Consider the two-dimensional, incompressible, turbulent jet from a slot, described by the time-averaged Navier-Stokes equations:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) - \frac{\partial}{\partial x}\overline{u'^2} - \frac{\partial}{\partial y}\overline{u'v'} \quad (1)$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right) - \frac{\partial}{\partial x}\overline{u'v'} - \frac{\partial}{\partial y}\overline{v'^2} \quad (2)$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0\,. \quad (3)$$

The following two lengths scales are defined:

- $\ell_x$ – the length in the $x$-direction over which the streamwise mean velocity $\bar{u}$ changes by a specified amount, say decreasing by a factor of 2;
- $\ell_y$ – the length in the $y$-direction over which $\bar{u}$ changes by the same amount.

It is observed experimentally that $\ell_y/\ell_x \ll 1$, i.e., the velocity changes much more quickly in $y$ than in $x$, i.e., it is 'thin'. The following rough scaling analysis is based upon two principal assumptions:

1. $\ell_y(x)/\ell_x(x) \ll 1$, i.e., the jet is locally (in $x$) 'thin', and
2. $Re = U\ell_y/\nu \gg 1$, i.e., the local (in $x$) Reynolds number of the flow is very high, where $U(x)$ is the peak mean streamwise velocity at a downstream location $x$.

A rough estimate will be made for each term in the time-averaged equations. To do this, an order-of-magnitude estimate for each dependent and independent variable will be needed. These estimates are the following for any arbitrary downstream location $x$.

- $\bar{u}\,,\,\Delta\bar{u} \sim U = \bar{u}|_{\text{max}}$. Here the symbol $\sim$ implies order-of-magnitude equality, $\bar{u}|_{\text{max}}$ is the maximum average axial velocity at the downstream location $x$, and $\Delta$ denotes a differential scale.
- $\bar{v}\,,\,\Delta v \sim V = ?$. That is, the scaling for the $y$-component of the mean velocity, $V$, is not known at this point.
- $\Delta x \sim \ell_x$
- $\Delta y \sim \ell_y$
- $\Delta p \sim \rho U^2$, a dynamic pressure, using a Bernoulli-like scaling for pressure.
- $\overline{u'v'} \sim \mathcal{U}^2 = ?$. The scaling for the Reynolds stresses is also unknown at this point.

Using this scaling in the continuity equation, Equation (3), gives

$$\frac{V}{\ell_y} \sim \frac{U}{\ell_x}\,,\ \text{or} \quad (4)$$

$$V \sim U\left(\frac{\ell_y}{\ell_x}\right) \ll U \quad (5)$$

i.e., $V$ is expected to be very small compared to $U$, which is consistent with the assumption that the jet is 'thin'.

Next, scaling is presented for each term in the $x$-component of the mean momentum equation, Equation (1):

$$\underbrace{\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial x}}_{U^2/\ell_x}}_{1} + \underbrace{\underbrace{\bar{v}\frac{\partial \bar{u}}{\partial y}}_{U^2/\ell_x}}_{1} = \underbrace{\underbrace{-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}}_{U^2/\ell_x}}_{1} + \nu\Bigg( \underbrace{\underbrace{\frac{\partial^2 \bar{u}}{\partial x^2}}_{\nu U/\ell_x^2}}_{(\nu/U\ell_y)(\ell_y/\ell_x)} + \underbrace{\underbrace{\frac{\partial^2 \bar{u}}{\partial y^2}}_{\nu U/\ell_y^2}}_{(\nu/U\ell_y)(\ell_x/\ell_y)} \Bigg) - \underbrace{\underbrace{\frac{\partial}{\partial x}\overline{u'^2}}_{\mathcal{U}^2/\ell_x}}_{\mathcal{U}^2/U^2} - \underbrace{\underbrace{\frac{\partial}{\partial y}\overline{u'v'}}_{\mathcal{U}^2/\ell_y}}_{(\mathcal{U}^2/U^2)(\ell_x/\ell_y)} \tag{6}$$

In the above equation, the first line below the equation represents the scaling of each term, while the second line represent the scaling, but divided by the common term $U^2/\ell_x$. From examining this last line, we see that the penultimate term on the right-hand side must be much less than the last term, since $\ell_x/\ell_y \gg 1$ by assumption; so the penultimate term can be neglected. Furthermore, the second term on the right-hand side must be small compared to 1 based upon the high Reynolds assumption, and so can be neglected. The third term on the right-hand side may or may not be small, since $\nu/U\ell_y \ll 1$ whereas $\ell_x/\ell_y \gg 1$. It will be assumed that this term is small and can be neglected, but this will have to be checked *a posteriori*, i.e., after the fact. This leaves the two terms on the left-hand side, the pressure term, and the last term on the right-hand side. For turbulence to have an effect on the flow, this last term must be of order 1, which indicates that the scaling for $\mathcal{U}^2$ must be:

$$\frac{\mathcal{U}^2}{U^2}\frac{\ell_x}{\ell_y} \sim 1\,, \text{ or } \frac{\mathcal{U}^2}{U^2} \sim \frac{\ell_y}{\ell_x}\,, \text{ or } \frac{\mathcal{U}}{U} \sim \left(\frac{\ell_y}{\ell_x}\right)^{1/2},$$

giving the scaling for $\mathcal{U}$. Therefore the $x$-component of the mean momentum equation reduces to:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{\partial}{\partial y}\overline{u'v'}\,. \tag{7}$$

Note that the inferences from this scaling are the following.

1. Viscous stresses are much smaller than the Reynolds stresses by the ratio of $1/Re$, which is typical of turbulent flows except very near boundaries.

2. The effect of the normal stress $\overline{u'^2}$ is much less than the effect of the shear stress $\overline{u'v'}$.

Next the $y$-component of the mean momentum equation is scaled in the same way, giving:

$$\underbrace{\underbrace{\bar{u}\frac{\partial \bar{v}}{\partial x}}_{(U^2/\ell_y)(\ell_y/\ell_x)^2}}_{(\ell_y/\ell_x)^2} + \underbrace{\underbrace{\bar{v}\frac{\partial \bar{v}}{\partial y}}_{(U^2/\ell_y)(\ell_y/\ell_x)^2}}_{(\ell_y/\ell_x)^2} = \underbrace{\underbrace{-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}}_{U^2/\ell_y}}_{1} + \nu\Bigg( \underbrace{\underbrace{\frac{\partial^2 \bar{v}}{\partial x^2}}_{\nu U\ell_y/\ell_x^3}}_{(\nu/U\ell_y)(\ell_y/\ell_x)^3} + \underbrace{\underbrace{\frac{\partial^2 \bar{v}}{\partial y^2}}_{\nu U/\ell_y\ell_x}}_{(\nu/U\ell_y)(\ell_y/\ell_x)} \Bigg) - \underbrace{\underbrace{\frac{\partial}{\partial x}\overline{u'v'}}_{\mathcal{U}^2/\ell_x}}_{(\ell_y/\ell_x)^2} - \underbrace{\underbrace{\frac{\partial}{\partial y}\overline{v'^2}}_{\mathcal{U}^2/\ell_y}}_{(\ell_y/\ell_x)} \tag{8}$$

Again the first line below the equation represents the scaling of each term, while the next line represents the scaling, but divided in this case by the common term $U^2/\ell_y$. From the original assumptions, the only term possible of order 1 is the pressure term; all the remaining terms must be much smaller. Thus the $y$-component of the mean momentum equation reduces to :

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}\,. \tag{9}$$

This implies that the mean pressure $\bar{p}$ is independent of $y$. This is due to the fact that the jet is 'thin', so that, at a given downstream location $x$, the pressure is uniform across the jet.

Therefore, based upon the assumptions that $\ell_y/\ell_x \ll 1$ and $Re \gg 1$, the mean momentum and continuity equations reduce to:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{\partial}{\partial y}\overline{u'v'} \tag{10}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} \tag{11}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0\,. \tag{12}$$

These equations are often referred to as the 'boundary layer equations', although they apply to jets and wakes as well as boundary layers. For the jet they are a fairly good approximation away from the near field of the jet, for about $x/D > 10$ to 20, depending on the nozzle characteristics.

The equations imply that $\bar{p}(x, y) \to \bar{p}(x)$, independent of $y$. This should hold then, for a given $x$, for $y$ values far from the jet. Therefore, for a pressure change (gradient) to occur in the jet, it must come from the ambient pressure field. If there is no appreciable pressure gradient in the ambient, which is the case for most jets and which will be assumed here, then $\bar{p}$ = constant everywhere. Then the equations reduce further to:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial}{\partial y}\overline{u'v'} \tag{13}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0\,. \tag{14}$$

To summarize the development of these equations, note the following:

1. The turbulent jet is assumed to be incompressible, two-dimensional, and steady state.
2. It is assumed that the jet is 'thin', i.e., $\ell_y/\ell_x \ll 1$.
3. Furthermore, it is assumed that the Reynolds number is large, i.e., $Re = U\ell_y/\nu \gg 1$.
4. The equations hold for roughly $x/D > 10 - 20$.
5. The ambient pressure is assumed to be uniform.
6. There are now 2 equations, but 3 unknowns, $\bar{u}$, $\bar{v}$, and $\overline{u'v'}$. So there is still a closure problem.

Note that if, instead of a two-dimensional jet, an axisymmetric jet is considered, and assuming that $\ell_r/\ell_x \ll 1$ and $Re = U\ell_r/\nu \gg 1$, where $\ell_r$ is a characteristic length scale in the radial direction, analogous to $\ell_y$ in the above discussion, then scaling arguments similar to the above lead to the following equations:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'v'}) \tag{15}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial (r\bar{v})}{\partial r} = 0\,. \tag{16}$$

Here $u$ and $v$ are the velocities in the axial ($x$) and radial ($r$) directions, respectively.