boundary layer equations:

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}) = 0$$

$$\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'v'})$$

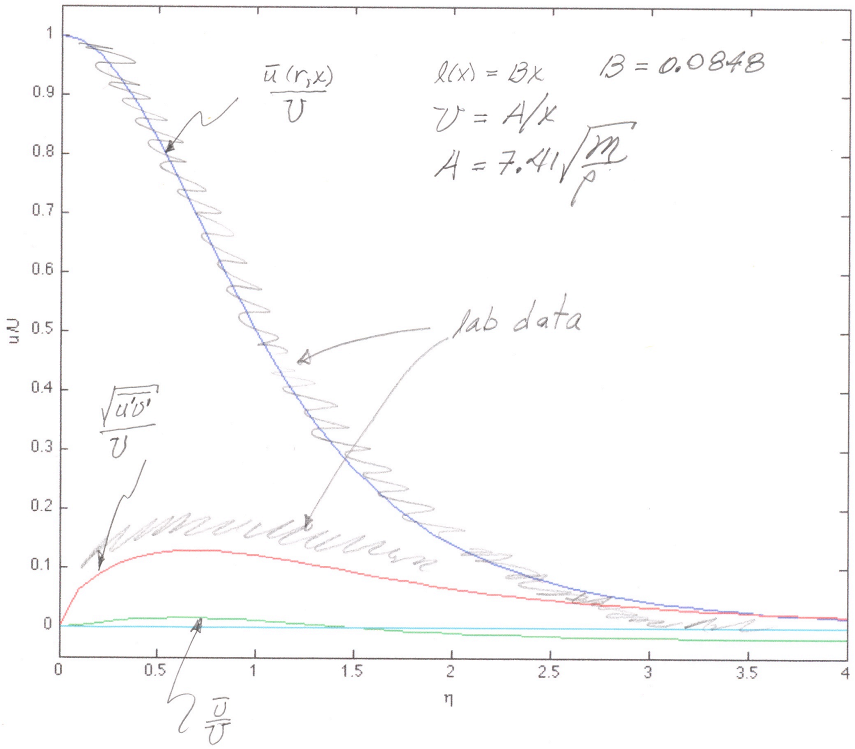

Figure 4: Plots of $\bar{u}(x,r)/U(x)$, $\bar{v}(x,r)/U(x)$, and $\sqrt{\overline{u'v'}}/U(x)$ as functions of $\eta = r/\ell(x)$.

- With the momentum flux $M(x)$ defined as

$$M(x) = 2\pi\rho \int_0^\infty r\bar{u}^2(x,r)dr\,,$$

then, for $x/D > 10$ to 20, $M(x)$ is constant in x. The momentum flux is one of the principal features defining a jet.

- On the other hand, with the mass flux $m(x)$ defined as

$$m(x) = 2\pi\rho \int_0^\infty r\bar{u}(x,r)dr\,,$$

then, for $x/D > 10$ to 20, $m(x)$ increases proportional to x. This is closely related to the entrainment process.

- With a closure assumption for the Reynolds stress $-\rho\overline{u'v'}$ in terms of a turbulent viscosity, solutions can be obtained for $\bar{u}$, $\bar{v}$, and $\overline{u'v'}$ giving fairly good agreement with the data. However, some adjustable constants have to properly chosen to obtain the agreement. This is indicative of the turbulence modeling found in the various commercial codes: to some degree they amount to sophisticated curve-fitting.