In turbulence modeling, the Reynolds stresses are usually written in terms of a turbulent (or "eddy") viscosity, i.e.,

$$-\overline{u'v'} = \nu_T \frac{\partial \overline{u}}{\partial y},$$

where the turbulent viscosity is defined by

$$\nu_T = c_T\, \upsilon_T \mathcal{L}_T .$$

Here c_T is a constant to be determined, and υ_T and $\mathcal{L}_T$ are characteristic velocity and length scales of the turbulence.

A popular and useful way to obtain υ and $\mathcal{L}_T$ is through obtaining the equations for the turbulent kinetic energy per unit mass, $\bar{k}$, and the turbulent dissipation rate per unit mass, ϵ, (both defined below). Once these are obtained, then they are related to the characteristic velocity and length scales as: $\upsilon^2 = \bar{k}$, and $\mathcal{L}_T = \bar{k}^{3/2}/\epsilon$.

To obtain a predictive equation for υ_T, consider first the Reynolds decomposition of the u-component of the velocity:

$$u = \overline{u} + u' .$$

Here $\overline{u}$ is the average, or mean, velocity, and u' represents the fluctuations about the average, the latter which is what contributes to the Reynolds stresses. Therefore in order to get an estimate for υ_T defining the turbulent viscosity, an estimate for u' is needed. Note that $\overline{u'} = 0$, so that its average cannot be used. The average that is useful is $\sqrt{\overline{u'^2}}$, the root-mean-square of u'. In the modeling, all three components are considered, in terms of the turbulent kinetic energy per unit mass, defined as

$$\overline{k} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right).$$

A dynamic equation for $\overline{k}$ is obtained in the following way. Here only the component $\frac{1}{2}\overline{u'^2}$ will be considered, as the derivation for the other two components is accomplished in the same manner. First consider the x-component of the momentum equation for an incompressible flow (neglecting the acceleration of gravity):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{1}$$

As shown in class, the time average of this equation is:

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right) - \frac{\partial}{\partial x}\overline{u'u'} - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial y}\overline{u'w'} . \tag{2}$$

An equation for u' can be obtained by first writing Equation (1) using the Reynolds decomposition for all the variables, and then subtracting Equation (2) from (1). Examples of terms to be dealt with are the following.

$$\frac{\partial}{\partial t}(\overline{u} + u') - 0 = \frac{\partial u'}{\partial t}$$

$$(\overline{u} + u')\frac{\partial}{\partial x}(\overline{u} + u') - \overline{u}\frac{\partial \overline{u}}{\partial x} = \overline{u}\frac{\partial \overline{u}}{\partial x} + u'\frac{\partial \overline{u}}{\partial x} + \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} - \overline{u}\frac{\partial \overline{u}}{\partial x} = u'\frac{\partial \overline{u}}{\partial x} + \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}$$

$$\frac{1}{\rho}\frac{\partial}{\partial x}(\overline{p}+p') - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} = \frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$\nu\frac{\partial^2}{\partial x^2}(\overline{u}+u') - \nu\frac{\partial^2 \overline{u}}{\partial x^2} = \nu\frac{\partial^2}{\partial x^2}u'$$

The result of this process is:

$$\begin{aligned}
\frac{\partial u'}{\partial t} &+ \overline{u}\frac{\partial u'}{\partial x} + \overline{v}\frac{\partial u'}{\partial y} + \overline{w}\frac{\partial u'}{\partial z} \\
&+ u'\frac{\partial \overline{u}}{\partial x} + v'\frac{\partial \overline{u}}{\partial y} + w'\frac{\partial \overline{u}}{\partial z} \\
&+ u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} \\
&= -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right) \\
&+ \frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}\,.
\end{aligned}$$

Now multiply this last equation by u' and average. Terms such as the following are obtained.

$$\overline{u'\frac{\partial u'}{\partial t}} = \frac{\partial}{\partial t}\frac{1}{2}\overline{u'^2} = 0 \quad \text{(stationary)}$$

$$\overline{u'\overline{u}\frac{\partial u'}{\partial x}} = \overline{u}\frac{\partial}{\partial x}\frac{1}{2}\overline{u'^2}$$

$$\overline{u'u'\frac{\partial \overline{u}}{\partial x}} = \overline{u'^2}\frac{\partial \overline{u}}{\partial x}$$

$$\overline{u'v'\frac{\partial u'}{\partial y}} = \overline{v'\frac{\partial}{\partial y}\frac{1}{2}u'^2}$$

$$\begin{aligned}
\nu\overline{u'\frac{\partial^2 u'}{\partial x^2}} &= \nu\overline{\frac{\partial}{\partial x}\left(u'\frac{\partial u'}{\partial x}\right)} - \nu\overline{\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}} \\
&= \nu\frac{\partial^2}{\partial x^2}\frac{1}{2}\overline{u'^2} - \nu\overline{\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}}
\end{aligned}$$

The final results is:

$$\begin{aligned}
\overline{u}\frac{\partial}{\partial x}\frac{1}{2}\overline{u'^2} &+ \overline{v}\frac{\partial}{\partial y}\frac{1}{2}\overline{u'^2} + \overline{w}\frac{\partial}{\partial z}\frac{1}{2}\overline{u'^2} \\
&+ \overline{u'^2}\frac{\partial \overline{u}}{\partial x} + \overline{u'v'}\frac{\partial \overline{u}}{\partial y} + \overline{u'w'}\frac{\partial \overline{u}}{\partial z} \\
&+ \frac{\partial}{\partial x}\overline{u'\frac{1}{2}u'^2} + \frac{\partial}{\partial y}\overline{v'\frac{1}{2}u'^2} + \frac{\partial}{\partial z}\overline{w'\frac{1}{2}u'^2} \\
&= -\frac{1}{\rho}\overline{u'\frac{\partial p'}{\partial x}} + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{2}\overline{u'^2} \\
&- \nu\left[\overline{\left(\frac{\partial u'}{\partial x}\right)^2} + \overline{\left(\frac{\partial u'}{\partial y}\right)^2} + \overline{\left(\frac{\partial u'}{\partial z}\right)^2}\right].
\end{aligned}$$

Here the fact that the fluctuating velocity field (u', v', w') satisfies the incompressible form of the conservation of mass was used.

When the same operations are carried out for the equations for v' and w', and the three equations are added together, we obtain the equation for $\overline{k}$:

$$\overline{u}\frac{\partial \overline{k}}{\partial x} + \overline{v}\frac{\partial \overline{k}}{\partial y} + \overline{w}\frac{\partial \overline{k}}{\partial z} = \mathcal{P} + \mathcal{D} - \epsilon\,. \tag{3}$$

Here $\mathcal{P}$, called the production term and usually resulting in an increase in $\overline{k}$, is given by:

$$\begin{aligned}
\mathcal{P} \;&=\; \overline{u'^2}\frac{\partial \overline{u}}{\partial x} + \overline{u'v'}\frac{\partial \overline{u}}{\partial y} + \overline{u'w'}\frac{\partial \overline{u}}{\partial z} \\
&+\; \overline{u'v'}\frac{\partial \overline{v}}{\partial x} + \overline{v'^2}\frac{\partial \overline{v}}{\partial y} + \overline{v'w'}\frac{\partial \overline{v}}{\partial z} \\
&+\; \overline{u'w'}\frac{\partial \overline{w}}{\partial x} + \overline{v'w'}\frac{\partial \overline{w}}{\partial y} + \overline{w'^2}\frac{\partial \overline{w}}{\partial z}\,.
\end{aligned}$$

The term $\mathcal{D}$ is called the turbulent diffusion term, and results in the turbulent kinetic energy $\overline{k}$ being moved around in space (from regions of high $\overline{k}$ to regions of low $\overline{k}$) by the turbulence without any overall gain or loss in $\overline{k}$, and is:

$$\mathcal{D} = -\frac{\partial}{\partial x}\left(\overline{p'u'} + \overline{u'k}\right) - \frac{\partial}{\partial y}\left(\overline{p'v'} + \overline{v'k}\right) - \frac{\partial}{\partial z}\left(\overline{p'w'} + \overline{w'k}\right).$$

Finally ϵ, the kinetic energy (pseudo-)dissipation rate, which is always positive, represents the conversion of mechanical energy into internal energy (heat) by the turbulence, and is given by:

$$\begin{aligned}
\epsilon \;&\doteq\; \nu\Bigg[\overline{\left(\frac{\partial u'}{\partial x}\right)^2} + \overline{\left(\frac{\partial u'}{\partial y}\right)^2} + \overline{\left(\frac{\partial u'}{\partial z}\right)^2} \\
&+\; \overline{\left(\frac{\partial v'}{\partial x}\right)^2} + \overline{\left(\frac{\partial v'}{\partial y}\right)^2} + \overline{\left(\frac{\partial v'}{\partial z}\right)^2} \\
&+\; \overline{\left(\frac{\partial w'}{\partial x}\right)^2} + \overline{\left(\frac{\partial w'}{\partial y}\right)^2} + \overline{\left(\frac{\partial w'}{\partial z}\right)^2}\Bigg].
\end{aligned}$$

Note that in the equation for $\overline{k}$ new modeling has to be introduced for the terms $\overline{p'u'}$ and $\overline{u'k}$, and similarly for terms with v' and w', in the expression for the turbulence diffusion $\mathcal{D}$. Also, ϵ must be treated.

In addition to this equation for $\overline{k}$, an equation for ϵ can also be obtained (by another even more tedious procedure). It is of the form:

$$\overline{u}\frac{\partial \epsilon}{\partial x} + \overline{v}\frac{\partial \epsilon}{\partial y} + \overline{w}\frac{\partial \epsilon}{\partial z} = \mathcal{P}_\epsilon + \mathcal{D}_\epsilon - \epsilon_\epsilon\,. \tag{4}$$

The terms on the right hand side have similar interpretations as for the corresponding terms in the equation for $\overline{k}$, i.e., $\mathcal{P}_\epsilon$ represents the production of ϵ, $\mathcal{D}_\epsilon$ its turbulent diffusion, and ϵ_ϵ its dissipation rate. Additional modeling is needed for each of these terms.

If $\overline{k}$ is known (i.e., has been computed), then the velocity scale can be identified as

$$\upsilon_T = \sqrt{\overline{k}}\,.$$

Also, given $\overline{k}$ and ϵ, a length scale can be defined as

$$\mathcal{L}_{\mathcal{T}} = \overline{k}^{3/2}/\epsilon \,,$$

the only dimensional combination of $\overline{k}$ and ϵ possible. So the turbulent viscosity is

$$\nu_T = c_\mu \, \upsilon_T \mathcal{L}_{\mathcal{T}} = c_\mu \, \overline{k}^{1/2} \overline{k}^{3/2}/\epsilon = c_\mu \, \overline{k}^2/\epsilon \,,$$

where the constant c_μ needs to be determined.

To close Equations (2), (3), and (4), some additional assumptions are needed. First of all, in the production term $\mathcal{P}$, there is the need to model velocity fluctuation products, i.e., the Reynolds stresses, e.g., $\overline{u'v'}$. As discussed earlier in class, the viscous stresses can be written in terms of the velocity gradients as:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} \,, \quad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} \,, \quad \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \,, \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \,, \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) .$$

In analogy with the viscous stresses, the following assumptions are made for the diagonal components of the Reynolds stresses per unit density:

$$-\frac{\rho\,\overline{u'^2}}{\rho} = -\frac{2}{3}\bar{k} + 2\nu_T\frac{\partial \bar{u}}{\partial x} \quad , \quad -\frac{\rho\,\overline{v'^2}}{\rho} = -\frac{2}{3}\bar{k} + 2\nu_T\frac{\partial \bar{v}}{\partial y} \quad , \quad -\frac{\rho\,\overline{w'^2}}{\rho} = -\frac{2}{3}\bar{k} + 2\nu_T\frac{\partial \bar{w}}{\partial z} \,.$$

Note that, using the conservation of mass,

$$\overline{u'^2} + \overline{v'^2} + \overline{w'^2} = 2\bar{k} - 2\nu_T\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) = 2\bar{k} \,,$$

as expected. Assumptions for the remaining Reynolds stress terms are the following.

$$-\frac{\rho\,\overline{u'v'}}{\rho} = \nu_T\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) \qquad -\frac{\rho\,\overline{u'w'}}{\rho} = \nu_T\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right) \qquad -\frac{\rho\,\overline{v'w'}}{\rho} = \nu_T\left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right) .$$

The fluxes in the diffusion term $\mathcal{D}$ are also modeled in analogy with molecular diffusion. The turbulent fluxes in the x-, y-, and z-directions are modeled as, respectively,

$$(\overline{u'p'} + \overline{u'k}) = -\frac{\nu_T}{\sigma_k}\frac{\partial \bar{k}}{\partial x} \quad , \quad (\overline{v'p'} + \overline{v'k}) = -\frac{\nu_T}{\sigma_k}\frac{\partial \bar{k}}{\partial y} \quad , \quad (\overline{w'p'} + \overline{w'k}) = -\frac{\nu_T}{\sigma_k}\frac{\partial \bar{k}}{\partial z} \,.$$

Here, in analogy with the Schmidt number, the turbulent Schmidt number is defined as $\sigma_k = \dfrac{\nu_T}{\nu_k}$, where ν_k is the turbulent diffusivity for $\bar{k}$. The constant σ_k thus needs to be determined.

In Equation (4) for ϵ, additional modeling is needed. To model the production of ϵ, i.e., $\mathcal{P}_\epsilon$, it is assumed that $\mathcal{P}_\epsilon = c_{\epsilon 1}(\epsilon/\bar{k})\mathcal{P}$. Here $c_{\epsilon 1}$ needs to be determined. The fluxes of ϵ in the x-, y-, and z-directions are modeled as, again in analogy with molecular diffusion:

$$-\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x} \quad , \quad -\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial y} \quad , \quad -\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial z}$$

where the constant σ_ϵ needs to be determined.

Finally, the dissipation rate of ϵ, i.e., ϵ_ϵ, is modeled as

$$\epsilon_\epsilon = c_{\epsilon 2}\epsilon^2/\bar{k}\,,$$

where the constant $c_{\epsilon 2}$ needs to be determined.

Numerous experiments have been carried out to determine the constants, and the generally accepted values are the following.

$$c_\mu \doteq 0.09 \quad , \quad c_{\epsilon 1} \doteq 1.44 \quad , \quad c_{\epsilon 2} \doteq 1.92 \quad , \quad \sigma_k \doteq 1.0 \quad , \quad \sigma_\epsilon \doteq 1.3\,.$$

The $\bar{k}$-ϵ model tends to work fairly well for simple flows, like two-dimensional jets and non-swirling axi-symmetric jets. For swirling jets, however, it does not work as well, since the swirling flow changes the dynamics, which is not incorporated into the $\bar{k}$-ϵ model. The same is true for flows in more complex geometries, or flows with additional effects, such as variable densities and/or compressibility. So one has to generally use the $\bar{k}$-ϵ model with caution. If it is used with a more complex flow, then the results may be, at most, qualitatively or semi-quantitatively correct.